The Size Ramsey Number of Graphs with Bounded Treewidth

Original Authors

Nina Kamcev, Anita Liebenau, David R. Wood, and Liana Yepremyan

Presenters

Yung-Zhung Chiu, Zhe-An Li, and Kai-Yuan Lai

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Introduction

Ramseyness

Definition. Let G, H and H' be graphs. We say that G is Ramsey for (H, H'), denoted by

$$G \to (H, H'),$$

if every red/blue-coloring of the edges of G yields either a red copy of H or a blue copy of H'. Let $G \to H$ denote that $G \to (H, H)$.

Ramsey Numbers

The usual and size Ramsey numbers are defined as follows.

Definition. For graphs H and H', the usual Ramsey number $r(H, H')$ is defined by

$$r(H, H') = \min\{|V(G)| : G \to (H, H')\},$$

and let $r(H) = r(H, H)$.

Definition. For graphs H and H', the size Ramsey number $\hat{r}(H, H')$ is defined by

$$\hat{r}(H, H') = \min\{|E(G)| : G \to (H, H')\},$$

and let $\hat{r}(H) = \hat{r}(H, H)$.

A Recurrence

Theorem. For any positive integers n and m,

$$r(K_n, K_m) \leq r(K_{n-1}, K_m) + r(K_n, K_{m-1}).$$

Corollary. For any positive integers n and m,

$$r(K_n, K_m) \leq \binom{n+m-2}{n-1}.$$

Asymptotics

The diagonal Ramsey numbers are bounded by

$$\sqrt{2}e^{-1}(1+o(1))n2^{n/2} \quad \leq \quad r(K_n) \quad \leq \quad n^{-\Theta(\log n/\log\log n)}4^n.$$

A Hard Challenge

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $r(K_5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $r(K_6)$. In that case, he believes, we should attempt to destroy the aliens.

— Joel Spencer

A Hard Challenge

Note that there are $2^{n(n-1)/2}$ different red/blue-coloring of K_n. Thus, it is hard to compute the Ramsey numbers simply via brute force since the time complexity is exponential.

Known Upper and Lower Bounds

m	1	2	3	4	5	6
$r(K_1, K_m)$	1					
$r(K_2, K_m)$	1	2				
$r(K_3, K_m)$	1	3	6			
$r(K_4, K_m)$	1	4	9	18		
$r(K_5, K_m)$	1	5	14	25	43–48	
$r(K_6, K_m)$	1	6	18	36–41	58–87	102–165

Table: Known Bounding Ranges for Ramsey Numbers $r(K_n, K_m)$

A Simple Observation

The size Ramsey number is bounded by the usual Ramsey number.

Theorem. For any graphs H and H',

$$\hat{r}(H, H') \leq \binom{r(H, H')}{2}.$$

It is known that if H and H' are complete graphs, then the equality holds.

Size Ramsey Number of Sparse Graphs

However, when H and H' are sparse, the size Ramsey number may be small. Currently we have the following results.

- Stars: $\hat{r}(K_{1,n}, K_{1,m}) = n + m - 1$.
- Paths: $\hat{r}(P_n) = \Theta(n)$, and $\hat{r}(P_n) < 137n$ when $n \to \infty$.
- Cycles: $\hat{r}(C_n) = \Theta(n)$.
- Trees: $\hat{r}(T) = \Theta(\beta(T))$ for any tree T, where $\beta(\cdot)$ is a parameter satisfying $\beta(T) \leq |V(T)|\Delta(T)$.

Terminologies

Strong Product

The strong product of two graphs is defined as follows.

Definition. Let G and H be graphs. The strong product of G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, where (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$ if

- $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or
- $u_1 u_2 \in E(G)$ and $v_1 = v_2$, or
- $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$.

Examples of Strong Products

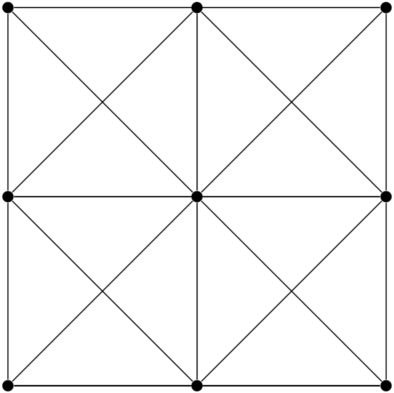

Figure: The strong product of P_3 and P_3.

Examples of Strong Products

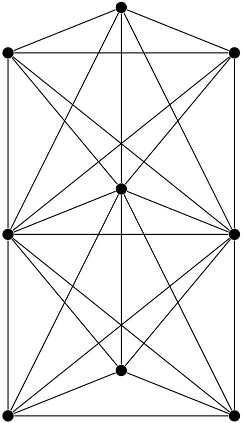

Figure: The strong product of P_3 and K_3.

Graph Decomposition

Definition. For graphs G and H, an H-decomposition of G is a graph $\mathcal{H} \cong H$ whose vertices, called bags, are subsets of $V(G)$, such that the following properties hold.

(a) For each $v \in V(G)$, the subgraph of $\mathcal{H}$ induced by $\{B \in V(\mathcal{H}) : v \in B\}$ is nonempty and connected.

(b) For each $uv \in E(G)$, $\{u, v\} \subseteq B$ for some $B \in V(\mathcal{H})$.

The width of $\mathcal{H}$ is the size of the largest bag of $\mathcal{H}$ minus one.

Tree Decomposition

Definition. A tree decomposition of a graph G is a T-decomposition for some tree T.

Definition. The treewidth of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G.

Tree Decomposition

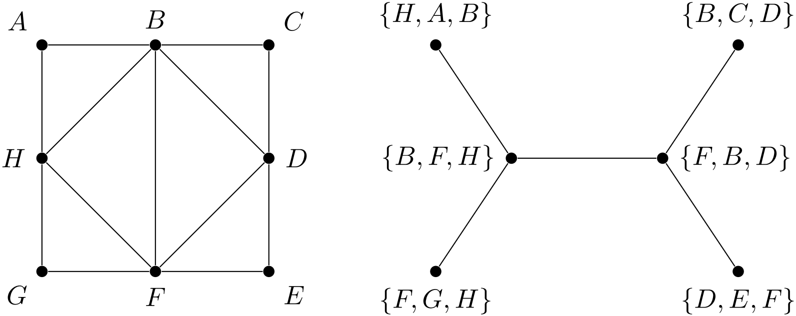

Figure: A graph G and a tree decomposition of G with width 2.

Treewidth

Given a graph G and a positive integer k, it is NP-complete to determine whether $\mathrm{tw}(G) \leq k$.

However, if k is a fixed constant, then one can easily recognize the graphs G with $\mathrm{tw}(G) \leq k$, and a tree decomposition of G with width k can be constructed in linear time.

Treewidth is a crucial and well-studied parameter of graphs since many NP-complete problems (e.g., maximum independent set) can be solved efficiently via dynamic programming on graphs of bounded treewidth.

Degeneracy

Definition. Let G be a graph, and let k be a positive integer. A graph G is said to be k-degenerate if every subgraph of G has minimum degree at most k.

Definition. The degeneracy of a graph G is the minimum positive integer k such that G is k-degenerate.

Note that all graphs of treewidth k are k-degenerate, while treewidth cannot be bounded in terms of degeneracy.

Lovász Local Lemma

If a large number of events are all independent of one another and each has probability less than 1, then with a positive probability none of these events occur.

The following lemma, called Lovász Local Lemma, generalizes this result by relaxing the independence condition slightly.

Lovász Local Lemma. Let $\mathcal{E}$ be a finite collection of events, and let d be a positive integer. If for each $E \in \mathcal{E}$, we have $\Pr(E) \leq 1/(4d)$ and there are at most d events in $\mathcal{E} \setminus \{E\}$ that are not independent from E, then

$$\Pr\left(\bigcup_{E \in \mathcal{E}} E\right) < 1.$$

Lovász Local Lemma

Lovász Local Lemma is a classic tool in probability graph theory since it can imply the existence of a specific object. However, the original proof of Lovász Local Lemma does not provide an explicit way to avoid all events in $\mathcal{E}$.

There was no constructive proof of Lovász Local Lemma found until Robin Moser and Gábor Tardos proposed an expected polynomial-time algorithm in 2010. This result won them the Gödel Prize in 2020.

Main Results

Main Results—Theorem 1

Theorem 1. For any positive integers k and d, there exists a constant c such that for any positive integer n, there exists a graph G with cn vertices and maximum degree c, such that

$$G \to (H, H')$$

holds for any graphs H and H' with n vertices, treewidth k and maximum degree d.

Corollary. For any positive integers k and d, there exists a constant c such that

$$\hat{r}(H) \leq c|V(H)|$$

for any graph H with $\mathrm{tw}(H) \leq k$ and $\Delta(H) = d$.

Main Results—Theorem 2

Theorem 2. For any positive integer k, there exists a tree T such that

$$G \nrightarrow T$$

for any k-degenerate graph G.

Proof of Theorem 1

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Proof Sketch of Theorem 1

The proof sketch of Theorem 1 is as follows.

- Reduce the problem to the case $T \boxtimes K_k$, where T is a tree.
- Obtain a regular graph H (which satisfies specific conditions) and define $G = H^3 \boxtimes K_r$, where $r = r(K_t)$ for some large constant t. Fix a red/blue-coloring ψ of G.
- Now we focus on the blue t-cliques in G. Construct an M-clique K_M, where M is the number of the blue t-cliques in G, and design a red/blue-coloring φ according to ψ and a parameter s.
- Consider the two different cases and show that both impliy the existence of a monochromatic copy of $T \boxtimes K_k$, finishing the proof.

Reduction

Let $\Delta(H)$ denote the maximum degree of vertices of H.

Lemma 1. For any graph H with $\mathrm{tw}(H) = k$ and $\Delta(H) = d$, there exists a tree T with $\Delta(T) \leq 18kd^2$ such that H is a subgraph of $T \boxtimes K_{18kd}$.

By Lemma 1, we can reduce Theorem 1 to the following case, where $\mathcal{T}_{n,d}$ denotes the collection of trees T with $|V(T)| = n$ and $\Delta(T) \leq d$.

Theorem. For any positive integers k and d, there exists a constant c such that for any positive integer n, there exists a cn-vertex graph G with $\Delta(G) \leq c$ such that for any $T_1, T_2 \in \mathcal{T}_{n,d}$, each red/blue-coloring of G contains either a red copy of $T_1 \boxtimes K_k$ or a blue copy of $T_2 \boxtimes K_k$.

A Key Tool

We pick specific numbers N, D and ϵ and apply the following lemma to obtain a regular graph H.

Lemma 2. For any positive integer d, any $\epsilon > 0$ and any even number $D > 100d^2/\epsilon^4$, there exists a constant c such that for any positive integers n and $N \geq cn$, there exists an N-vertex D-regular graph H with the following properties.

(a) If S and T are disjoint subsets of $V(H)$ and each contains at least $2N/\sqrt{D}$ vertices, then there is an edge of H connecting S and T.

(b) Each induced subgraph of H on at least ϵN vertices contains all trees T with $|V(T)| \leq n$ and $\Delta(T) \leq d$.

Let $r = r(K_t)$ with $t = (64kd)^{k(d^2+d)}$. We claim that $G = H^3 \boxtimes K_r$ satisfies the theorem.

Cliques

Now fix a red/blue-coloring ψ of the edges of G.

By $G = H^{(3)} \boxtimes K_r$, each $v \in V(H)$ corresponds to an r-clique $[v]$, which contains a monochromatic t-clique $[v]^*$ since

$$K_r \to K_t.$$

Let G^* be the subgraph of G induced by the blue t-cliques, and let M denote the number of the blue t-cliques. Suppose that $M \geq N/2$ without loss of generality.

Another Coloring

Now that we have M blue t-cliques, we can construct the (G^*, ψ, s)-coloring φ of K_M according to the following rule to describe the pattern of the red/blue-colored edges between these blue t-cliques. Here $s = k(d^2 + d)$ is a constant.

Definition. Let G be a graph with vertices partitioned into $V_1, V_2, \ldots, V_M$, and let ψ be a red/blue-coloring of $E(G)$. The (G, ψ, s)-coloring φ of K_M is defined such that for each pair of $i, j \in \{1, 2, \ldots, M\}$, the edge ij is colored

- blue if there is a blue copy of $K_{s,s}$ between V_i and V_j in G;
- red otherwise.

Note that if $uv \in E(G)$, then the edges between the blue t-cliques $[u]^*$ and $[v]^*$ forms a complete bipartite $K_{t,t}$.

Two Possible Cases

Following is an important lemma.

Lemma 3. Let n, δ, q be integers and let $M \geq 20n\delta q$. Fix a red/blue-coloring φ of the edges of K_M. Then K_M contains either

(a) a blue copy of every tree in $\mathcal{T}_{n,\delta}$, or

(b) a red copy of a complete q-partite graph where each part has size at least $M/(5\delta q)$.

We apply Lemma 3 with $\delta = d^2$ and $q = 2k+1$ and consider the two possible cases as follows.

The First Case

Case 1. Assume that K_M contains a blue copy of every tree in $\mathcal{T}_{n,d^2}$. Then we can obtain a blue copy of $T_2 \boxtimes K_k$ by applying Lemma 4.

Lemma 4. Let T be a rooted tree in $\mathcal{T}_{k,d}$, and let T' be the truncation of T. Let G be a graph with vertices partitioned into $V_1, V_2, \dots, V_M$. Fix a (G, ψ, s)-coloring of K_M with $s = d(k + k^2)$. Suppose that $G[V_i]$ is a blue clique and $|V_i| \geq s$ for each i. If K_M contains a blue copy of T', then G contains a blue copy of $T \boxtimes K_k$.

Here we truncate a rooted tree by deleting all vertices with positive and even depth and connecting the vertices that were connected by a two-edge path.

The Second Case

Case 2. Now assume that K_M contains a red copy of a complete $(2k+1)$-partite graph where each of the parts $V_0, V_1, \ldots, V_{2k}$ has size at least $M/(5d^2(2k+1))$.

By the construction of H and Kőnig's Theorem, we can find a set $S \subseteq V_0$ with $|S| \geq \epsilon N$ such that

(a) for $1 \leq i \leq 2n$, there exists an injection $\mu_i : S \to V_i$ such that v and $\mu_i(v)$ are adjacent in H for each $v \in S$, and

(b) the subgraph of H induced by S contains a copy $\widetilde{T}_1$ of T_1.

To finish the proof, we aim to find a red copy of $T_1 \boxtimes K_k$ in G^* using this copy $\widetilde{T}_1$ in H.

Construction

Root $\widetilde{T}_1$ at an arbitrary vertex. For each vertex v in $\widetilde{T}_1$,

- let $S_v = \{\mu_i(v) : 1 \leq i \leq k\}$ if v is at even depth, and
- let $S_v = \{\mu_i(v) : k+1 \leq i \leq 2k\}$ otherwise.

Recall that each edge connecting distinct parts V_i and V_j is red. It follows that the union of S_v for all $v \in V(\widetilde{T}_1)$ induces a red copy F of $T_1 \boxtimes K_k$ in K_M. Thus, it suffices to find an isomorphism between F and a subgraph of G^*.

Bounding the Number of Blue Edges

Note that we have the following facts.

- Each edge of F is an edge of H^3.
- If $uv \in E(H^3)$ is a red edge in K_M, then there is no blue copy of $K_{s,s}$ between the t-cliques $[u]^*$ and $[v]^*$ in G^*.

Lemma 5 bounds the number of blue edges that connects $[u]^*$ and $[v]^*$.

Lemma 5. An n-vertex graph containing no $K_{s,s}$ subgraph has at most $(s-1)^{1/s}n^{2-1/s} + (s-1)$ edges.

Let F' be the subgraph of G^* containing all red edges connecting $[u]^*$ and $[v]^*$ in G^* for each $uv \in E(F)$.

Blowup

The t-blowup of a graph F is the strong product of F and an t-vertex empty graph, with each $v \in V(F)$ corresponding to an independent set $[v]$ of size t.

We finish the proof by applying Lemma 6, which is implied by Lovász Local Lemma.

Lemma 6. Let t be a positive integer, and let F be a graph with $\Delta(F) = d$. If F' is a spanning subgraph of the t-blowup of F such that for each edge $uv \in E(F)$, there are at least $(1 - 1/(8d))t^2$ edges connecting $[u]$ and $[v]$ in F', then F is isomorphic to a subgraph of F'.

Since F and F' satisfy the assumption, we conclude that F, which is a red copy of $T_1 \boxtimes K_k$, is isomorphic to a subgraph of F', finishing the proof.

Proof of Theorem 2

Main Results—Theorem 2

Theorem 2. For any positive integer k, there exists a tree T such that

$$G \nrightarrow T$$

for any k-degenerate graph G.

Proof of Theorem 2

We show that a k-degenerate graph G is not Ramsey for the complete $(k+1)$-ary tree T with height $k+1$.[†]

Let $V(G) = \{1, 2, \ldots, n\}$ such that for each $u \in V(G)$, there are at most k neighbors v of u with $v < u$. Fix a proper vertex coloring $\phi : V(G) \to \{1, 2, \ldots, k+1\}$. Color an edge uv with $u < v$ red if $\phi(u) < \phi(v)$ and blue otherwise.

Then

- each monochromatic (either red or blue) monotone path in G has length at most k, and
- each copy of T in G must contain a monotone path of length $k+1$.

Thus, $G \nrightarrow T$.

[†] This proof was unpublished and was independently discovered by M. Geißer, J. Rollin, and P. Stumpf.

References

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