Exercises*

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1. Prove that if the weights on the edges of a connected graph are distinct, then there is a unique minimum spanning tree.
2. Prove or disprove that if unique, the shortest edge is included in any minimum spanning tree.
3. Let e be a minimum-weight edge in a graph G. Show that e belongs to some minimum spanning tree of G.
4. Let e be a maximum-weight edge on some cycle of $G = (V, E)$ and $G' = (V, E - \{e\})$. Show that a minimum spanning tree of G' is also a minimum spanning tree of G.
5. Devise an algorithm to determine the smallest change in edge cost that causes a change of the minimum spanning tree.
6. Given a graph, its minimum spanning tree, and an additional vertex with its associated edges and edge costs, devise an algorithm for rapidly updating the minimum spanning tree.
7. Prove or disprove that Borůvka's, Kruskal's, and Prim's algorithms still apply even when the weights may be negative.
8. Devise an algorithm to find a *maximum* spanning tree of a given graph. How efficient is your algorithm?
9. Devise an algorithm to find a minimum spanning forest, under the restriction that a specified subset of the edges must be included. Analyze its running time.
10. Prove or disprove that, if unique, the shortest edge is included in any shortest-paths tree.
11. Show that Kruskal's, Prim's, and Dijkstra's algorithms still apply even when the problem statement requires the inclusion of specific edges.
12. Apply the Bellman-Ford algorithm to Figure 1, and show how it detects the negative cycle in the graph.
13. Given a directed or undirected graph, devise an $O(m+n)$ time algorithm that detects whether there exists a cycle in the graph.
14. Devise an algorithm that determines the number of different shortest paths from a given source to a given destination.

*An excerpt from the book "Spanning Trees and Optimization Problems," by Bang Ye Wu and Kun-Mao Chao (2004), Chapman & Hall/CRC Press, USA.

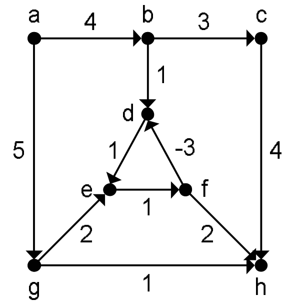

Figure 1: A directed graph with a negative-weight cycle.

15. Give an example of a graph G and a vertex v, such that the minimum spanning tree of G is exactly the shortest-paths tree rooted at v.

16. Give an example of a graph G and a vertex v, such that the minimum spanning tree of G is very different from the shortest-paths tree rooted at v.

17. Let T be the shortest-paths tree of a graph G rooted at v. Suppose now that all the weights in G are increased by a constant number. Is T still the shortest-paths tree?

18. Devise an algorithm that finds a shortest path from u to v for given vertices u and v. If we solve the shortest-paths tree problem with root u, we solve this problem, too. Can you find an asymptotically faster algorithm than the best shortest-paths tree algorithm in the worst case?

19. Compare Dijkstra's algorithm with Prim's algorithm.

20. If the input graph is known to be a directed acyclic graph, can you improve the performance of the Bellman-Ford algorithm?

21. What is an MRCT of a complete graph with unit length on each edge? Prove your answer.

22. What is an MRCT of a complete bipartite graph $K_{m,n}$ with unit length on each edge? Prove your answer.

23. What is the routing cost of an n-vertex path with unit cost on each edge?

24. Show that the problem of finding a minimum routing cost path visiting each vertex exactly once is NP-hard. (Hint: Consider the Hamiltonian path problem.)

25. Design an algorithm for finding a centroid of a tree. What is the time complexity of your algorithm?

26. Give a tree with two centroids.

27. Let $r : V \to Z^+$ be a vertex weight and define the r-centroid of a tree T to be the vertex c such that if we remove c from T, the total vertex weight of each component is no more than half of the total vertex weight. Generalize the algorithm in Exercise 25 to find a r-centroid of a tree.

28. Show that for a tree with positive edge lengths, the median coincides with the centroid.

29. Design an algorithm to find a δ-separator of a tree. What is the time complexity?

30. Let $T = (V, E, w)$ be a tree with $V = \{v_i | 1 \le i \le n\}$ and $r(v_i) = i$.

    (a) What is the p.r.c. cost if $T = (v_1, v_2, \ldots, v_n)$ is a path and each edge has unit length?

    (b) What is the p.r.c. cost if T is a star centered at v_1 and $w(v_1, v_i) = i$ for each $1 < i \le n$?

31. Find the s.r.c. cost for each of the two trees in the previous problem.

32. What is the r-centroid of the path in Exercise 30?

33. For any three points in the Euclidean plane, what is the length of the Steiner minimal tree?

34. What is the rectilinear distance between two points $(10, 20)$ and $(30, 50)$?

35. What are the radius, diameter, and center of a path consisting of n vertices and weighted edges?