A 2-approximation of the MRCT (SPT)

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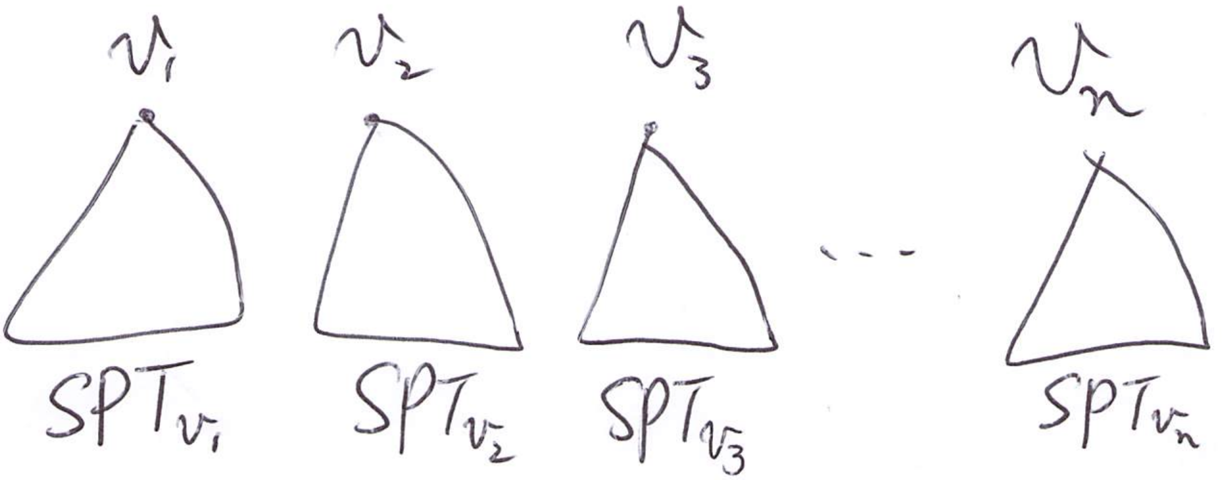
@2021

Thm. A shortest-paths tree rooted at the median of a graph is a 2-approximation of an MRCT of the graph.

MRCT : Minimum routing cost spanning tree

median : the vertex with minimum total distance
(r) to all vertices

$$r \leftarrow \arg\min_{v \in V} \boxed{f(v) = \sum_{u \in V} d_G(v, u)}$$



① $\sum_{u \in V} d_{SPT_{v_1}}(v_1, u) = \sum_{u \in V} d_G(v_1, u)$, ...

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$\hat{T}$: optimal MRCT

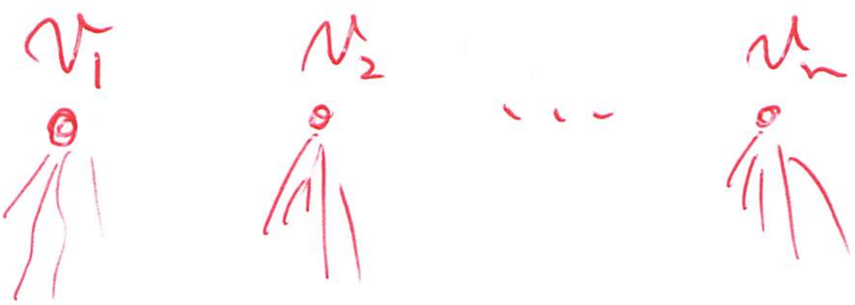
$$C(\hat{T}) = \underline{\sum_{u,v \in V} d_{\hat{T}}(u,v)}$$

$$\geq \underline{\sum_{u,v \in V} d_G(u,v)} \quad \left(\because \hat{T} \text{ is a subgragh of } G\right)$$

$$= \sum_u \boxed{\sum_v d_G(u,v)} \quad (\text{for all } u \in V) \;\; \geq \sum_v d_G(r,v)$$

median

$$\geq n \sum_v d_G(r,v)$$



n stars

← minimum star

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r: the median of G

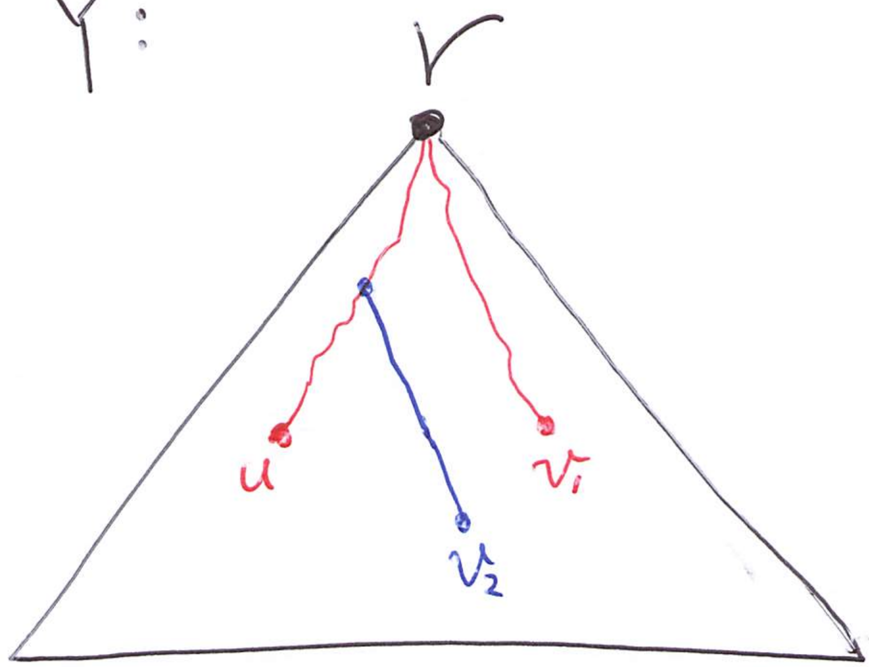
Y: a shortest-paths tree rooted at r

$$d_Y(r, v) = d_G(r, v) \quad \forall v \in V$$

$$d_Y(u, v) \leq d_Y(u, r) + d_Y(r, v)$$

for any $u, v \in V$

Y:



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$$C(Y) = \sum_{u,v} d_Y(u,v)$$

$$\leq \sum_{u}\sum_{v} \Big(\underbrace{d_Y(u,r)}_{\text{For any } u,\ \text{“}\sum_v\text{”} = n \text{ times}} + \underbrace{d_Y(r,v)}_{\text{For any } v,\ \text{“}\sum_u\text{”} = n \text{ times}} \Big)$$

$$= n \sum_{u} d_Y(u,r) + n \sum_{v} d_Y(r,v)$$

$$= 2n \sum_{v} d_Y(r,v)$$

$$= 2n \sum_{v} d_G(r,v)$$

$$\leq 2C(\hat{T})$$

$\boxed{Y \text{ is a 2-approximation of } \hat{T}}$

$\boxed{\text{Check Examples I, II, III}}$

Solution decomposition

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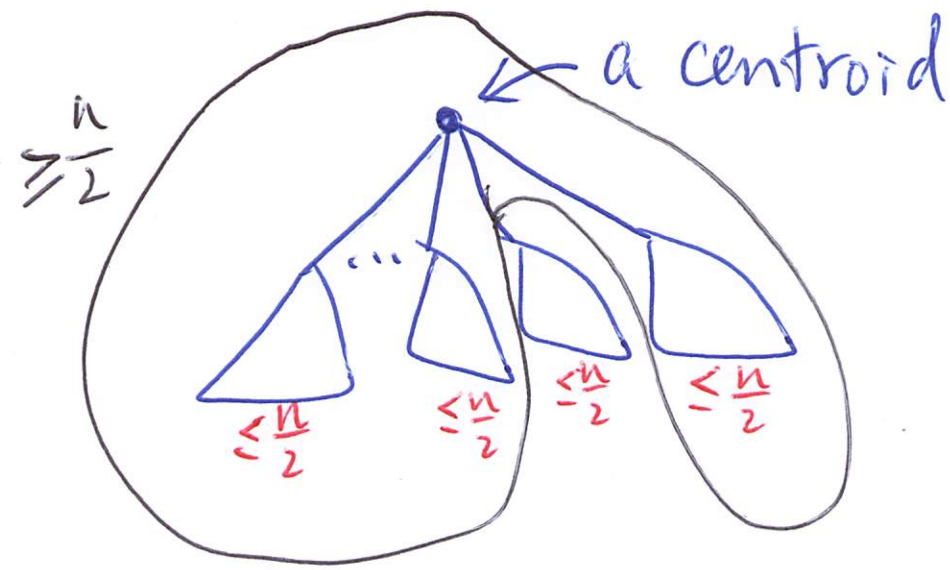
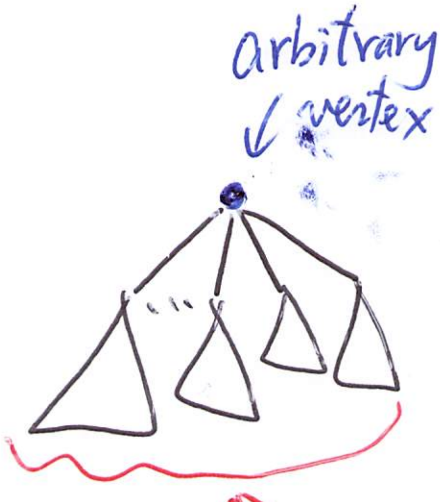
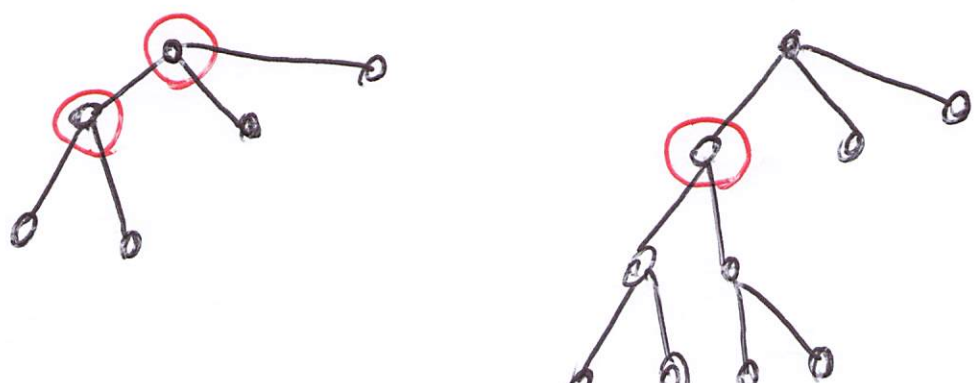
X : an optimal solution

Decompose $X \longrightarrow \underbrace{Y}_{\text{an approximation}}$





At most one branch with $> \frac{n}{2}$ vertices.



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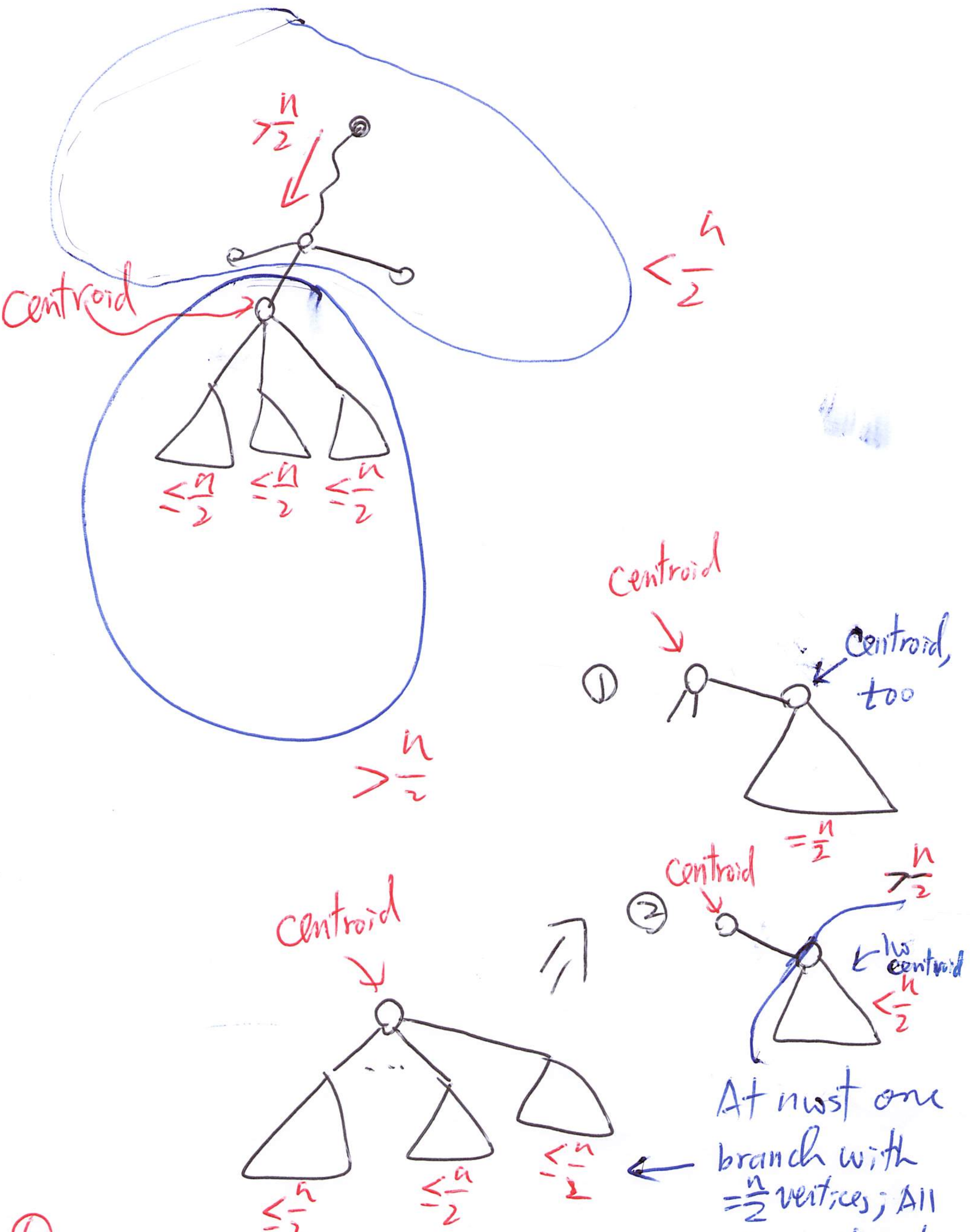

At most one branch with $=\frac{n}{2}$ vertices; All others: $<\frac{n}{2}$ vertices

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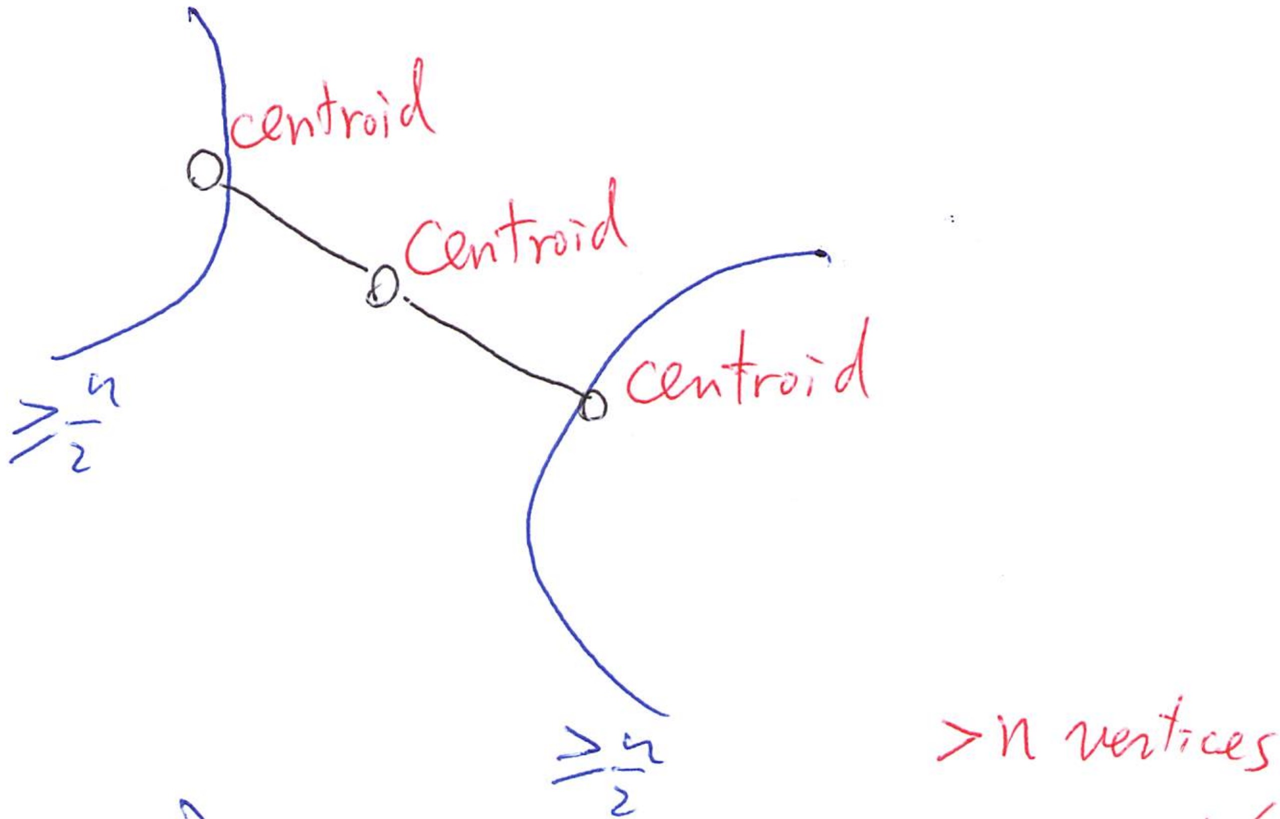
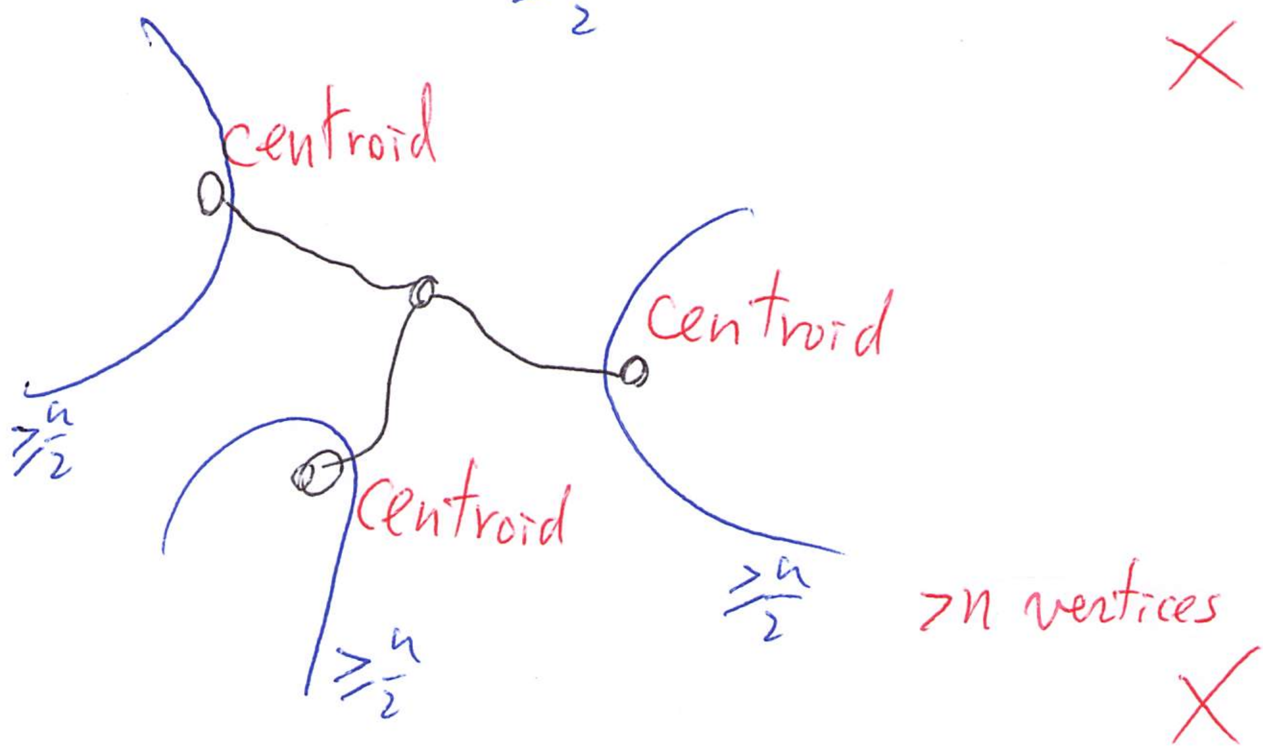
A tree can have at most two centroids.





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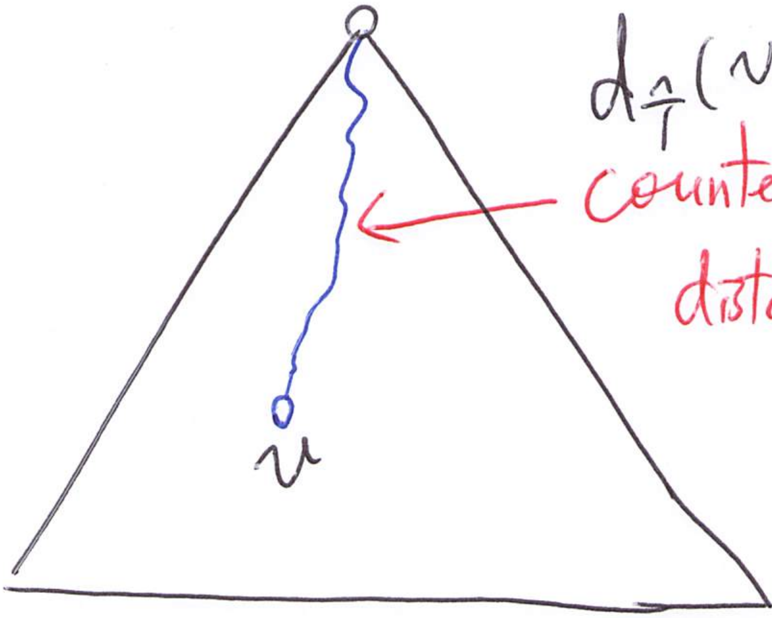
$\hat{T}$:

x: centroid



$d_{\hat{T}}(v, x)$

counted in the total distance at least $\frac{n}{2} \times 2$ times

$\Downarrow$

$$C(\hat{T}) \geq n \sum_{v} d_{\hat{T}}(v, x)$$

Let Y be a shortest-paths tree rooted at the centroid x.

$$\begin{aligned}
C(Y) &= \sum_{u} \sum_{v} d_Y(u, v) \\
&\leq \sum_{u} \sum_{v} d_Y(u, x) + \sum_{u} \sum_{v} d_Y(x, v) \\
&= 2n \sum_{v} d_Y(v, x) \\
&= 2n \sum_{v} d_G(v, x) \leq 2n \sum_{v} d_{\hat{T}}(v, x) \leq 2C(\hat{T})
\end{aligned}$$

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We have shown that a shortest-paths tree rooted at the centroid of an optimal MRCT is a 2-approximation solution.

The question is that we don't have an optimal MRCT at hand, not to mention its centroid.

Try all vertices, find shortest-paths tree rooted at each vertex and the best of them is guaranteed to be a 2-approximation of $\hat{T}$.

$$c(\square) \le c(Y) \le 2c(\hat{T})$$ #