A reduction to the metric case

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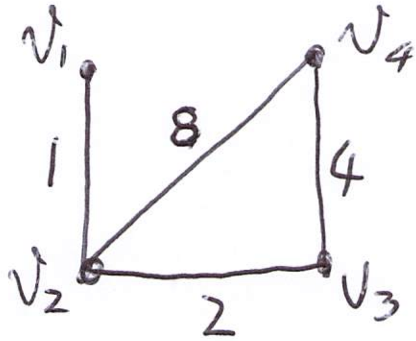
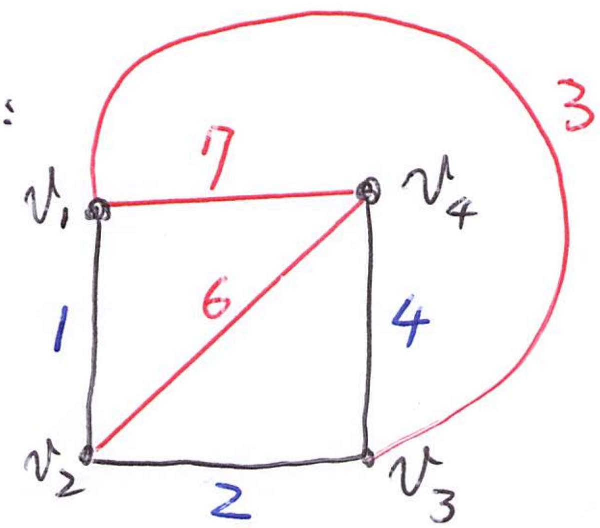
▲ The metric closure of a graph $G=(V,E,w)$ is the complete graph $\overline{G}=(V, V\times V, \overline{w})$, where $\overline{w}(u,v)=d_G(u,v)$ for all $u,v\in V$.

G:



$\overline{G}$:



Since $\overline{w}(u,v)\le w(u,v)$ for all $u,v\in V$,

$c(mrct(\overline{G}))\le c(mrct(G))$.

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Surprisingly, it can be shown that

$$C(mrct(\bar{G})) \geq C(mrct(G)).$$

Therefore, $C(mrct(\bar{G})) = C(mrct(G))$.

An edge (a,b) is called a bad edge if

$(a,b) \notin E$ or $w(a,b) > \bar{w}(a,b)$.

executed at most $O(n^2)$ times.

Algorithm: REMOVE_BAD
Input: A spanning tree T of $\bar{G}$.
Output: A spanning tree Y of G such that $C(Y) \leq C(T)$.

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        Compute all-pairs shortest paths of G.
(I)     while there exists a bad edge in T
            Pick a bad edge (a, b). Root T at a.
            /* assume SP_G(a, b) = (a, x, ..., b) and y is the parent of x */
            if b is not an ancestor of x then
                Y* ← T ∪ (x, b) − (a, b); Y** ← Y* ∪ {(a, x)} − {(x, y)};
            else
                Y* ← T ∪ (a, x) − (a, b); Y** ← Y* ∪ {(b, x)} − {(x, y)};
            if C(Y*) < C(Y**) then Y ← Y* else Y ← Y**
(II)        T ← Y
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$$C(Y) \leq C(T)$$

$$\| $$

$$\min\{C(Y^*), C(Y^{**})\}$$

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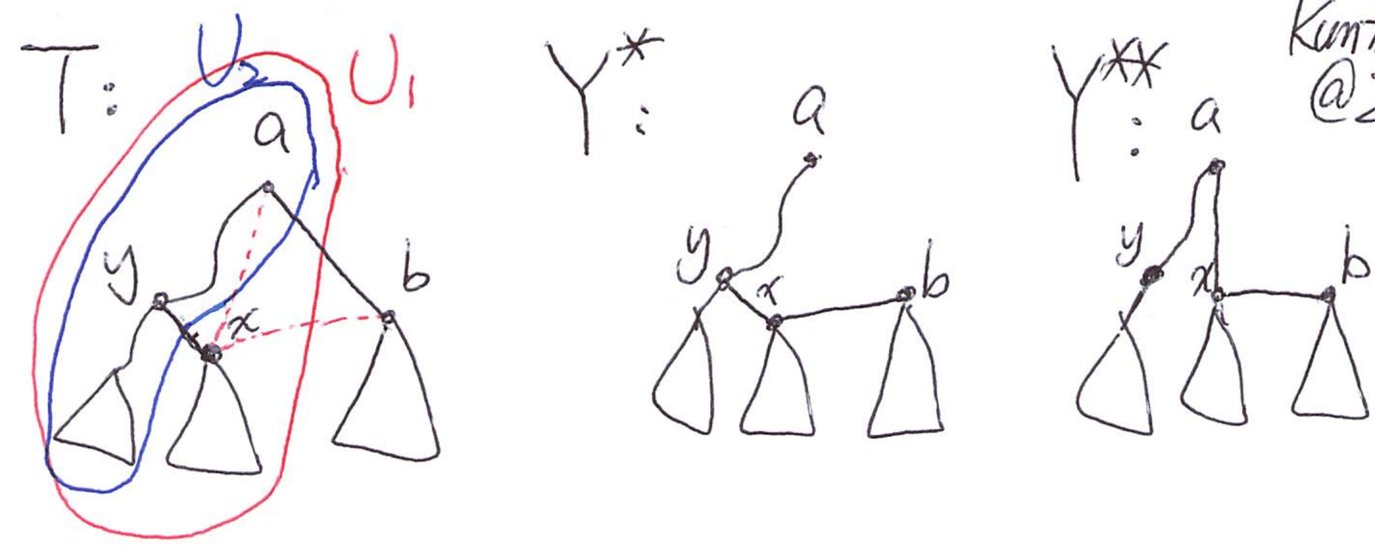

$$U_1 = V(T_a) - V(T_b)$$

$$U_2 = V(T_a) - V(T_b) - V(T_x)$$

If $C(Y^*) > C(T)$

$$\Rightarrow \left(C(Y^*) - C(T)\right)/2 = \sum_{u \in U_1} \sum_{v \in V(T_b)} \left(d_T(u,x) + \bar{w}(x,b) - \left(d_T(u,a) + \bar{w}(a,b)\right)\right) > 0$$

$$\Rightarrow \sum_{u \in U_1} \sum_{v \in V(T_b)} \left(d_T(u,x) - d_T(u,a) - \bar{w}(a,x)\right) > 0$$

$$\Rightarrow |V(T_b)| \sum_{u \in U_1} \left(d_T(u,x) - d_T(u,a) - \bar{w}(a,x)\right) > 0$$

$$\Rightarrow \sum_{u \in U_1} \left(d_T(u,x) - d_T(u,a)\right) > |U_1| \cdot \bar{w}(a,x)$$

$$\Rightarrow \sum_{u \in U_2} \left(d_T(u,x) - d_T(u,a)\right) > |U_1| \, \bar{w}(a,x)$$

3 $$\Rightarrow \sum_{u \in U_2} \left(d_T(u,a) - d_T(u,x)\right) < -|U_1| \, \bar{w}(a,x)$$

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$$\left(C(Y^{**}) - C(T)\right)/2$$

$$= \sum_{u \in U_2} \sum_{v \in V(T_x)} \left(d_T(u,a) + \overline{w}(a,x) - d_T(u,x) \right)$$

$$+ \underbrace{\sum_{u \in V(T_x)} \sum_{v \in V(T_b)} \left(d_T(u,x) + \overline{w}(x,b) - \left(d_T(u,a) + \overline{w}(a,b)\right)\right)}_{\leq 0}$$

$$\leq \sum_{u \in U_2} \sum_{v \in V(T_x)} \left(d_T(u,a) - d_T(u,x) + \overline{w}(a,x) \right)$$

$$= |V(T_x)| \left(\underbrace{\sum_{u \in U_2} \left(d_T(u,a) - d_T(u,x) \right)}_{< -|U_1| \cdot \overline{w}(a,x)} + \underbrace{\overline{w}(a,x)}_{= |U_2| \overline{w}(a,x)} \right)$$

$$< \underbrace{|V(T_x)|}_{>0} \cdot \underbrace{\left(|U_2| - |U_1|\right)}_{<0} \cdot \underbrace{\overline{w}(a,x)}_{\geq 0}$$

$$\leq 0$$

It has been shown that

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$$\min\{C(Y^*), C(Y^{**})\} \le C(T).$$

* For any spanning tree T of $\overline{G}$, it can be transformed into a spanning tree Y of G, in $O(n^3)$ time such that $C(Y) \le C(T)$.
* If there is a $(1+\varepsilon)$-approximation algorithm for ΔMRCT (for $\overline{G}$), then there is a $(1+\varepsilon)$-approximation algorithm for MRCT (for G).
* The transformation from G to $\overline{G}$ (metric closure of G) can be done in time $\underbrace{O(n \log n + m)}_{\text{Dijkstra's algorithm}} * n = O(n^2 \log n + mn)$.