S: a minimal δ-separator

Kun-Mao Chao
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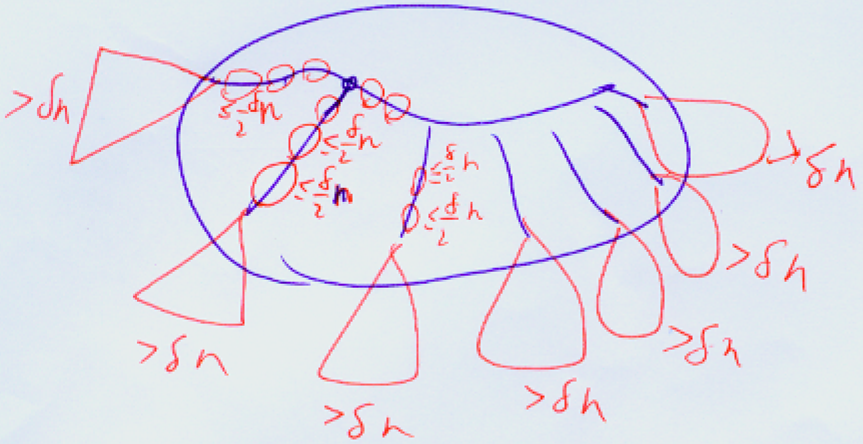


h: # of leaves of S $\quad (\because h(\delta n+1) \leq n \Rightarrow h < \frac{1}{\delta})$

h': # of internal nodes of S with degree ≥ 3

$$h' \leq h-2 \qquad (\because 3h' + h \leq 2(h'+h-1))$$

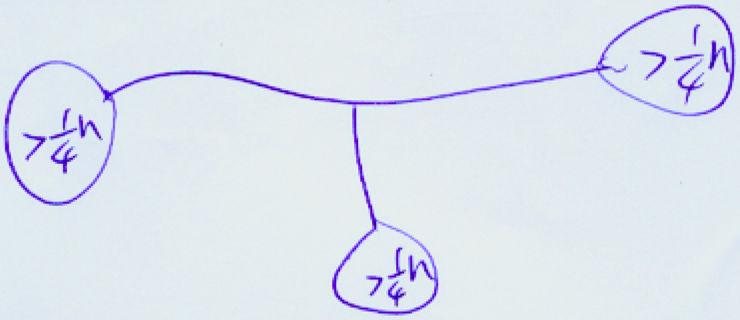
h'': # of additional cutting nodes

$$h'' \leq \left\lceil \frac{n-h\delta n}{\frac{\delta n}{2}} \right\rceil - 1 = \left\lceil \frac{2}{\delta} \right\rceil - 2h - 1$$

$$h+h'+h'' \leq h+h-2+\left\lceil \frac{2}{\delta} \right\rceil - 2h - 1$$

$$= \left\lceil \frac{2}{\delta} \right\rceil - 3$$

a minimal $\frac{1}{4}$-separator



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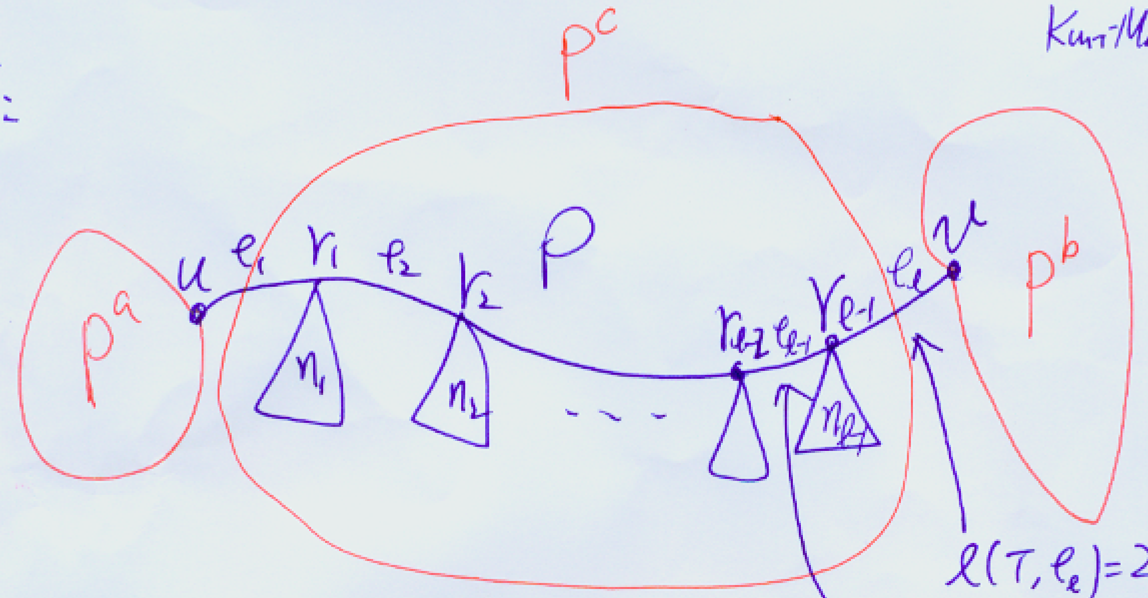
T:



$\ell(T, e_\ell) = 2 \times (p^a + p^c) \times p^b$

$\ell(T, e_{\ell-1}) = 2 \times (p^a + (p^c - n_{\ell-1})) \times (p^b + n_{\ell-1})$
$= 2\big((p^a + p^c) \times p^b + (p^a - p^b) \cdot n_{\ell-1} + (p^c - n_{\ell-1}) \cdot n_{\ell-1}\big)$

Assume that $p^a \geq p^b$

$p^a + p^b + p^c = n$

$$\sum_{e \in P} \ell(T, e) \cdot w(e) = 2 \sum_{j=1}^{\ell} \Big(p^a + p^c - \sum_{j'=j}^{\ell-1} n_{j'}\Big) \cdot \Big(p^b + \sum_{j'=j}^{\ell-1} n_{j'}\Big) w(e_j)$$

$$= 2 \Big(\sum_{j=1}^{\ell} (p^a + p^c) \cdot p^b \cdot w(e_j) + (p^a - p^b) \underbrace{\sum_{j=1}^{\ell} \sum_{j'=j}^{\ell-1} n_{j'} \cdot w(e_j)}_{\sum_{j=1}^{\ell-1} n_j \cdot d_T(u, r_j)}$$
$$+ \sum_{j=1}^{\ell} \underbrace{\Big(p^c - \sum_{j'=j}^{\ell-1} n_{j'}\Big)}_{\geq 0} \cdot \underbrace{\sum_{j'=j}^{\ell-1} n_{j'}}_{\geq 0} \cdot \underbrace{w(e_j)}_{\geq 0} \Big)$$

$$\geq 2 \Big((p^a + p^c) \cdot p^b \cdot w(P) + (p^a - p^b) \sum_{j=1}^{\ell-1} n_j d_T(u, r_j) \Big)$$

For $q \in \bigcup_{j=1}^{\ell-1} VB(T, P, r_j)$, $d_T(q, P)$ will be counted at least $2(1-\delta)n$ times.

the routing cost  (X)

$$C(X) \geq 2(1-\delta) n \sum_{q \in \triangle\triangle\triangle} d_T(q, P) + 2 \Big((p^a + p^c) \cdot p^b \, w(P) + (p^a - p^b) \sum_{j=1}^{\ell-1} n_j d_T(u, r_j) \Big)$$

X':

Kun-Mao Chang

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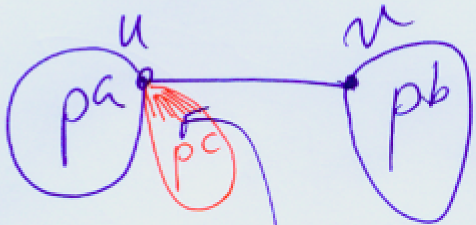
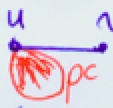


$VB(T,P,r_j)\quad 1\le j\le l-1$

$q\in VB(T,P,r_j)$

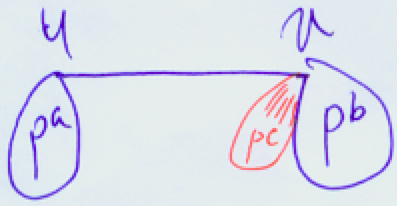
$$d_{X'}(u,q)\le d_T(q,P)+d_T^P(q,u)$$

The routing cost of  (X')

$$\le 2\Big((p^a+p^c)\cdot p^b\cdot \underbrace{w((u,v))}_{\le w(P)} + (n-1)\Big(\sum_{q\in\bigcup_{j=1}^{l-1}VB(T,P,r_j)} d_T(q,P)+\sum_{j=1}^{l-1} n_j\cdot d_T(u,r_j)\Big)\Big)$$

$$\le 2\Big(p^a p^b w(P) + n\sum d_T(q,P) + \boxed{p^b p^c w(P) + n\cdot\sum_{j=1}^{l-1} n_j d_T(u,r_j)}\Big)$$

X'':



$w(P)-d_T(u,r_j)$

$\parallel$

$d_T(r_j,v)$

$\parallel$

$q\in VB(T,P,r_j),\ 1\le j\le l-1,\ d_{X''}(q,v)\le d_T(q,P)+d_T^P(q,v)$

The routing cost of  (X'')

$$=2\Big((p^b+p^c)\cdot p^a\,\underbrace{w((u,v))}_{\le w(P)} + (n-1)\Big(\sum_{q\in\bigcup_{j=1}^{l-1}VB(T,P,r_j)} d_T(q,P)+\sum_{j=1}^{l-1} n_j\cdot(w(P)-d_T(u,r_j))\Big)\Big)$$

$$\le 2\Big(p^a p^b w(P) + n\sum d_T(q,P) + \boxed{p^a p^c w(P) + n\cdot\Big(p^c\cdot w(P) - \sum_{j=1}^{l-1} n_j d_T(u,r_j)\Big)}\Big)$$

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$\min\{C(X'), C(X'')\}$

$$\leq 2\Big(p^a p^b \omega(p) + n\sum d_T(q,P) + \frac{p^a}{p^a+p^b}\cdot\Big(p^b p^c \omega(p) + n\sum_{j=1}^{\ell-1} n_j \cdot d_T(u,r_j)\Big)$$

$$+ \frac{p^b}{p^a+p^b}\Big(p^a p^c \omega(p) + n\Big(p^c \omega(p) - \sum_{j=1}^{\ell-1} n_j \cdot d_T(u,r_j)\Big)\Big)\Big)$$

$$= 2\Big(n\sum d_T(q,P) + \frac{1}{p^a+p^b}\Big(p^a \cdot p^b \underbrace{(n-p^c)}_{= p^a+p^b} + 2p^a \cdot p^b \cdot p^c + np^b p^c\Big)\omega(p)$$

$$+ \frac{n(p^a-p^b)}{p^a+p^b}\cdot\sum_{j=1}^{\ell-1} n_j\, d_T(u,r_j)\Big)$$

$\omega(p)$'s coefficient

$$\frac{\min\{C(X'), C(X'')\}}{C(X)} \leq \max\left\{\frac{1}{1-\delta}, \frac{\dfrac{np^b(p^a+p^c)+p^a p^b p^c}{p^a+p^b}}{(p^a+p^c)p^b}, \frac{\dfrac{n(p^a-p^b)}{p^a+p^b}}{p^a-p^b}\right\}$$

$$= \max\left\{\frac{1}{1-\delta}, \frac{n}{p^a+p^b} + \frac{p^a \cdot p^c}{(p^a+p^b)(p^a+p^c)}\right\}$$

$\because p^c \leq \frac{\delta n}{2}$

$$\frac{n}{p^a+p^b} + \frac{p^a \cdot p^c}{(p^a+p^b)(p^a+p^c)}$$

$$\leq \frac{n}{p^a+p^b} + \frac{p^c}{p^a+p^b} = \frac{n+p^c}{p^a+p^b} = \frac{n+p^c}{n-p^c} \leq \frac{n+\frac{\delta}{2}n}{n-\frac{\delta}{2}n} = \frac{2+\delta}{2-\delta} = 1+\frac{2\delta}{2-\delta}$$

$$= 1+\frac{\delta}{1-\frac{\delta}{2}} \leq 1+\frac{\delta}{1-\delta} = \frac{1}{1-\delta}$$

Integer k

Take $k = \frac{2}{\delta} - 3$.

$\Rightarrow \delta = \frac{2}{k+3}$

$$\frac{1}{1-\delta} = \frac{1}{1-\left(\frac{2}{k+3}\right)} = \frac{k+3}{k+1}$$

An optimal k-star of a metric graph is a $\frac{k+3}{k+1}$ approximation of an MRCT.

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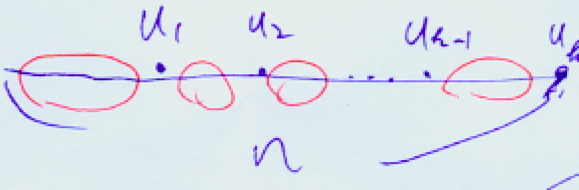
k-star : $(\underset{\substack{\uparrow \\ \text{internal nodes}}}{S}, \underset{\substack{\uparrow \\ \text{tree topology}}}{T}, \underset{\substack{\uparrow \\ \text{leaves}}}{L})$

$\Downarrow \binom{n}{k}$

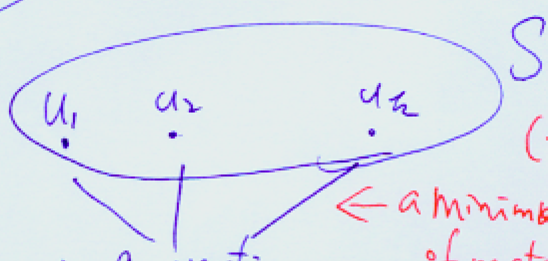
k internal nodes.

k^{k-2} possible tree topologies.



$u_1, u_2, \ldots, u_k$

$\binom{n-1}{k-1}$



$O(n^3)$-time (the assignment Problem)

← a minimum-cost way of matching which obeys the degree constraints.

$n-k$ vertices.