Uniform edge-partition of a tree

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@2019

T: a tree

$E(T)$: the edgeset of T

$e(T)$: the number of edges of T

$\overset{\shortparallel}{n}$ $\quad 1 \leq k \leq e(T)$

A k-tuple $(T_1, T_2, \cdots, T_k)$ is a k-split of T if

(1) each T_i is a connected subgraph of T;

(2) T_i and T_j are edge disjoint for $T_i \neq T_j$;

(3) the union $T_1 \cup T_2 \cup \cdots \cup T_k = T$

The ratio of a k-split $(T_1, T_2, \cdots, T_k)$ of T is defined as $\dfrac{\max_i\{e(T_i)\}}{\min_i\{e(T_i)\}}$.

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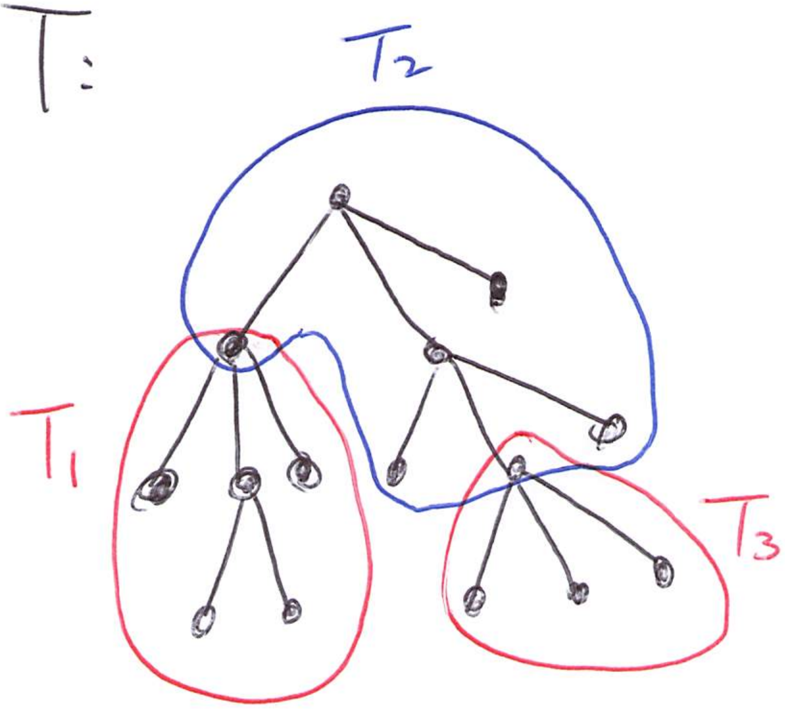
T:



3-split

$$T_1 \cup T_2 \cup T_3 = T$$

$e(T_1) = 5$

$e(T_2) = 6$ ← max

$e(T_3) = 3$ ← min

The ratio is $\frac{e(T_2)}{e(T_3)} = 2$.

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$1 \le r \le e(T)$

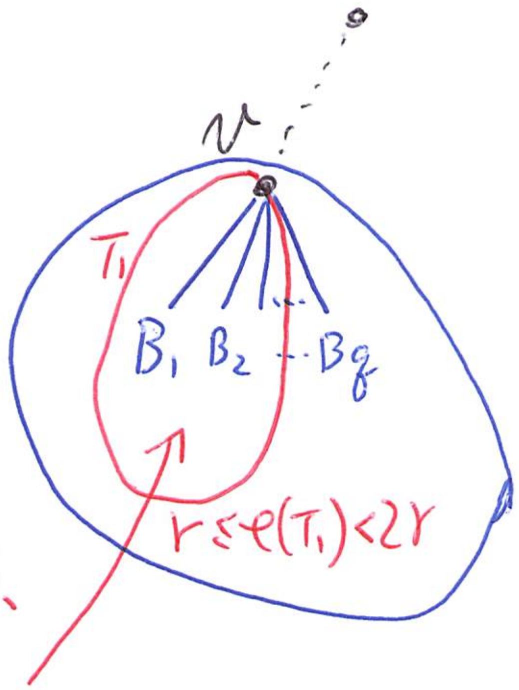
v: $e(T_v) \ge r$ & $e(T_u) < r$ for any child u.

2-split (T_1, T_2)

$$\underline{T_1} \downarrow$$

$r \le e(T_1) < 2r$

T:



$e(T_v) \ge r$

$e(B_1) < r$

$e(B_2) < r$

$\vdots$

$e(B_q) < r$

Find $j \le q$ s.t.

$$\sum_{i=1}^{j-1} e(B_i) < r$$

$$\begin{gathered}\sum_{i=1}^{j} e(B_i) \ge r \\ \shortparallel \\ \underbrace{\sum_{i=1}^{j-1} e(B_i)}_{<r} + \underbrace{e(B_j)}_{<r} < 2r\end{gathered}$$

post-order traversal in linear time

Take $r = \frac{n}{3}$

$\frac{n}{3} \le e(T_1) < \frac{2n}{3}$

$\Rightarrow \frac{n}{3} < e(T_2) \le \frac{2n}{3}$

$\le \frac{2}{3}n$

$$\frac{\max\{e(T_1), e(T_2)\}}{\min\{e(T_1), e(T_2)\}} \le 2$$

$\ge \frac{1}{3}n$

Therefore, for any tree T, there is a 2-split of T with ratio at most 2.

Algorithm Simple-Split

Input: A tree T and an integer $k \leq e(T)$.

Output: A k-split of T.

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1:    Initiate an empty queue $Q$ of trees, and insert $T$ into $Q$.
2:    For $i \leftarrow 1$ to $k-1$ do
2.1:      Choose a tree $Y$ in $Q$ with maximum number of edges.
2.2:      Find a 2-split $(Y_1, Y_2)$ of $Y$ with ratio at most two.
2.3:      Remove $Y$ from $Q$.
2.4:      Insert $Y_1$ and $Y_2$ into $Q$.
3:    Output the $k$ trees in $Q$ as the $k$-split of $T$.
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$Q:$ (T^1)

(T_1^2, T_2^2)

(T_1^3, T_2^3, T_3^3)

$\vdots$ 2-splitting the maximum T_i

$(T_1^k, T_2^k, \cdots, T_k^k)$

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The simple-split algorithm starts at the 1-split (T) and repeatedly computes the (i+1)-split from the i-split by 2-splitting the maximum subgraph.

The resulting k-split is with ratio at most 3.

M_i : the maximum number of edges at the i-th iteration

m_i : minimum 〃

$\overset{\text{Initially}}{\boxed{\frac{M_1}{m_1} = 1}}$ Suppose that $\frac{M_i}{m_i} \le 3$ for some i.

$$M_{i+1} \le M_i \leftarrow Y \begin{array}{l} \nearrow Y_1 \\ \searrow Y_2 \end{array}$$

$$m_{i+1} = \min\{\underset{\geq \frac{M_i}{3}}{m_i}, \underset{\geq \frac{M_i}{3}}{e(Y_1)}, \underset{\geq \frac{M_i}{3}}{e(Y_2)}\} \Rightarrow \frac{M_{i+1}}{m_{i+1}} \le \frac{M_i}{\frac{M_i}{3}} = 3$$

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The i-th iteration takes $O(M_i)$ time.

$$M_i \le 3\underset{\sim}{m_i} \le 3 \cdot \frac{(n-M_i)}{\underbrace{i-1}_{\text{Average of the rest}}} \qquad (T_1^i, T_2^i, \cdots, T_i^i)$$

$$\Rightarrow M_i \le \frac{3n}{i+2}$$

$$c\left(\frac{3n}{1+2} + \frac{3n}{2+2} + \frac{3n}{3+2} + \cdots + \frac{3n}{k+2}\right)$$

$$= 3cn\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{k+2}\right)$$

$$< 3c\underset{=}{n}\underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)}_{O(\log k)}$$

$O(n \log k)$ time.