Anti-Ramsey number of edge-disjoint rainbow spanning trees

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Overview

Introduction

Introduction

Definition 1

An edge-colored graph G is called *rainbow* if every edge of G receives a different color.

Definition 2

anti-Ramsey problem: fins the anti-Ramsey number $AR(n, \mathcal{G})$ in an edge-coloring of K_n containing no rainbow copy of any graph in class $\mathcal{G}$.

Definition 3

$r(n, t)$: the maximum number of colors in an edge-coloring K_n not having t edge-disjoint rainbow spanning trees.

Previous Works

- anti-Ramsey number for perfect matchings is $\binom{n-3}{2} + 2$ for $n \geq 14$. [HY12]
- The maximum number of colors in an edge-coloring of K_n $(n \geq 4)$ with no rainbow spanning tree is $\binom{n-2}{2} + 1$. [BV01]
- $r(n, 2) = \binom{n-2}{2} + 2$ for $n \geq 6$. [S A07]
- $$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n > 2t + \sqrt{6t - \frac{23}{4}} + \frac{5}{2} \\ \binom{n}{2} - t & \text{for } n = 2t \end{cases}$$

  . [S J16b]

Previous Works

Also a conjecture:

Conjecture 1

$r(n, t) = \binom{n-2}{2} + t$ *for* $n \geq 2t + 2 \geq 6$.

.[S J16b] This paper proves this conjecture.

Theorem 1

Combining with these three results ([BV01], [S A07], [S J16b]), we have

Theorem 1

For all positive integer t,

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n \geq 2t+2 \\ \binom{n-1}{2} & \text{for } n = 2t+1 \\ \binom{n}{2} - t & \text{for } n = 2t \end{cases}$$

Remark 1

If $n < 2t$, K_n doesn't have enough edges for t edge-disjoint spanning trees.

Theorem 2

When $t = 1$, [CH17] showed that at determining the largest rainbow spanning forest of a graph can be solved by applying the Matroid Intersection Theorem.

Theorem 2

Theorem 2

An edge-colored connected graph G has a rainbow spanning tree if and only if for every $2 \leq k \leq n$ and every partition of G with k parts, at least $k-1$ different colors are represented in edges between partition classes.

[Sch03], [Suz06], [CH17]

Theorem 3

By generalizing theorem 2 to t color-disjoint rainbow spanning tree, by [Sch03],

Theorem 3

An edge-colored multigraph G has t pairwise color-disjoint rainbow spanning trees if and only if for every partition P of $V(G)$ into $|P|$ parts, at least $t(|P|-1)$ distinct colors are represented in edges between partition classes.

Nash-Walliam-Tutte Theorem

Remark 2

Nash-Williams-Tutte Theorem: A multigraph contains t edge-disjoint spanning trees if and only if for every partition P of its vertex set, it has at least $t(|P|-1)$ cross-edges. Theorem 3 implies the Nash-Williams-Tutte Theorem by assigning every edge of the multigraph a distinct color.

[Nas61], [Tut61].

Theorem 4

Theorem 3 can be also generalized to extending edge-disjoint rainbow spanning forests to edge-disjoint rainbow spanning trees.
Let G be an edge-colored multigraph and $F_1, \cdots, F_t$ be t edge-disjoint rainbow spanning forests.

Definition 4

A extension from $F_1, \cdots, F_t$ to $T_1, \cdots, T_t$ which is t rainbow spanning trees in G is color-disjoint if all edges in $\cup_i(E(T_i)\backslash E(F_i))$ have distinct colors and these colors are different from the colors appearing in the edges of $\cup_i E(F_i)$.

Theorem 4

By using metroid methods again or graph theoretical arguments, we have

Theorem 4

A family of t edge-disjoint rainbow spanning forests $F_1. \cdots, F_t$ has a color-disjoint extension in G if and only if for every partition P of G into $|P|$ parts,

$$|c(cr(P, G'))| + \sum_{i=1}^{t} |cr(P, F_i)| \geq t(|P| + 1)$$

, where G' is the spanning subgraph of G by removing all edges with colors appearing in some F_i and $c(cr(P, G'))$ be the set of colors appearing in the edges of G' crossing the partition P.

Proof of Theorem 3

Proof of Theorem 3

We will prove theorem by using matroid and graph theoretical arguments.

Matroid

A matroid is defined as $M = (E, \mathcal{I})$, where E is the ground set and $\mathcal{I} \subseteq 2^E$ is a set containing subsets of E that satisfy

- if $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
- if $A \subseteq \mathcal{I}$, $B \subseteq \mathcal{I}$ and $|A| > |B|$, then $\exists a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

Given a matroid $M = (E, \mathcal{I})$, the rank function $r_M : 2^E \to \mathrm{N}$ is defined as $r_M(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}$.

Matroid

- The graphic matroid of a graph G is the matroid $M = (E, \mathcal{I})$ where $E = E(G)$ and $\mathcal{I}$ is the set of forests in G.
- The partition matroid of a graph G is the matroid $M = (E, \mathcal{I})$ where $E = E(G)$ and $\mathcal{I}$ is the set of rainbow subgraphs of G.
- Given k matroids $\{M_i = (E_i, \mathcal{I}_i)\}_{i\in[k]}$, the union of the k matroids is a matroid $M = (E, \mathcal{I}) = (\bigcup_{i=1}^{k} E_i, \{I_1 \cup \cdots \cup I_k : I_i \in \mathcal{I}_i \text{for all } i \in [k]\})$. This matroid has rank function

$$r(S) = \min_{T\subseteq S}\left(|S \setminus T| + \sum_{i=1}^{k} r_{M_i}(T \cap E_i)\right)$$

. [Edm68] [Nas67]

Matroid

- Given two matroid $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ on the same ground set with rank function r_1, r_2 respectively. The Matroid Intersection Theorem shows that

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq E} (r_1(U) + r_2(E \setminus U))$$

. [Edm70]

Proof of Theorem 3 using Matroid

[Sch03]

The forward direction is clear.

It remains to show that if for every partition P of $V(G)$ into $|P|$ parts, at least $t(|P|-1)$ distinct colors are represented in edges between partition classes, then there exist t edge-disjoint rainbow spanning trees in G.

Proof of Theorem 3 using Matroid

Given an edge-colored graph G, let $M = (E, \mathcal{I})$ be the graphic matroid of G and $M' = (E, \mathcal{I}')$ be the partition matroid of G. Let $M^t = M \vee M \vee \cdots \vee M = (E, \mathcal{I}^t)$, where we take t copies of M, which contains the union of t forests. By matroid union theorem, we obtain that

$$r_{M^t}(S) = \min_{T \subseteq S} \left(|S \setminus T| + t \cdot r_M(T)\right)$$

.

Proof of Theorem 3 using Matroid

By the Matroid Intersection Theorem, we have

$$\begin{aligned}\max_{I\in\mathcal{I}^t\cap\mathcal{I}'}|I| &= \min_{U\subseteq E}(r_{M^t}(U)+r_{M'}(E\setminus U))\\ &= \min_{U\subseteq E}(\min_{T\subseteq U}(|U\setminus T|+t\cdot r_M(T))+r_{M'}(E\setminus U))\end{aligned}$$

Let $T\subseteq U\subseteq E$ be an arbitrary chosen. Observe that $t\cdot r_M(T)=t(n-q(T))$, where $q(T)$ is the number of connected components of $G[T]$.

Proof of Theorem 3 using Matroid

Now we claim that

$$|U \setminus T| + r_M(E \setminus U) \geq r_M(E \setminus T) \geq t(q(T)-1)$$

For any color c appearing in some edge $e \in E \setminus T$, if $e \in E \setminus U$, then the color c is counted in $r_M(E \setminus U)$; if $e \in U$, then that color is counted in $|U \setminus T|$. In particular, at least $t(q(T)-1)$ distinct colors are represented in edges between connected components of T, thus in $E \setminus T$.

Proof of Theorem 3 using Matroid

It follows that

$$|U \setminus T| + t \cdot r_M(T) + r_M(E \setminus U) \geq t(q(T)-1) + t(n-q(T)) \geq t(n-1)$$

which implies that $\max_{I \in \mathcal{I}^t \cap \mathcal{I}} |I| \geq t(n-1)$. By definition, we then have t edge-disjoint rainbow spanning trees.

Proof of Theorem 3 using graph theoretical arguments

Definition 5

$V(G), E(G)$: the set of the vertex and the edge of G.

Definition 6

$||G||$: $|E(G)|$.

Definition 7

$c(E)$: the set of colors that appear in E.
$c(e)$: the color of edge e.

Definition 8

A color c has multiplicity k in G if the number of edges with color c in G is k.
The color multiplicity of an edge in G is the multiplicity of the color of the edge in G.

Proof of Theorem 3 using graph theoretical arguments

For any partition P of the vertex set $V(G)$ and a subgraph H of G, let $|P|$ denote the number of parts in the partition P and let $cr(P, H)$ denote the set of crossing edges in H whose end vertices belong to different parts in the partition P. When $H = G$, we also write $cr(P, G)$ as $cr(P)$. Given two partitions $P_1 : V = \cup_i V_i$ and $P_2 : V = \cup_j V'_j$, let the intersection $P_1 \cap P_2$ denote the partition given by $V = \bigcup_{i,j} V_i \cap V'_j$.

Proof of Theorem 3 using graph theoretical arguments

Given a spanning disconnected subgraph H, there is a natural partition P_H associated to H, which partitions V into its connected components.
WLOG, we abuse our notation $cr(H)$ to denote the crossing edges of G corresponding to this partition P_H.
Recall we want to show that an edge-colored multigraph G has t color-disjoint rainbow spanning trees if and only if for any partition P of $V(G)$ (with $|P| \geq 2$), $|c(cr(P))| \geq t(|P| - 1)$.

Proof of Theorem 3 using graph theoretical arguments

For one direction, suppose that G contains t pairwise color-disjoint rainbow spanning trees $T_1, \cdots T_t$, then all edges in these trees have distinct colors. For any partition P of the vertex set V, each tree contributes at least $|P| - 1$ crossing edges, thus t trees contribute at least $t(|P| - 1)$ crossing edges and the colors of these edges are all distinct.

Proof of Theorem 3 using graph theoretical arguments

For the other direction, assume G satisfies inequality
$|c(cr(P))| \geq t(|P|-1)$.
We will use a contradiction to prove that G contains t pairwise color-disjoint rainbow spanning trees.
Assume G does not contain t pairwise color-disjoint rainbow spanning trees, and $\mathcal{F}$ be the collection of all families of t color-disjoint rainbow spanning forests.

Proof of Theorem 3 using graph theoretical arguments

Consider the process:

$C' \leftarrow \bigcup_{j=1}^{t} c(cr(F_j))$
while $C' \neq \emptyset$
  for each color x in C'
    for j in $1 \cdots t$
      if x appears in F_j
        delete the edge in color x from F_j
  $C' \leftarrow \bigcup_{j=1}^{t} c(cr(F_j)) - C'$

We use $F_j^{(i)}$ to denote the rainbow spanning forest F_j after i iterations of the while loop. Specially, $F_j^{(\infty)}$ is the resulting rainbow spanning forest of F_j after the process. Also, C_i denote the set C' after the i-th iteration of the while loop.

Proof of Theorem 3 using graph theoretical arguments

This procedure is deterministic, thus $\{F_j^{(i)} : j \in [t], i > 0\}$ is unique for a fixed family $\{F_1, \cdots, F_t\}$. We can define a preorder on $\mathcal{F}$:
The family $\{F_j\}_{j=1}^t$ is less than or equal to family $\{F'_j\}_{j=1}^t$ if there is a positive integer l such that

- For $1 \leq i < l$, $\sum_{j=1}^t \| F_j^{(i)} \| = \sum_{j=1}^t \| F_j'^{(i)} \|$.
- $\sum_{j=1}^t \| F_j^{(l)} \| < \sum_{j=1}^t \| F_j'^{(l)} \|$

Proof of Theorem 3 using graph theoretical arguments

Since G is finite, so is $\mathcal{F}$. Thus there exists a maximal element $\{F_1, \cdots, F_t\} \in \mathcal{F}$. Run the deterministic process on $\{F_1, \cdots, F_t\}$. The goal is to construct a common partition P by refining $cr(F_j)$ so that $|c(cr(P))| < t(|P| - 1)$. We will show that all forests in $\{F_j^{(\infty)} : j \in [t]\}$ admit the same partition P.

Claim 1

$$\bigcup_{j=1}^{t} c\left(cr\left(F_j^{(i)}\right)\right) \subseteq \left(\bigcup_{j=1}^{t} c\left(cr\left(F_j^{(i-1)}\right)\right)\right) \cup \left(\bigcup_{j=1}^{t} c(F_j^{(i)})\right)$$

Proof of Theorem 3 using graph theoretical arguments

Assume there is a contradiction: $x \in \bigcup_{j=1}^{t} c(cr(F_j^{(i)})) \setminus \bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}))$ and there is no edge with color x in all $F_1^{(i)}, \cdots, F_t^{(i)}$.
Let e be the edge such that $c(e) = x$ and $e \in cr(F_s^{(i)})$ for some $s \in [t]$.
Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}))$, it follows that $F_s^{(i-1)} + e$ contains a rainbow cycle, which passes through e and another edge $e' \in F_s^{(i-1)}$ joining two distinct components of $F_s^{(i)}$.
Considering a new family of rainbow spanning forest $\{F'_1, \cdots, F'_t\}$ where $F'_j = F_j$ for $j \neq s$ and $F'_s = F_s - e' + e$.

Proof of Theorem 3 using graph theoretical arguments

The color-disjoint property is reserved since the $c(e)$ is not in any F_j.
Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}))$, $F_s'^{(i)}$ will have one fewer component than $F_s^{(i)}$. Thus we have

$$\sum_{j=1}^{t} \|F_j^{(k)}\| = \sum_{j=1}^{t} \|F_j'^{(k)}\|, \forall k < i$$

$$\sum_{j=1}^{t} \|F_j'^{(i)}\| > \sum_{j=1}^{t} \|F_j^{(i)}\|$$

which contradicts our maximality assumption of $\{F_i : i \in [t]\}$.

Proof of Theorem 3 using graph theoretical arguments

Claim 1 implies that for each $x \in Ci$, there is an edge e of color x in exactly one of the forests in $\{F_j^{(i)} : j \in [t]\}$. Thus removing that edge in the next iteration will increase the sum of number of partitions exactly by 1. Thus we have that

$$\sum_{j=1}^{t} |P_{F_j^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_j^{(i)}}| + |C_i|$$

It then follows that

$$\sum_{j=1}^{t} |P_{F_j^{(\infty)}}| = \sum_{P_{F_j}} | + \sum_{i} |C_i|$$

$$= \sum_{P_{F_j}} | + |\bigcup_{j=1}^{t} c(cr(F_j^{(\infty)}))|$$

Proof of Theorem 3 using graph theoretical arguments

Finally set the partition $P = \bigcap_{j=1}^{t} P_{F_j^{(\infty)}}$. We claim $P_{F_j^{(\infty)}} = P, \forall j$. This is because all edges in $cr(P_{F_j^{(\infty)}}) \cap \bigcup_{k=1}^{t} E(F_k^{(\infty)})$ have been already removed. We then have

$$t|P| = \sum_{j=1}^{t} |P_{F_j^{(\infty)}}| = \sum_{P_{F_j}} | + | \bigcup_{j=1}^{t} c(cr(F_j^{(\infty)}))| = \sum_{j=1}^{t} |P_{F_j}| + |c(cr(P))|$$
$$\geq t + 1 + |c(cr(P))|$$

We obtain

$$|c(cr(P))| \leq t(|P| - 1) - 1$$

Contradiction.

Corollary 1

Corollary 1

The edge-colored complete graph K_n has t color-disjoint rainbow spanning trees if the number of edges colored with any fixed color is at most $n/(2t)$.

Proof of Corollary 1

Suppose K_n does not have t color-disjoint rainbow spanning trees, then there exists a partition P of $V(K_n)$ into r parts $(2 \leq r \leq n)$ such that the number of distinct colors in the crossing edges of P is at most $t(r-1)-1$. Let m be the number of edges crossing the partition P. It follows that

$$m \leq (t(r-1)-1) \cdot \frac{n}{2t} \leq \frac{n}{2}(r-1) - \frac{n}{2t}$$

Proof of Corollary 1

On the other hand,

$$m \geq \binom{n}{2} - \binom{n-(r-1)}{2}$$

Hence we have

$$\binom{n}{2} - \binom{n-(r-1)}{2} \leq \frac{n}{2}(r-1) - \frac{n}{2t}$$

which implies

$$(n-r)(r-1) \leq -\frac{n}{t}$$

which contradicts that $2 \leq r \leq n$.

Proof of Theorem 4

Proof of Theorem 4

Recall that we want to show that any t edge-disjoint rainbow spanning forests $F_1, \cdots, F_t$ have a color-disjoint extension to edge-disjoint rainbow spanning trees in G if and only if

$$|c(cr(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \geq t(|P| - 1)$$

where G' is the spanning subgraph of G by removing all edges with colors appearing in some F_j.

Proof of Theorem 4

The forward direction is also trivial. We will show that the condition

$$|c(cr(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \geq t(|P| - 1)$$

implies the existance of a color-disjoint extension to edge-disjoint rainbow spanning trees.
The proof is similar to the proof of Theorem 3.

Proof of Theorem 4

Consider a set of edge-maximal forests $F_1^{(0)}, \cdots, F_t^{(0)}$ which is a color-disjoint extension of $F_1, \cdots, F_t$. From $\{F_j^{(0)}\}$, we delete all edges in $\{F_j^{(0)}\}$ of some color c appearing in $\bigcup_{j=1}^{t} c(cr(F_j^{(0)}, G'))$ to get a new set $\{F_j^{(1)}\}$. Repeat this process until we reach a stable set $\{F_j^{(\infty)}\}$. Since we only delete edges in G', we have $E(F_j) \subseteq E(F_j^{(\infty)}$ for each $1 \leq j \leq t$. The edges and colors in $\bigcup_{j=1}^{t} E(F_j)$ will not affect the process.

Proof of Theorem 4

A similar claim still holds:

Claim 2

$$\bigcup_{j=1}^{t} c\left(cr\left(F_j^{(i)}, G'\right)\right)$$

$$\subseteq \left(\bigcup_{j=1}^{t} c\left(cr\left(F_j^{(i-1)}, G'\right)\right)\right) \cup \left(\bigcup_{j=1}^{t} c\left(E(F_j^{(i-1)}) \cap E(G')\right)\right)$$

Proof of Theorem 4

Let $C_i = \left(\bigcup_{j=1}^{t} c(cr(F_j^{(i)}, G'))\right) \setminus \left(\bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}, G'))\right)$, then we have:

$$\sum_{j=1}^{t} |P_{F_j^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_j^{(i)}}| + |C_i|$$

.

It follows that

$$\sum_{j=1}^{t} |P_{F_j^{(\infty)}}| = \sum_{j=1}^{t} |P_{F_j^{(0)}}| + \sum_{i} |C_i|$$
$$= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |\bigcup_{j=1}^{t} c(cr(F_j^{(\infty)}, G'))|$$

. Set the partition $P = \bigcap_{j=1}^{t} P_{F_j^{(\infty)} \setminus E(F_j)}$. Clearly all the edges in $cr(P, G')$ are removed. All possible edges remaining in G that cross the partition P are exactly the edges in $\bigcup_{j=1}^{t} cr(P, F_j)$.

Proof of Theorem 4

We have

$$
\begin{aligned}
t|P| &= \sum_{j=1}^{t} |P_{F_j^{(\infty)}}| + \sum_{j=1}^{t} |cr(P, F_j)| \\
&= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |\bigcup_{j=1}^{t} c(cr(F_j^{(\infty)}, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \\
&= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |c(cr(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \\
&\geq t + 1 + |c(cr(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)|
\end{aligned}
$$

.

Proof of Theorem 4

We obtain a contradiction:

$$|c(cr(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \leq t(|P| - 1) - 1$$

.

Proof of Theorem 1

Proof of Theorem 1

Recall Theorem 1
For all positive integer t,

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n \geq 2t+2 \\ \binom{n-1}{2} & \text{for } n = 2t+1 \\ \binom{n}{2} - t & \text{for } n = 2t \end{cases}$$

Lower bound for $r(n, t)$

See [S J16a] (Lemma 5.1) for more detail about the lower bound for $r(n, t)$

Technical Lemma

Lemma 1

Let G be an edge-colored graph with s colors $c_1, c_2, \ldots, c_s$ and $|V(G)| = n = 2t + 2$ where $t \geq 3$. For color c_i, let m_i be the number of edges of color c_i. Suppose $\sum_{i=1}^{s}(m_i - 1) = 3t$ and $m_i \geq 2$ for all $i \in [s]$. Then, we can construct t edge-disjoint rainbow forest $F_1, \ldots, F_t$ in G such that if we define $G_0 = G - \bigcup_{i=1}^{t} E(F_i)$, then $|E(G_0)| \leq 2t + 1$ and $\Delta(G_0) \leq t + 1$.

Proof of Technical Lemma

Consider two cases: $m_1 \geq 2t + 2$ and $m_1 \leq 2t + 1$.

Proof of Technical Lemma Case 1

Note that $\sum_{i=2}^{s}(m_i - 1) = 3t - (m_1 - 1) \leq t - 1$. Thus, $s \leq t$.
Let $d_i(v)$ be the number of edges in color c_i and incident to v in the current graph G. We construct the edge-disjoint rainbow forests $F_1, F_2, \ldots, F_t$ in two rounds.

Proof of Technical Lemma Case 1 First Round

In the first round, we greedily extract edges only in color c_1.
For $i = 1, \ldots, t$, at step i, pick a vertex v with maximum $d_1(v)$ (pick arbitrarily if tie). Pick an edge in color c_1 incident to v, assign it to F_i, and delete it from G.
We claim that after the first round, $d_1(v) \leq t + 1$ for any vertex v.

Proof of Technical Lemma Case 1 First Round

Proof of $d_1(v) \leq t+1$:
Suppose $d_1(v) \geq t+2$. Since $n-1-(t+2) < t$, it follows that there exists another vertex u with $d_1(u) \geq d_1(v) - 1 \geq t+1$. This implies $m_1 \geq t + d_1(v) + d_1(u) - 1 \geq 3t+2$.
However, $m_1 - 1 \leq \sum_{i=1}^{s}(m_i - 1) = 3t$, which gives us the contradiction.

Proof of Technical Lemma Case 1 Second Round

In second round:

- Greedily extract edges not in color c_1.
- For $i = 1, \ldots, t$. In the i-th step, among all vertices v with at least one neighboring edge not in color c_1, pick a vertex v with maximum vertex degree $d(v)$ (pick arbitrarily if tie). Pick an edge incident to v and not in color c_1, assign it to F_i, and delete it from G.

If we succeed with selecting t edges not in color c_1 in the second round, we claim $d(v) \leq t + 1$ for any vertex v.

Proof of Technical Lemma Case 1 Second Round

Proof $d(v) \leq t+1$ for any vertex v.
Suppose not, if $d(v) \geq t+2$, then there's another vertex u with $d(u) \leq d(v) - 1 \leq t+1$. It implies $\sum_{i=1}^{s} m_i \geq 2t + d(u) + d(v) - 1 \geq 4t + 2$. However, since $s \leq t$, we have $\sum_{i=1}^{s} m_i \leq 3t + s \leq 4t$. Contradiction.
Therefore, $d(v) \leq t+1$. Moreover, $|E(G_0)| \leq 4t - 2t \leq 2t$

Proof of Technical Lemma Case 1

If the process stops at step $i = l < t$, then all remaining edges in G_0 must be color 1. Thus, by the previous claim, $\Delta(G_0) \leq t + 1$. Moreover, $|E(G_0)| \leq m_1 - t \leq (3t + 1) - t = 2t + 1$.
In both cases above, $F_1, \ldots, F_t$ are edge-disjoint rainbow forests!

Proof of Technical Lemma Case 2

Claim: There exists t edge-disjoint rainbow forests $F_1, F_2, \ldots, F_t$, such that $\Delta(G_0) \leq t+1$.
Proof: For $j = 1, 2, \ldots, t$, we'll construct a rainbow forest F_j by selecting a rainbow set of edges, such that after deleting these edges from G, $\Delta(G_0) \leq 2t+1-j$. Notice that when $j = t$, we will have $\Delta(G_0) \leq t+1$.

Proof of Technical Lemma Case 2

For step j, WLOG let $v_1, v_2, \ldots, v_t$ be the vertices with degree $2t + 2 - j$ and let $c_1, c_2, \ldots, c_m$ be the set of colors of edges incident $v_1, v_2, \ldots, v_t$ in G.
If there's no such vertex, simply pick an edge incident to the max-degree vertex and assign it to F_j.
Otherwise, we will construct an auxiliary bipartite graph $H = A \cup B$ where $A = \{v_1, v_2, \ldots, v_l\}$ and $B = \{c_1, c_2, \ldots, c_m\}$ and $v_x c_y \in E(H)$ iff there's an edge of color c_y incident to v_x.

Proof of Technical Lemma Case 2

We claim that there exists a perfect matching of A in H.
Suppose not, then by Hall's theorem, there exists a set of vertices $A' = \{u_1, u_2, \ldots, u_k\} \subseteq A$ such that $|N(A')| < |A'| = k$ where $k \geq 2$.
WLOG, suppose $N(A) = \{c'_1, c'_2, \ldots, c'_q\}$ where $q \leq k-1$. Let m'_i be the number of edges of color c'_i remaining in G.
Note that $k \neq 2$ since otherwise we will have on color with at least $2 \times (2t+2-j) - 1 \geq 2t+3$ edges, which contradicts our assumption in this case.
Notice that for every $i \in [k]$, u_i has at least $(2t+2-j)$ edges incident to it. Moreover, at least $j-1$ edges are already deleted from G in previous steps.

Proof of Technical Lemma Case 2

Therefore, we have
$\frac{k(2t+2-j)}{2} \leq \sum_{i=1}^{q} m_i' \leq (\sum_{i=1}^{q}(m_i' - 1)) + (k-1) \leq 3t - (j-1) + (k-1)$.
It follows that $k \leq 2 + \frac{2t}{2t-j} \leq 4$.
Similarly, using another way of counting the edges incident to some $u_i (i \in [k])$, we have $k(2t+2-j) - \binom{k}{2} \leq 3t - (j-1) + (k-1)$. Which implies that $t(2k-3) \leq \frac{k(k-3)}{2} + j(k-1) \leq \frac{k(k-3)}{2} + t(k-1)$.
It follows that $t \leq \frac{k(k-3)}{2(k-2)}$. Since $k \leq 4$ and $k > 2$, we obtain that $t \leq 1$, which contradicts our assumption that $t \geq 2$.
Thus, by contradiction, there exists a matching of A in H.

Proof of Technical Lemma Case 2

This implies that there exists a rainbow set of edges E_j that cover all vertices with degree $2t + 2 - j$ in step j. We can then find a maximally acyclic subset F_j of E_j such that F_j is a rainbow forest and every vertex of degree $2t + 2 - j$ is adjacent to some edge in F_j. Delete edgs of F_j from G andwe have $\Delta(G_0) \leq 2t + 1 - j$. As a result, after t steps, we obtain t edge-disjoint rainbow forests $F_1, F_2, \ldots, F_t$ and $\Delta(G_0) \leq t + 1$.
This finishes the proof of the claim.

Proof of Technical Lemma Case 2

Now let $\{F_1, F_2, \ldots, F_t\}$ be an edge-maximal set of t edge-disjoint rainbow forests that satisfies $\Delta(G_0) \leq t+1$. We claim that $|E(G_0)| \leq 2t+1$. Suppose not, i.e., $|E(G_0)| \geq 2t+2$. It follows that $\sum_{i=1}^{t} |E(F_i)| \leq 6t-(2t+2) < 4t$, i.e. there exists a $j \in [t]$ such that F_j has at most three edges.

Since F_j is edge maximal, none of the edges in G_0 can be added to F_j. We have three cases: $|E(F_j)| = 1, 2, 3$.

Proof of Technical Lemma Case 2a

Case 2a: $|E(F_j)| = 1$. It then follows that all edges in G_0 have the same color (call it c_1') as the single edge in F_j. Thus, we have a color with multiplicity at least $2t + 3$, which contradicts that $m_1 < 2t + 2$.

Proof of Technical Lemma Case 2b

Case 2b: $|E(F_j)| = 2$. Similarly, we have that at least $2t + 1$ edges in G_0 that share the same color (call them c_1', c_2') as edges in F_j. It follows that $m_1 + m_2 \geq 2t + 3$. Similar to Case 1, in this case, we have $s \leq t + 1$ and $|E(G)| = 3t + s \leq 4t + 1$. Since $|E(G_0)| \geq 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \leq (4t + 1) - (2t + 2) = 2t - 1$. Hence, there exists some F_k such that $\sum_{i=1}^{t} |E(F_i)| \leq (4t + 1) - (2t + 2) = 2t - 1$. Hence, there exists some F_k such that $|E(F_k)| \leq 1$ and we are done by Case 2a.

Proof of Technical Lemma Case 2c

Case 2c: $|E(F_j)| = 3$. Similarly, we have that at least $2t - 1$ edges in G_0 share the same colors (call them c_1', c_2', c_3') as edges in F_j. It follows that $m_1 + m_2 + m_3 \geq 2t + 2$. By the inequality, we have that $s \leq t + 4$ and $|E(G)| \leq 4t + 4$. Since $|E(G_0)| \geq 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \leq 2t + 2$. Since $t \geq 3$ by our assumption, there exists a $k \in [t]$ such that $|E(F_k) \leq 2|$ and we are done by Case 2b and Case 2c.

Proof of Technical Lemma Case 2

Therefore, by contradiction, we have that $|E(G_0)| \leq 2t + 1$ an we're done.

Theorem 1 where $n = 2t + 2$

Proposition 1

For any $n = 2t + 2\ 6$*, we have* $r(n, t) = \binom{n-2}{2} + t = 2t^2$

Theorem 1 where $n = 2t + 2$

Note that the lower bound is shown by Proposition 1. For the upper bound, we will assume that $t \geq 3$ since the case when $t = 2$ is implied by the result of [S A07]. We will show that any coloring of K_{2t+2} with $2t^2 + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this edge-colored graph G. Let m_i be the multiplicity of the color c_i in G. WLOG, say the first s colors have multiplicity at least 2, that is, $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$.

Theorem 1 where $n = 2t + 2$

Let G_1 be the spanning subgraph of G consisting of all edges with color multiplicity greater than 1 in G. Let G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s}(m_i - 1) = \binom{n}{2} - (2t^2 + 1) = 3t$$

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 3t + s \leq 6t$$

Theorem 1 where $n = 2t + 2$

By Lemma 1, it follows that we can construct t edge-disjoint rainbow spanning forests $F_1, \cdots, F_t$ in G such that if we define $G_0 = E(G_1) - \bigcup_{i=1}^{t} E(F_i)$, then

$$|E(G_0)| \leq 2t + 1 \text{ and } \Delta(G_0) \leq t + 1$$

Now we show that $F_1, \cdots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P. We will verify

$$|c(cr(P, G_2))| + \sum_{i=1}^{t} |cr(P, F_i)| \geq t(|P| - 1)$$

Theorem 1 where $n = 2t + 2$

We will first verify the case when $3 \leq |P| \leq n$. Note that

$$|c(cr(P, G_2))| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1) \geq \binom{n}{2} - 2(t+1) - \binom{n - |P| + 1}{2} - t(|$$

We want to show that the right hand side of the above inequality is nonnegative. Note that the function on the right hand side is concave downward with respect to $|P|$. Thus it is sufficient to verify it at $|P| = 3$ and $|P| = n$. When $|P| = 3$, we have

$$\binom{n}{2} - (2t + 1) - \binom{n - 2}{2} - 2t = 0$$

when $|P| = n$, we have

$$\binom{n}{2} - (2t + 1) - t(n - 1) = 0$$

Theorem 1 where $n = 2t + 2$

It remains to verify for $|P| = 2$. By Theorem 4, we have $|E(G_0)| \leq 2t + 1$. If each part of P contains at least 2 vertices, then we have

$$\begin{aligned}
&|c(cr(P, G_2))| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1) \\
&\geq \binom{n}{2} - |E(G_0)| - (\binom{n-2}{2} + 1) - t \\
&\geq \binom{n}{2} - (2t + 1) - (\binom{n-2}{2} + 1) - t \\
&= t - 1 \geq 0
\end{aligned}$$

Otherwise, P is of the form $V(G) = \{v\} \cup B$ for some $v \in V(G)$ and $B = V(G) \setminus \{v\}$. By Lemma 1, we have $d_{G_0} \leq t + 1$. Thus,

$$\begin{aligned} &|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1) \\ \geq\ &(n-1) - d_{G_0}(v) - t \geq 2t + 1 - (t+1) - t = 0 \end{aligned}$$

Therefore, by Theorem 4, $F_1, \cdots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees.

Theorem 1 where $n \geq 2t + 3$

Proposition 2

For any $n \geq 2t + 2 \geq 6$, *we have* $r(n, t) = \binom{n-2}{2} + t$

Theorem 1 where $n \geq 2t + 3$

Again, the lower bound is due to Proposition 1. For the upper bound, we will show that every edge-coloring of K_n with exactly $\binom{n-2}{2} + t + 1$ distinct colors has t edge-disjoint spanning trees. Call this edge-colored graph G. Given a vertex v, we define $D(v)$ to be the set of colors C such that every edge with colors in C is incident to v. Given a vertex v and a set of colors C, define $\Gamma(v, C)$ as the set of edges incident to v with colors in C. For ease of notation, we let $\Gamma(v) = \Gamma(v, D(v))$.

Theorem 1 where $n \geq 2t + 3$

For fixed t, we will prove the theorem by induction on n. The base case is when $n = 2t + 2$, which is proven in Proposition 2. Let’ s now consider the theorem when $n \geq 2t + 3$.

Theorem 1 where $n \geq 2t + 3$ Case 1

Case 1: there exists a vertex $v \in V(G)$ with $|\Gamma(v)| \geq t$ and $|D(v)| \leq n-3$. In this case, we set $G = G - \{v\}$. Note that G is an edge-colored complete graph with at least $\binom{n-2}{2} + t + 1 - (n-3) = \binom{n-3}{2} + t + 1$ distinct colors. Moreover $|G| \geq 2t + 2$. Hence by induction, there exists t edge-disjoint rainbow spanning trees in G. Note that by our definition of $D(v)$, none of the colors in $D(v)$ appear in $E(G)$. Moreover, since $|\Gamma(v)| \geq t$, we can extend the t edge-disjoint rainbow spanning trees in G to G by adding one edge in $\Gamma(v)$ to each of the rainbow spanning trees in G.

Theorem 1 where $n \geq 2t + 3$ Case 2

Case 2: Suppose we are not in Case 1. We first claim that there exists two vertices $v_1, v_2 \in V(G)$ such that $|\Gamma(v_1)| \leq t-1$ and $|\Gamma(v_2)| \leq t$. Otherwise, there are at least $n-1$ vertices u with $|\Gamma(u)| \geq t$. Since we are not in Case 1, it follows that all these vertices u also satisfy $|D(u)| \geq n-2$. Hence by counting the number of distinct colors in G, we have that

$$\frac{(n-1)(n-2)}{2} \leq \binom{n-2}{2} + t + 1$$

which implies that $n \leq t + 3$, giving us the contradiction.

Theorem 1 where $n \geq 2t + 3$ Case 2

Now suppose $|\Gamma(v_1)| \leq t{-}1$ and $|\Gamma(v_2)| \leq t{-}1$. Let $D = D(v_1) \cup D(v_2)$. Add new colors to D until $|\Gamma(v_1, D)| \geq t$, $|\Gamma(v_2, D)| \geq t + 1$ and $|D| \geq t + 1$. Call the resulting color set S. Note that $t + 1 \leq |S| \leq 2t + 1 \leq n - 2$. Now let $G \; = G{-}\{v_1, v_2\}$ and delete all edges of colors in S from G . We claim that G has t color-disjoint rainbow spanning trees. By Theorem 3, it is sufficient to verify the condition that for any partition P of $V(G')$,

$$|c(cr(P, G'))| \geq t(|P| - 1)$$

Theorem 1 where $n \geq 2t+3$ Case 2

Observe

$$
\begin{aligned}
&|c(cr(P, G'))| - t(|P| - 1) \\
&\geq |c(E(G'))| - \binom{n-1-|P|}{2} - t(|P| - 1) \\
&\geq \binom{n-2}{2} + t + 1 - |S| - \binom{n-1-|P|}{2} - t(|P| - 1) \\
&\geq \binom{n-2}{2} + t + 1 - (n-2) - \binom{n-1-|P|}{2} - t(|P| - 1)
\end{aligned}
$$

Note the expression above is concave downward as a function of $|P|$. It is sufficient to check the value at 2 and $n-2$.

Theorem 1 where $n \geq 2t+3$ Case 2

When $|P| = 2$, we have

$$|c(cr(P, G'))| - t(|P| - 1) \geq \binom{n-2}{2} + t + 1 - (n-2) - \binom{n-3}{2} - t = 0$$

When $|P| = n - 2$, we have

$$\begin{aligned} c(cr(P, G'))| - t(|P| - 1) &\geq \binom{n-2}{2} + t + 1 - (n-2) - t(n-3) \\ &= \frac{(n-4)(n-2t-3)}{2} \\ &\geq 0 \end{aligned}$$

Here we use the assumption $n \geq 2t+3$ in the last step.

Theorem 1 where $n \geq 2t + 3$ Case 2

Now it remains to extend the t color-disjoint spanning trees we found to G by using only the colors in S. Let $e_1, \cdots, e_k$ be the edges in G incident to v_1 with colors in S. Let $e_1 , \cdots , e'_l$ be the edges in $G \setminus \{v_1\}$ incident to v_2 with colors in S. With our selection of S, it follows that $k, l \geq t$. Now construct an auxiliary bipartite graph H with partite sets $A = \{e_1, \cdots, e_k\}$ and $B = \{e'1, \cdots, e'_l\}$ such that $e_i e'_j \in E(H)$ if and only if e_i, e'_j have different colors in G.

Theorem 1 where $n \geq 2t + 3$ Case 2

We claim that there is a matching of size t in H. Let M be the maximum matching in H. WLOG, suppose $e_1 e_1', \cdots, e_m e_m' \in M$ where $m < t$. It follows that $\{e_j : m < j \leq k\} \cup \{e_j' : m < j \leq l\}$ all have the same color (otherwise we can extend the matching). WLOG, they all have color x. Now observe that for every matched edge $e_i e_i'$, exactly one of the two end vertices must be in color x. Otherwise, we can extend the matching by pairing e_i with e_t' and e_t with e_i'. This implies that H has at most t colors, which contradicts that $|S| \geq t + 1$. Hence there is a matching of size t in H. Since none of the edges in G have colors in S, it follows that we can extend the t color-disjoint rainbow spanning trees in G to t edge-disjoint rainbow spanning trees in G.

Theorem 1 where $n = 2t + 1$

Proposition 3

For positive integers $t \geq 1$ *and* $n = 2t + 1$, $r(n, t) = \binom{n-1}{2} = 2t^2 = t$.

Theorem 1 where $n = 2t + 1$

The lower bound is due to proposition 1.
We prove that any edge-coloring of K_{2t+1} with $2t^2 - t + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this graph G.
The proof approach is similar to the case when $n = 2t + 2$. Let m_i be the multiplicity of the color c_i in G.

Theorem 1 where $n = 2t + 1$

WLOG, say the first s colors have multiplicity ≥ 2, which is $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$.
Let G_1 be the spanning subgraph consisting of all edges whose color multiplicity is greater than 1 in G, and G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s}(m_i - 1) = \binom{n}{2} - (2t^2 - t + 1) = 2t - 1 \tag{1}$$

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 2t - 1 + s \leq 4t - 2$$

.

Theorem 1 where $n = 2t + 1$

Claim 3

We can construct t edge-disjoint rainbow forests $F_1, \cdots, F_t$ in G_1 such that if we let $G_0 = G_1 \setminus \bigcup_{i=1}^{l} E(F_i)$, then $|E(G_0)| \leq t$.

To prove the claim, we consider two cases.

Case 1: $m_1 \geq t + 2$.
By equation 1, we have that $s \leq (2t - 1) - (t + 1) + 1 = t - 1$. We construct t edge-disjoint rainbow forests $F_1, \cdots, F_t$ as follows: First take t edges of color c_1 and add one edge to each of $F_1, \cdots, F_t$. Next, pick one edge from each of the remaining $s - 1$ colors and add each of them to a distinct F_i.
Clearly, we can obtain t edge-disjoint rainbow forests in this way.
Furthermore,

$$|E(G_0)| \leq 2t - 1 + s - (t + s - 1) = t$$

, which proves the claim.

Theorem 1 where $n = 2t + 1$

Case 2: $m_1 < t + 2$.
Let $F_1, \cdots, F_t$ be the edge-maximal family of rainbow spanning forests in G_1.
Let $G_0 = G_1 \setminus \bigcup_{i=1}^{t} E(F_i)$. Support $|E(G_0)| > t$, then

$$\sum_{i=1}^{t} |E(F_i)| \leq 2t - 1 + s - (t + 1) = t + s - 2$$

. Since $s \leq 2t - 1$, it follows that there exists some j such that $|E(F_j)| \leq 2$.

Theorem 1 where $n = 2t + 1$

Case 2a: $|E(F_j)| = 1$.
Since $\{F_1, \cdots, F_t\}$, is edge-maximal and $|E(G_0)| \geq t + 1$, it follows that all edges in G_0 share the same color (call it c_1') as the single edge in F_j. Thus $m_1 \geq t + 2$ which contradicts that $m_1 < t + 2$.

Case 2b: $|E(F_j)| = 2$.
Similarly, at least t edges in G_0 share the same colors (named as c'_1, c'_2) as the two edges in F_j. It follows that $m_1 + m_2 \geq t + 2$, hence $s \leq t + 1$.
Since $|E(G_0)| \geq t + 1$, it follows

$$\sum_{i=1}^{t} |E(F_i)| \leq 2t - 1 + s - (t + 1) = t + s - 2 \leq 2t - 1$$

, thus there exists some forest with only one edge, in which case we are done in Case 2a.
Thus, by contradiction, we have $|E(G_0)| \leq t$, and the proof is completed.

Theorem 1 where $n = 2t + 1$

Now we show that $F_1, \cdots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P, we will verify

$$|c(cr(P), G_2) + \sum_{i=1}^{t} |cr(P, F_i)| \geq t(|P| - 1)$$

.

We have

$$|c(cr(P), G_2))| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1)$$
$$\geq \binom{n}{2} - t - \binom{n - |P| + 1}{2} - t(|P| - 1)$$

. Note that the function on right is concave downward on |P|. We can verify it at $|P| = 2$ and $|P| = n$.

Theorem 1 where $n = 2t + 1$

When $|P| = 2$, we have

$$\binom{n}{2} - t - \binom{n-1}{2} - t = n - 1 - 2t \geq 0$$

.

When $|P| = n$, we have

$$\binom{n}{2} - t - t(n-1) = 0$$

.

By theorem 4, $F_1, \cdots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees.

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