A PTAS for ΔMRCT

metric

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- PTAS: Polynomial Time Approximation Scheme

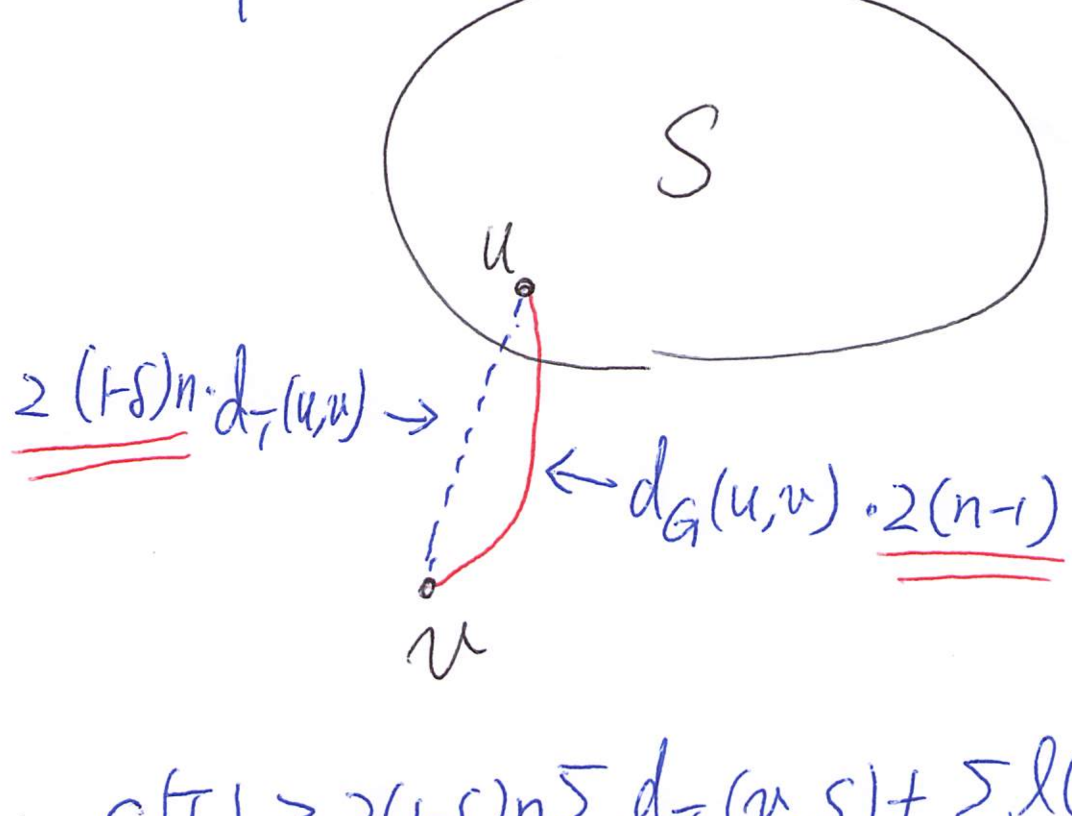
No way!

a. S: a minimal δ-separator of T

b. For each vertex not in S, its lowest ancestor is known.

Y:

$$\frac{C(Y)}{C(T)} \le \frac{1}{1-\delta}$$



$$C(T) \geq 2(1-\delta)n\sum_{v\in V} d_T(v,S) + \sum_{e\in E(S)} l(T,e)\,w(e)$$

$$C(Y) = 2(n-1)\sum_{v\in V} d_G(v,S) + \sum_{e\in E(S)} l(T,e)\,w(e)$$

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* k-star: a tree with no more than k internal nodes.

δ-spine

* A δ-separator can be cut into several δ-paths in a way that the ( $\leq \frac{1}{2}\delta n$) total number of cut nodes and leaves is bounded by $\lceil \frac{2}{\delta} \rceil - 3$.

* There exists a $(\lceil \frac{2}{\delta} \rceil - 3)$-star Y converted from a tree T satisfying

$$C(Y) \leq \left(1 + \frac{\delta}{1-\delta}\right) C(T).$$

* By taking $\delta = \frac{2}{k+3}$, we conclude that an optimal k-star of a metric graph is a $1 + \frac{\frac{2}{k+3}}{1-\frac{2}{k+3}} = 1 + \frac{\frac{2}{k+3}}{\frac{k+1}{k+3}} = 1 + \frac{2}{k+1} = \frac{k+3}{k+1}$ approximation of an MRCT.

S: a minimal δ-separator

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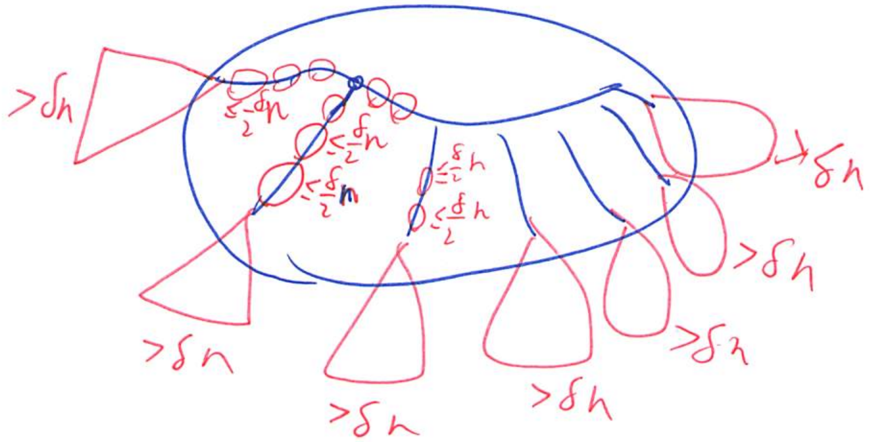


h: # of leaves of S $\quad (\because h(\delta n+1) \le n \Rightarrow h < \frac{1}{\delta})$

h': # of internal nodes of S with degree ≥ 3

$h' \le h-2 \quad (\because \underline{\underline{3}}h' + h \le 2(h'+h-1))$

total number of degrees excluding those nodes with degree 2

Notice that a tree with n nodes has $n-1$ edges, and each edge contributes "two" to the total number of degrees.

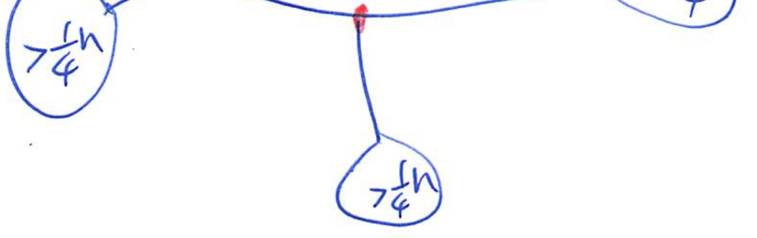
h'': # of additional cutting nodes

$$h'' \le \left\lceil \frac{n - h\delta n}{\frac{\delta n}{2}} \right\rceil - 1 = \left\lceil \frac{2}{\delta} \right\rceil - 2h - 1$$

$$h + h' + h'' \le h + h - 2 + \left\lceil \frac{2}{\delta} \right\rceil - 2h - 1$$

$$= \left\lceil \frac{2}{\delta} \right\rceil - 3$$

a minimal $\frac{1}{4}$-separator



$$\left\lceil \frac{2}{\delta} \right\rceil - 3 = \left\lceil \frac{2}{\frac{1}{4}} \right\rceil - 3 = 5$$

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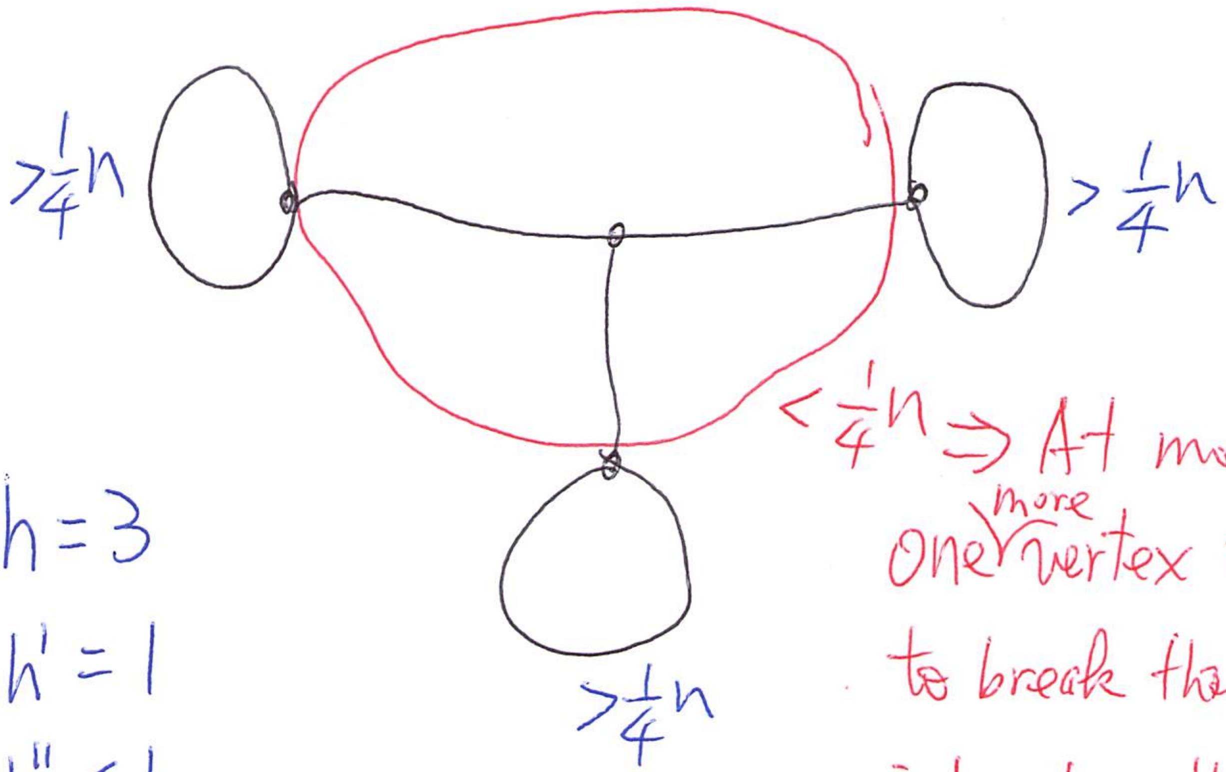
A minimal $\frac{1}{4}$-separator



$<\frac{1}{4}n \Rightarrow$ At most one more vertex is needed to break the separator into $\frac{1}{4}$-paths, where each $\frac{1}{4}$-path has at most $\frac{1}{8}n$ vertices.

$h = 3$

$h' = 1$

$+)\ h'' \leq 1$

≤ 5

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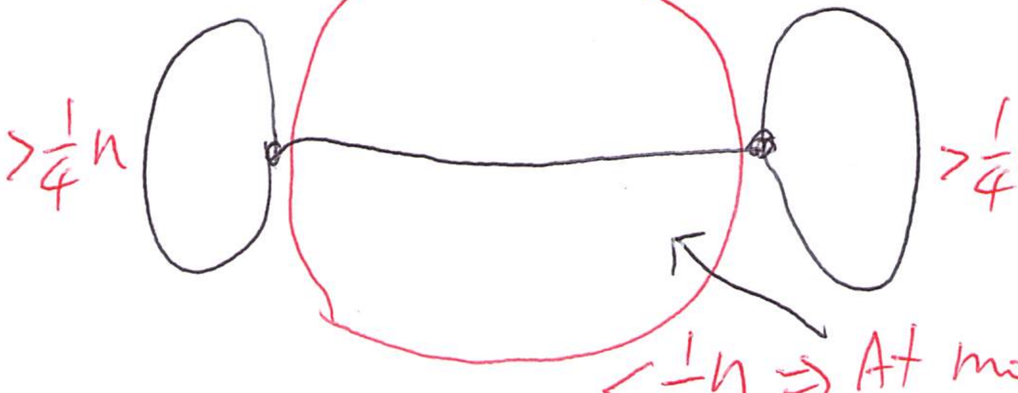
degenerated case



$h = 2$

$h' = 0$

$+)\ h'' \leq 3$

≤ 5

$<\frac{1}{2}n \Rightarrow$ At most three more vertices is needed to break the separator into $\frac{1}{4}$-paths.

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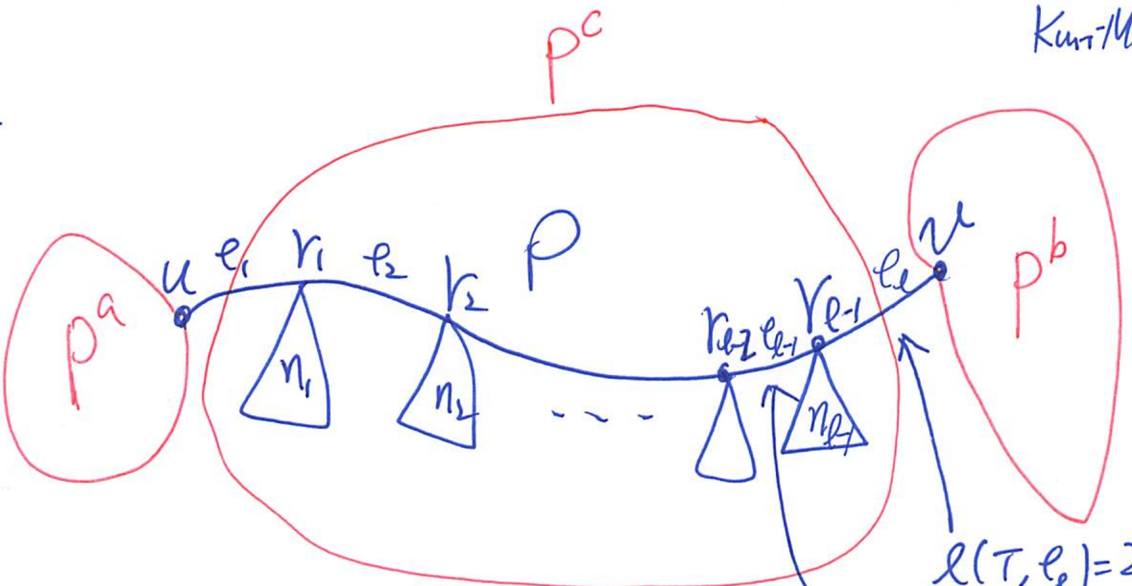
T:



$\ell(T, e_\ell) = 2\times(p^a+p^c)\times p^b$

$\ell(T, e_{\ell-1}) = 2\times(p^a+(p^c-n_{\ell-1}))\times(p^b+n_{\ell-1})$
$= 2((p^a+p^c)\times p^b + (p^a-p^b)\cdot n_{\ell-1} + (p^c-n_{\ell-1})\cdot n_{\ell-1})$

Assume that $p^a \geq p^b$

$p^a+p^b+p^c=n$

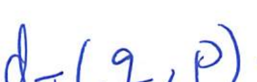
$$\sum_{e\in P}\ell(T,e)\cdot w(e) = 2\sum_{j=1}^{\ell}\left(p^a+p^c-\sum_{j'=j}^{\ell-1} n_{j'}\right)\cdot\left(p^b+\sum_{j'=j}^{\ell-1}n_{j'}\right)w(e_j)$$

$$=2\left(\sum_{j=1}^{\ell}(p^a+p^c)\cdot p^b\cdot w(e_j) + (p^a-p^b)\underbrace{\sum_{j=1}^{\ell}\sum_{j'=j}^{\ell-1} n_{j'}\cdot w(e_j)}_{\sum_{j=1}^{\ell-1} n_j\cdot d_T(u,r_j)} + \sum_{j=1}^{\ell}\underbrace{\left(p^c-\sum_{j'=j}^{\ell-1}n_{j'}\right)}_{\geq 0}\cdot\underbrace{\sum_{j'=j}^{\ell-1}n_{j'}}_{\geq 0}\cdot\underbrace{w(e_j)}_{\geq 0}\right)$$

$n_{\ell-1}(w(e_{\ell-1})+w(e_{\ell-2})+\dots)$
$+n_{\ell-2}(w(e_{\ell-2})+w(e_{\ell-3})+\dots)$
$+\dots$
$+n_1\cdot w(e_1)$

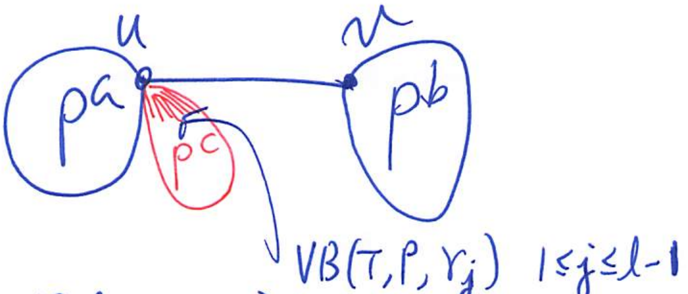
$$\geq 2\left((p^a+p^c)\cdot p^b\cdot w(P) + (p^a-p^b)\sum_{j=1}^{\ell-1} n_j d_T(u,r_j)\right)$$

For $q\in\bigcup_{j=1}^{\ell-1}VB(T,P,r_j)$, $d_T(q,P)$ will be counted at least $2(1-\delta)n$ times.

the routing cost  (X)

$$C(X)\geq 2(1-\delta)n\sum_{q} d_T(q,P) + 2\left((p^a+p^c)\cdot p^b\, w(P) + (p^a-p^b)\sum_{j=1}^{\ell-1} n_j d_T(u,r_j)\right)$$

X':



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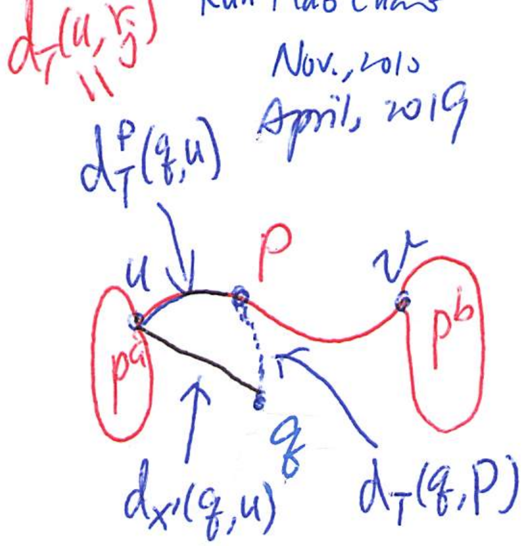


$q \in VB(T,P,r_j)$

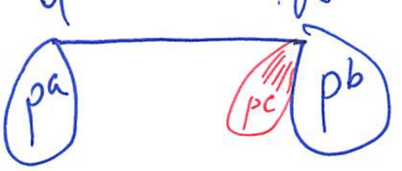
$$d_{X'}(q,u) \le d_T(q,P) + d_T^P(q,u)$$

The routing cost of  (X')

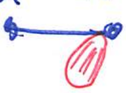
$$\le 2\Big((p^a+p^c)\cdot p^b \cdot \underbrace{w((u,v))}_{\le w(P)} + (n-1)\Big(\sum_{q\in\bigcup_{j=1}^{l-1} VB(T,P,r_j)} d_T(q,P) + \sum_{j=1}^{l-1} n_j \cdot d_T(u,r_j)\Big)\Big)$$

$$\le 2\Big(p^a p^b w(P) + n\sum d_T(q,P) + \boxed{p^b p^c w(P) + n\cdot\sum_{j=1}^{l-1} n_j d_T(u,r_j)}\Big)$$

X'':



$$q \in VB(T,P,r_j),\ 1\le j\le l-1,\ d_{X''}(q,v) \le d_T(q,P) + \overbrace{d_T^P(q,v)}^{w(P)-d_T(u,r_j)\,=\,d_T(r_j,v)\,=}$$

The routing cost of  (X'')

$$\le 2\Big((p^b+p^c)\cdot p^a\, \underbrace{w((u,v))}_{\le w(P)} + (n-1)\Big(\sum_{q\in\bigcup_{j=1}^{l-1} VB(T,P,r_j)} d_T(q,P) + \sum_{j=1}^{l-1} n_j\cdot(w(P)-d_T(u,r_j))\Big)\Big)$$

$$\le 2\Big(p^a p^b w(P) + n\sum d_T(q,P) + \boxed{p^a p^c w(P) + n\cdot\Big(p^c\cdot w(P) - \sum_{j=1}^{l-1} n_j d_T(u,r_j)\Big)}\Big)$$

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$$\boxed{\min\{A,B\} \le \frac{x}{x+y}A + \frac{y}{x+y}B \quad x,y>0}$$

$$\min\{C(X'), C(X'')\}$$

$$\le 2\Big(p^a p^b \omega(p) + n\sum d_T(q,P)\Big) + \frac{p^a}{p^a+p^b}\cdot\Big(p^b p^c \omega(p) + n\sum_{j=1}^{\ell-1} n_j d_T(u,r_j)\Big)$$

$$+ \frac{p^b}{p^a+p^b}\Big(p^a p^c \omega(p) + n\big(p^c\omega(p) - \sum_{j=1}^{\ell-1} n_j \cdot d_T(u,r_j)\big)\Big)\Big)\Big)$$

$$= 2\Big(n\sum d_T(q,P) + \frac{1}{p^a+p^b}\big(p^a \cdot p^b \underbrace{(n-p^c)}_{= p^a+p^b} + 2p^a\cdot p^b\cdot p^c + np^b p^c\big)\omega(p)$$

$$+ \frac{n(p^a-p^b)}{p^a+p^b}\cdot\sum_{j=1}^{\ell-1} n_j\, d_T(u,r_j)\Big)$$

$\omega(p)$'s coefficient

$$\frac{\min\{C(X'), C(X'')\}}{C(X)} \le \max\left\{\frac{1}{1-\delta},\ \frac{\frac{np^b(p^a+p^c)+p^a p^b p^c}{p^a+p^b}}{(p^a+p^c)p^b},\ \frac{\frac{n(p^a-p^b)}{p^a+p^b}}{p^a-p^b}\right\}$$

$$\frac{1}{1-\delta} \ge \frac{1}{1-\frac{1}{2}\delta} = \frac{n}{n-\frac{1}{2}\delta n} \ge \frac{n}{p^a+p^b}$$

$(\because p^c \le \frac{1}{2}\delta n)$

$$= \max\left\{\frac{1}{1-\delta},\ \frac{n}{p^a+p^b} + \frac{p^a\cdot p^c}{(p^a+p^b)(p^a+p^c)}\right\}$$

$\because p^c \le \frac{\delta n}{2}$

$$\frac{n}{p^a+p^b} + \frac{p^a\cdot p^c}{(p^a+p^b)(p^a+p^c)} \leftarrow \frac{p^a}{p^a+p^c} \le 1 \qquad = \frac{1}{1-\delta}$$

$$\le \frac{n}{p^a+p^b} + \frac{p^c}{p^a+p^b} = \frac{n+p^c}{p^a+p^b} = \frac{n+p^c}{n-p^c} \le \frac{n+\frac{\delta}{2}n}{n-\frac{\delta}{2}n} = \frac{2+\delta}{2-\delta} = 1+\frac{2\delta}{2-\delta}$$

$$= 1+\frac{\delta}{1-\frac{\delta}{2}} \le 1+\frac{\delta}{1-\delta} = \frac{1}{1-\delta}$$

$$\frac{1+x}{1-x}$$

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Integer k

Take $k = \frac{2}{\delta} - 3$.

$$\Rightarrow \delta = \frac{2}{k+3}$$

$$\frac{1}{1-\delta} = \frac{1}{1-\left(\frac{2}{k+3}\right)} = \frac{k+3}{k+1}$$

An optimal k-star of a metric graph is a $\frac{k+3}{k+1}$-approximation of an MRCT.

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k-star : (S, T, L)

internal nodes, tree topology, leaves.

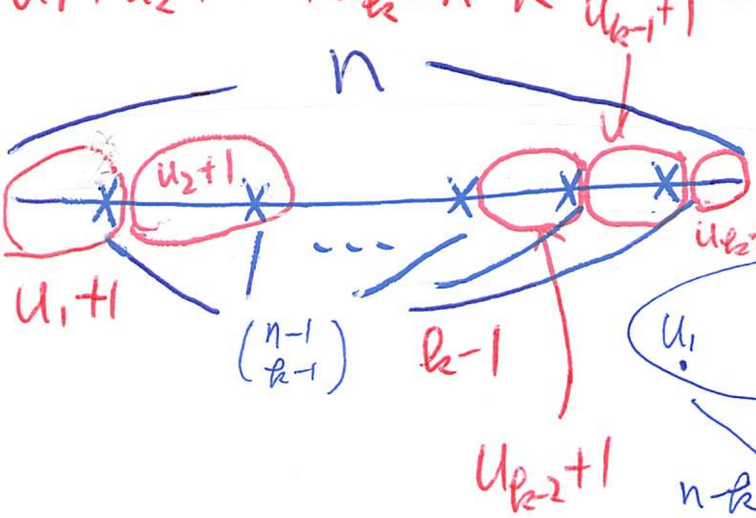
$v_1 \; v_2 \; \cdots\cdots \; v_n$

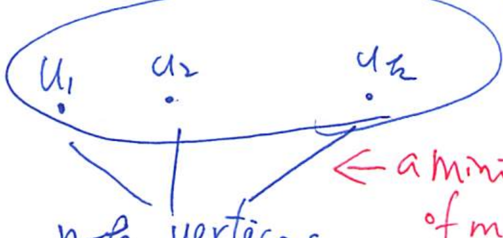
$\Downarrow \binom{n}{k}$

$u_1 + u_2 + \cdots + u_k = n - k$

k internal nodes.

k^{k-2} possible tree topologies.





$O(n^3)$-time (the assignment Problem)

← a minimum-cost way of matching which obeys the degree constraints.