Tree Eccentricities, diameters, and radii   Kun Mao Chen
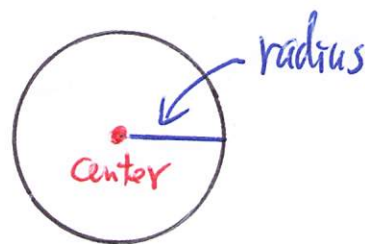
$G = (V, E, \omega)$

ω: positive edge weights

$U \subseteq V$

$$\underline{\underline{D_G}}(v, U) = \max_{u \in U} \{ d_G(v, u) \}$$

The *eccentricity* of v is the maximum of the distance to any vertex in the graph, i.e. $\max_{u \in V} \{ d_G(v, u) \}$ or
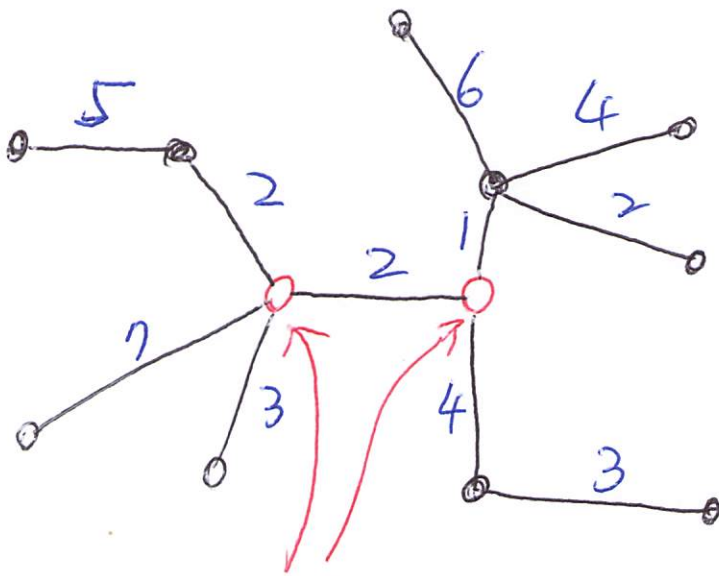
$$D_G(v, V).$$

Diameter: $\max_{v \in V} \{ \max_{u \in V} \{ d_G(v, u) \} \} = \max_{u,v} d_G(u,v)$

↑ also referred to as a path

Radius: $\min_{v \in V} \{ \max_{u \in V} \{ d_G(v, u) \} \}$

① Center: the vertex with the minimum eccentricity

5
6
4
2
1
2
2
7
3
4
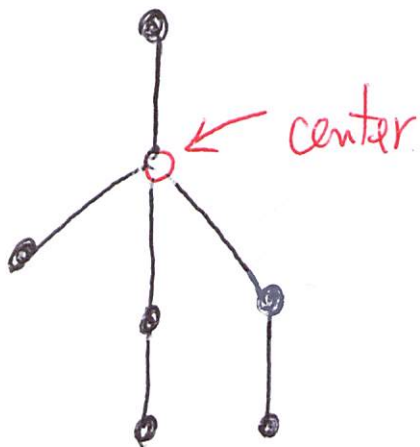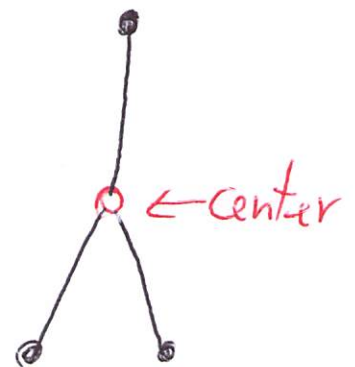3

Centers

radius = 9
diameter = 16

← center

radius = 2
diameter = 4

Centers
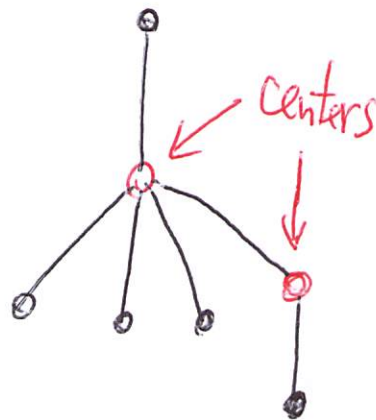
radius = 2
diameter = 3

← center

radius = 1
diameter = 2

②

For an unweighted tree T,

$$2 \times radius - 1 \leq diameter \leq 2 \times radius.$$
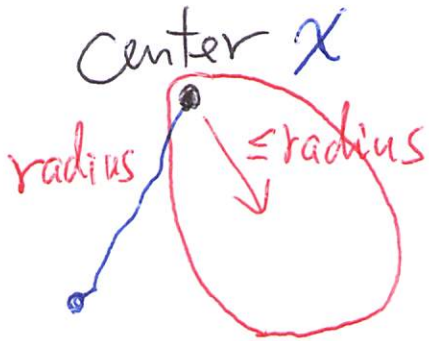
center x

radius $\leq radius$

$$d_T(u,v) \leq d_T(u,x) + d_T(x,v)$$
$$\leq radius + radius$$
$$= 2 \times radius$$
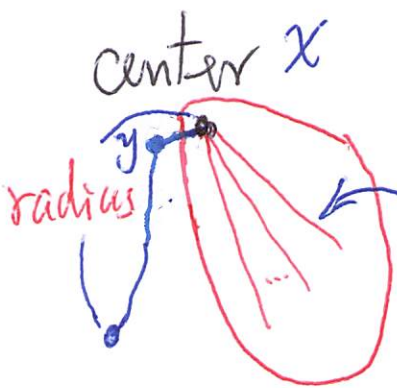
$\Rightarrow diameter \leq 2 \times radius$
$(\max\limits_{u,v} d_T(u,v))$

center x

y

radius

At least one branch $\geq radius - 1$

(Otherwise, we have a smaller radius by selecting y as the center.)

$\Rightarrow diameter \geq radius + radius - 1$
$$= 2 \times radius - 1$$

For positive weighted tree T,

$$2 \times radius - \max_{e \in E}\{w(e)\} \leq diameter \leq 2 \times radius$$

$$d_T(u,v) \leq d_T(u,x) + d_T(x,v)$$
$$\leq 2 \times radius$$

center

Center x

radius

y
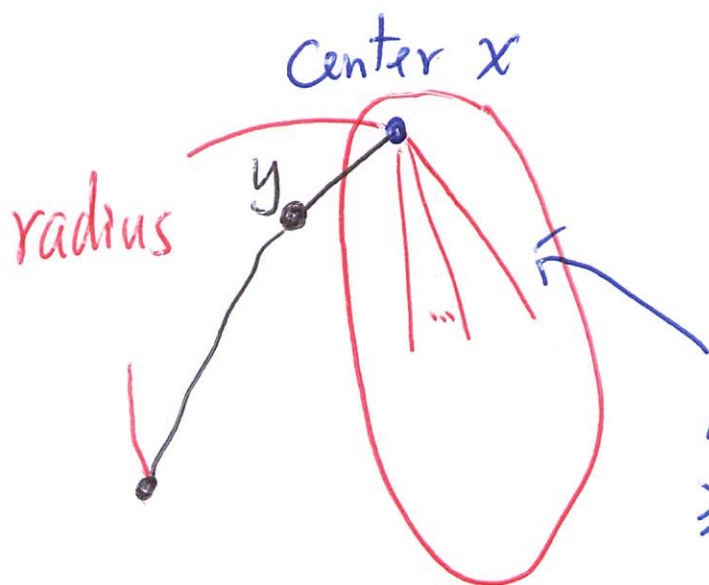
At least one branch
$$\geq radius - w((x,y)) \geq radius - \max_{e \in E}\{w(e)\}$$
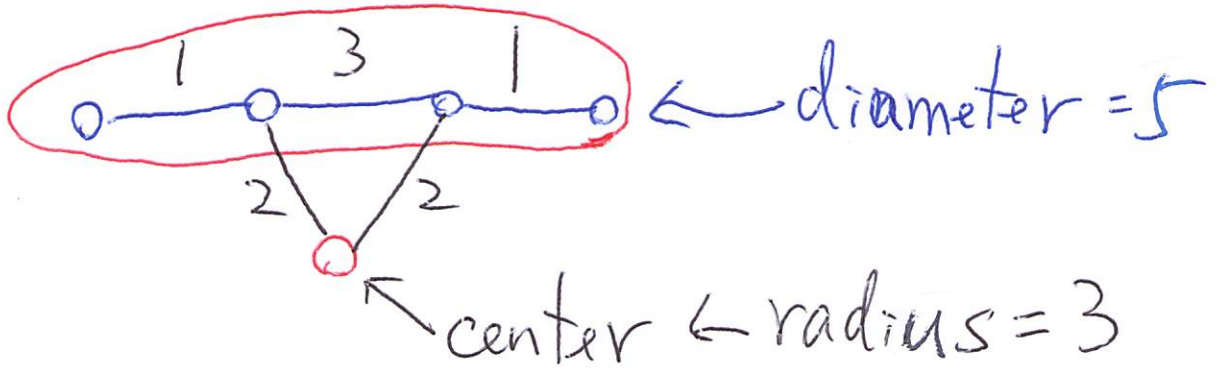
(Otherwise, we have a smaller radius by selecting y as the center.)

$$\Rightarrow diameter \geq radius + radius - \max_{e \in E}\{w(e)\}$$
$$= 2 \times radius - \max_{e \in E}\{w(e)\}$$

④

Examples (centers not in the diameter)   Kun-Mao Chao @2021

Weighted graph:

0 --1-- o --3-- o --1-- 0 ← diameter = 5

2   2

center ← radius = 3

unweighted graph:

← diameter = 3

All three vertices are centers.
radius = 2

5 4 4 4 4 5 ← diameter = 5

4   4

4

eccentricity   center
radius = 3

4

④-1
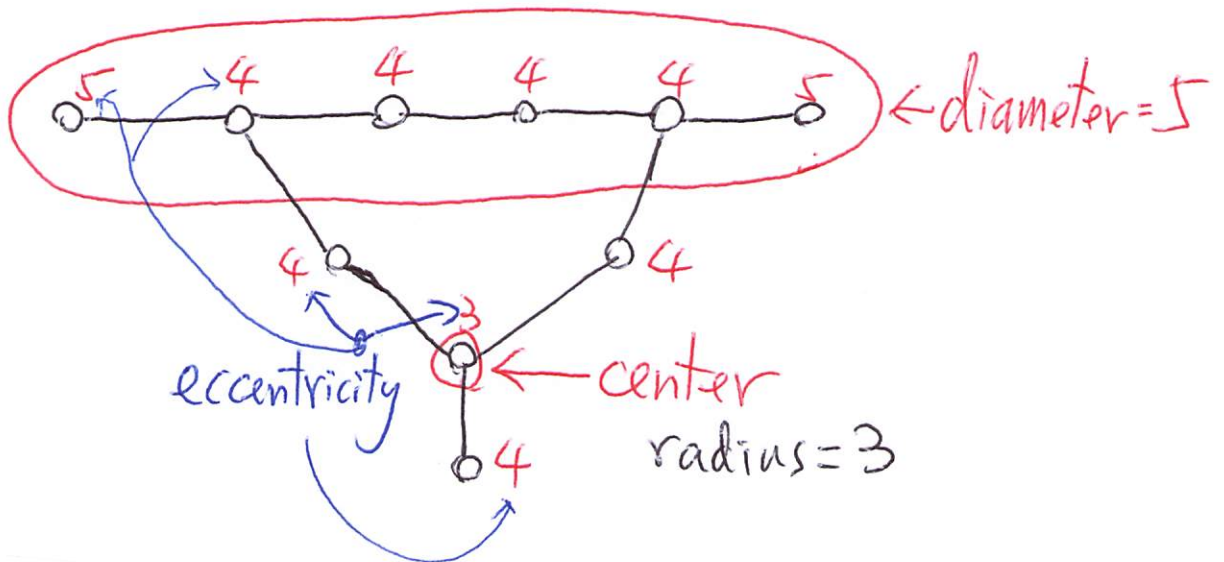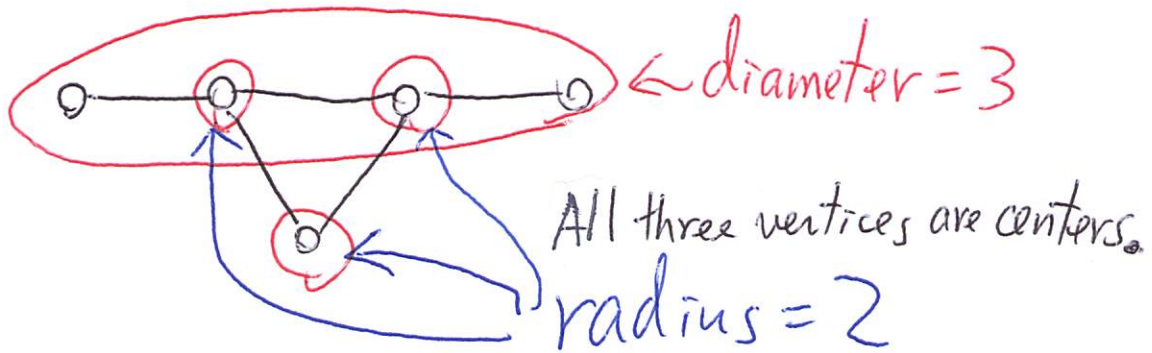
T_r : a subtree rooted at r

T_r :
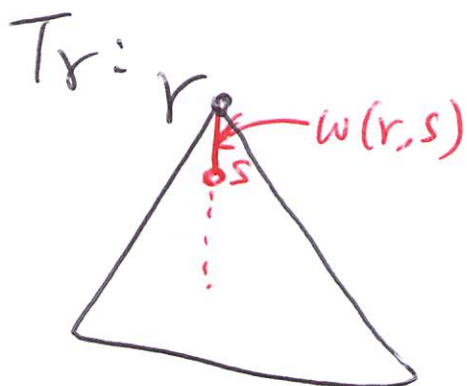
$$D_{T_r}(r, V(T_r)) =$$
$$\max_{s \in child(r)} \left\{ D_{T_r}(s, V(T_s)) + w(r, s) \right\}$$

We can root the tree at any given vertex v and compute v's eccentricity in linear time by the above recurrence relation. (Postorder traversal or recursive algorithm)

bottom-up

$$C(x,y,z) = SP_T(x,y) \cap SP_T(x,z) \cap SP_T(y,z)$$
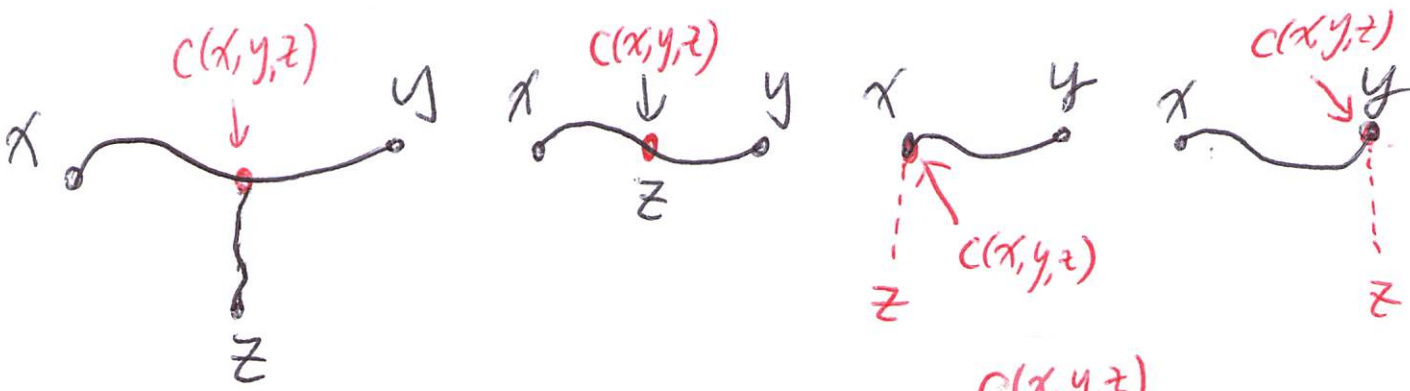
$$d_T(x, C(x,y,z))$$

$$= \frac{1}{2}\left(d_T(x,y) + d_T(x,z) - d_T(y,z)\right)$$

$$= \frac{1}{2}\left(d_T(x,y) + d_T(x,z) + d_T(y,z)\right) - d_T(y,z)$$

⑥
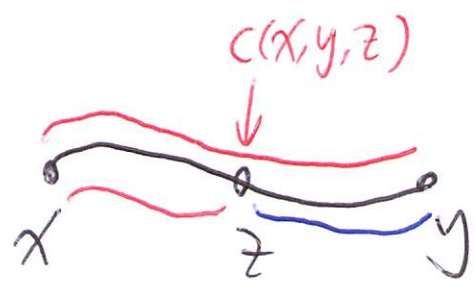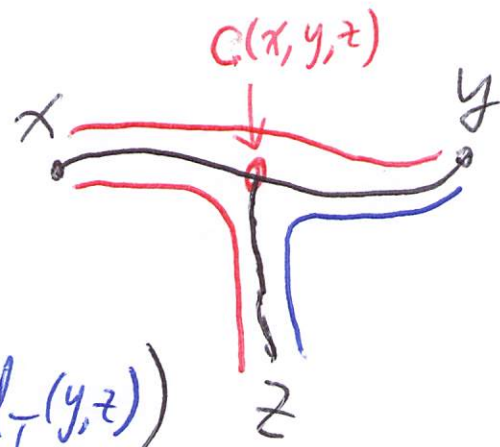
$SP_T(v_1, v_2)$: diameter

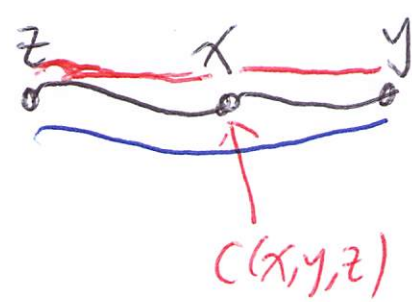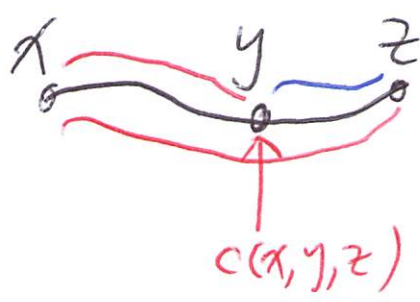r : a vertex on $SP_T(v_1, v_2)$

diameter

For any vertex x, $d_T(x, r) \le \max\{d_T(r, v_1), d_T(r, v_2)\}$.

pf. If $d_T(x, r) > \max\{d_T(r, v_1), d_T(r, v_2)\}$,

then $SP_T(x, v_1)$ or $SP_T(x, v_2)$ is a path

longer than the diameter.

$SP_T(x, v_2) >$
$SP_T(v_1, v_2)$ $\Leftarrow x$

$x \Rightarrow SP_T(x, v_1) >$
$SP_T(v_1, v_2)$

$\Rightarrow SP_T(x, v_1) > SP_T(v_1, v_2)$
$SP_T(x, v_2) > SP_T(v_1, v_2)$

⑧
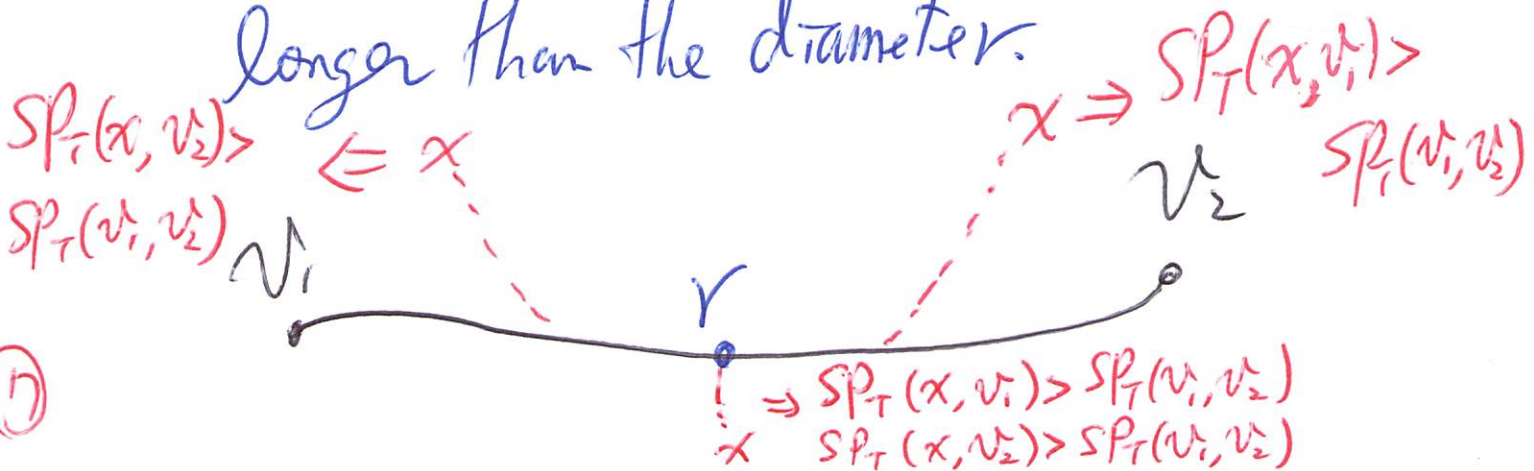
$SP_T(v_1, v_2)$: diameter

r : a vertex not on the diameter

$$u = c(r, v_1, v_2)$$
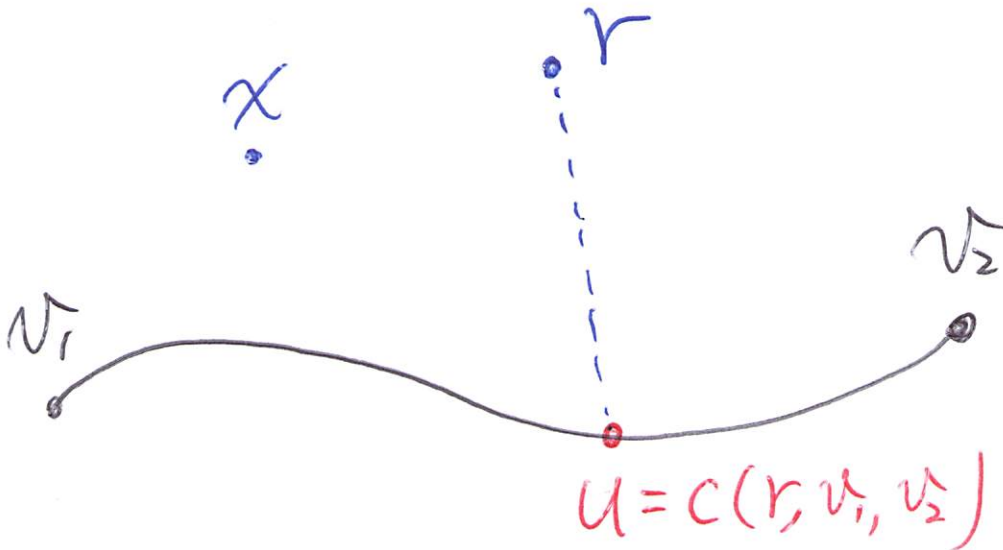
Wlog, let $d_T(u, v_1) \geq d_T(u, v_2)$.

on the diameter

$$d_T(x, r) \leq d_T(x, u) + d_T(u, r)$$
$$\leq d_T(v_1, u) + d_T(u, r)$$
$$= d_T(v_1, r)$$

⭑For any vertex, either on the diameter or not, one of the endpoints of a diameter

⑧ must be the farthest vertex.

$SP_T(v_1, v_2)$: diameter

r: any vertex

$\Rightarrow$ v_3 must be an endpoint of a diameter.

v_3: the vertex farthest to r

r

v_1

v_2

$u_1 = c(r, v_1, v_2)$

Wlog, let $d_T(u_1, v_1) \geq d_T(u_1, v_2)$.

r
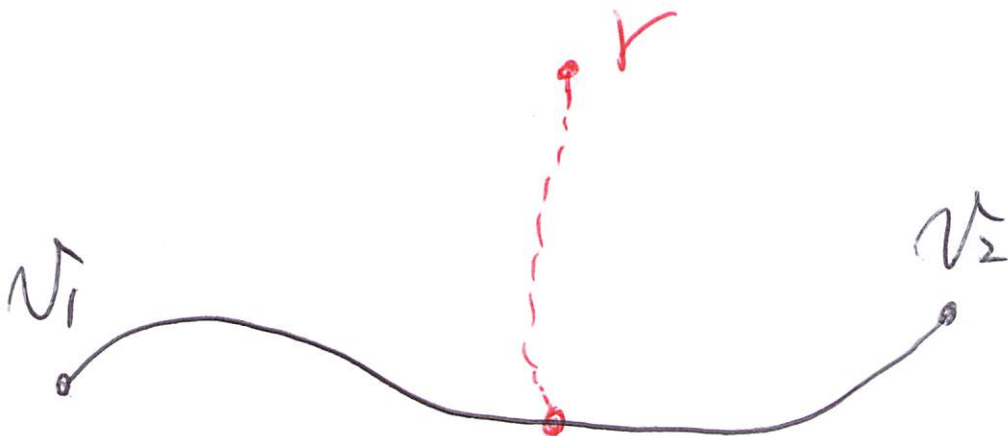
$d_T(v_1, v_3)$
$= d_T(v_1, u_2) + d_T(u_2, v_3)$
$= 2 d_T(v_1, u_2)$
$> 2 d_T(v_1, u_1)$
$\geq d_T(v_1, v_2)$

v_3

$u_2 = c(r, v_1, v_3)$ v_2

v_1

v_3

(ii)
u_2

u_1

(i)
$c(r, v_3, v_2)$
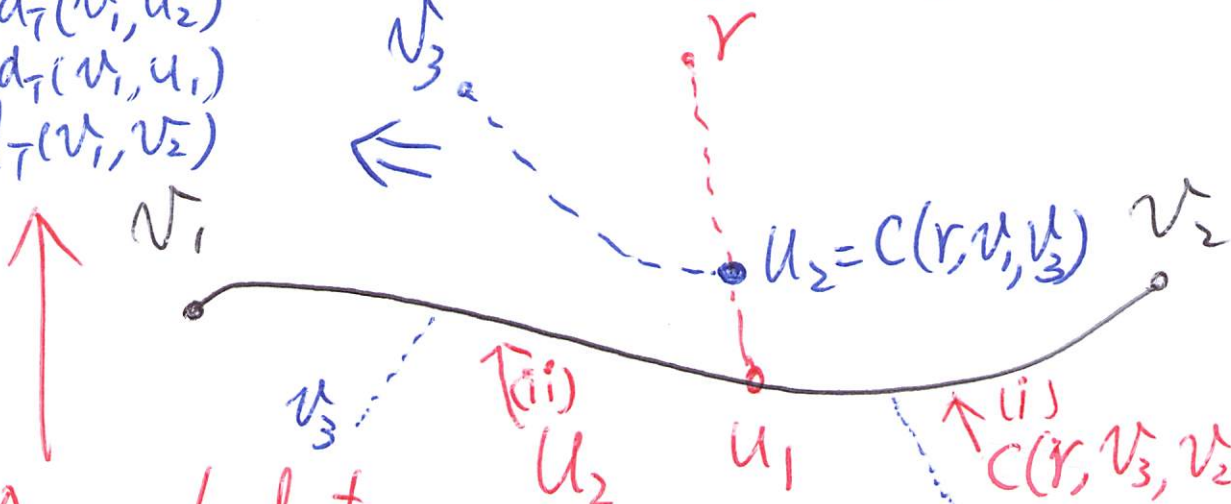
A contradiction.

$\Rightarrow u_2$ is on $SP_T(v_1, u_1)$.

v_3

If $c(r, v_3, v_2) \neq u_1$, then (i) $SP_T(v_1, v_3)$ is a diameter.

(9)

Otherwise, (ii) $SP_T(v_3, v_2)$ is a diameter.

Algorithm: ECCENT(T_r)
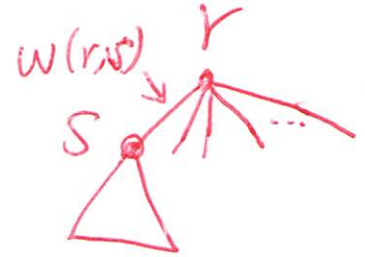**Input:** A tree $T_r = (V, E, w)$ rooted at r.
Output: The eccentricity of r in T_r.
1:     if r is a leaf **then**
         return 0;
2:     **for** each child s of r **do**
         compute ECCENT(T_s) recursively;
3:     return $\max_{s \in child(r)}\{\text{ECCENT}(s) + w(r, s)\}$.

$w(r,s)$ r
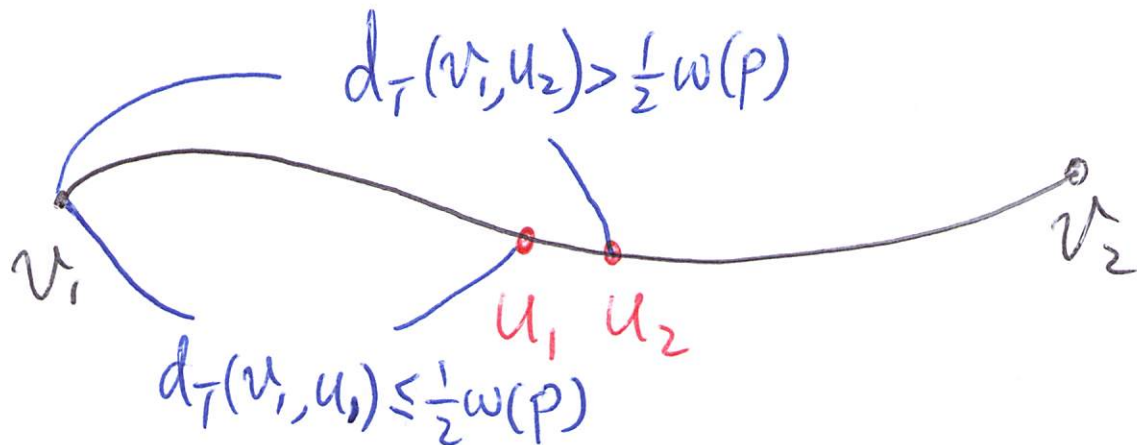
s

Algorithm: TREEDIAMETER
Input: A tree $T = (V, E, w)$.
Output: The diameter of T.
1:     Root T at an arbitrary vertex r.
2:     Use ECCENT to find the farthest vertex v to r.
3:     Root T at v.
4:     Use ECCENT to find the eccentricity of v.
5:     Output the eccentricity of v as the diameter of T.

an endpoint of a diameter
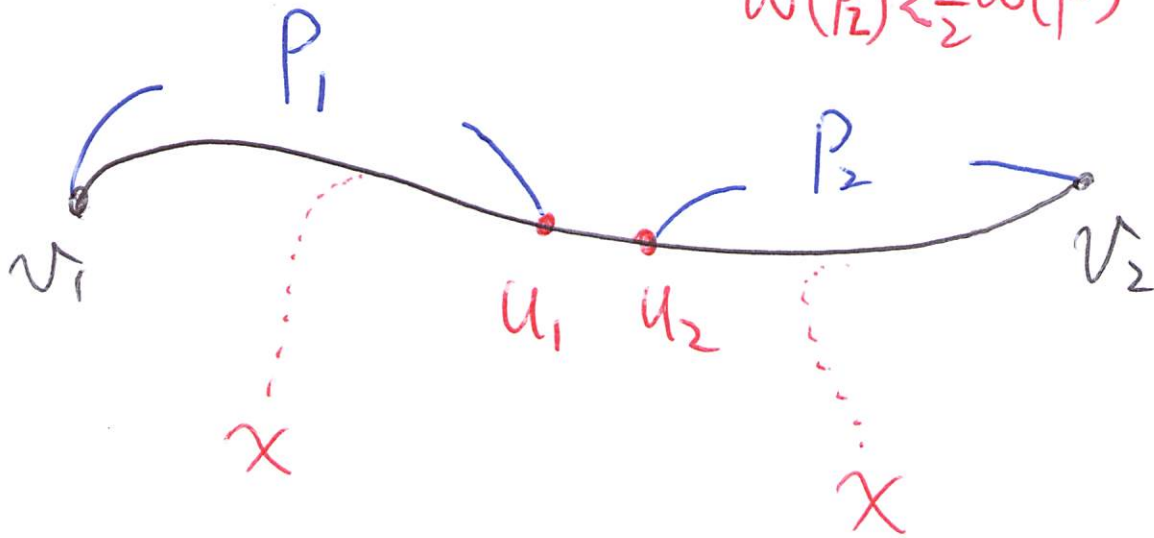
$$P = SP_T(v_1, v_2) = \text{diameter}$$

$$d_T(v_1, u_2) > \tfrac{1}{2} w(P)$$

v_1

$u_1 \quad u_2$

v_2

$$d_T(v_1, u_1) \leq \tfrac{1}{2} w(P)$$

The eccentricity of u_1 is $d_T(u_1, v_2)$.

The eccentricity of u_2 is $d_T(u_2, v_1)$.

$P:$

$W(P_1) \leq \frac{1}{2} W(P)$

$W(P_2) < \frac{1}{2} W(P)$

P_1

P_2

V_1

$u_1 \quad u_2$

V_2

X

X

For any vertex connected to P at a vertex in P_1, $d_T(x, V_2) > d_T(u_1, V_2)$.

$\longleftarrow$ u_1's eccentricity

For any vertex connected to P at a vertex in P_2, $d_T(x, v_1) > d_T(u_2, v_1)$.

$\longrightarrow$ u_2's eccentricity

$\Rightarrow \min \{ d_T(u_1, v_2), d_T(u_2, v_1) \}$ is the radius.
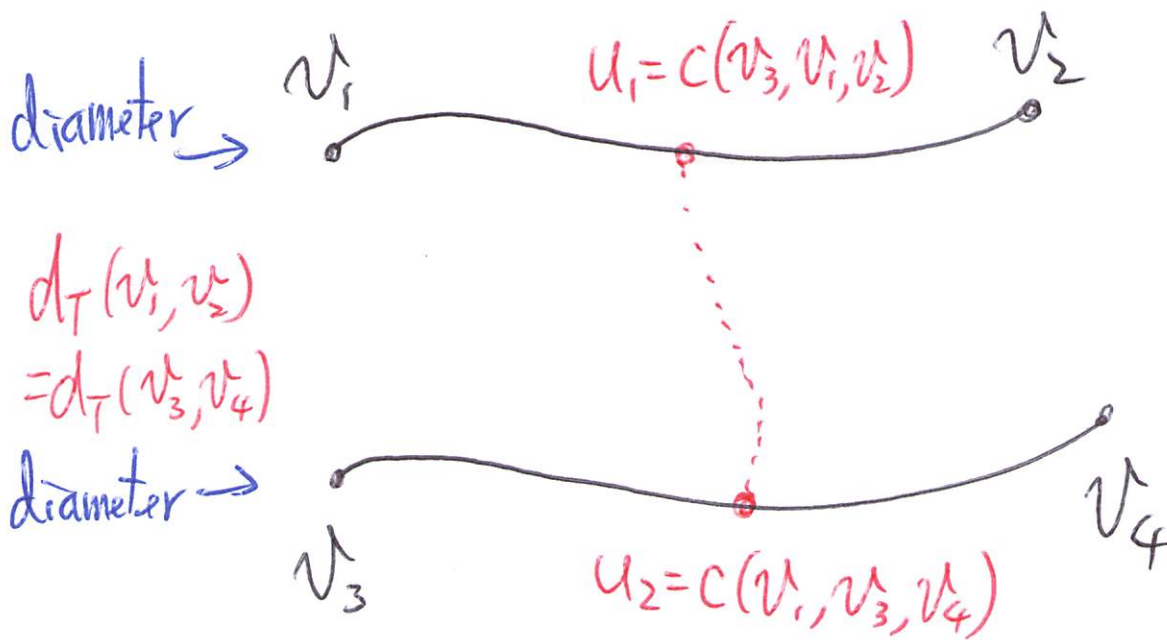
One of them must be a center.

We have proved the following theorem. Kun-Mao Chao@2019

★ Thm. The <u>diameter</u>, <u>radius</u>, and <u>center</u> of a tree can be computed <u>in linear time</u>.

Two diameters of a tree cannot be disjoint.

diameter → v_1 $u_1 = C(v_3, v_1, v_2)$ v_2

$d_T(v_1, v_2)$
$= d_T(v_3, v_4)$

diameter → v_3 $u_2 = C(v_1, v_3, v_4)$ v_4

$$d_T(v_1, v_4) + d_T(v_2, v_3)$$

$$= d_T(v_1, v_2) + d_T(v_3, v_4) + 2d_T(u_1, u_2) \quad >0$$

$$> 2 d_T(v_1, v_2)$$

$$\Rightarrow \underline{d_T(v_1, v_4) > d_T(v_1, v_2)} \text{ or } \underline{d_T(v_2, v_3) > d_T(u_1, u_2)}$$

A contradiction!

One of them is longer than the diameter.

⑫

All diameters of a tree share at least one common vertex.