A note on Eccentricities, diameters, and radii

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Let $G = (V, E, w)$ be a graph and $U \subset V$. By $D_G(v, U)$, we denote the maximum distance from vertex v to any vertex in U. For a vertex v, the *eccentricity* of v is the maximum of the distance to any vertex in the graph, i.e., $\max_{u \in V}\{d_G(v, u)\}$ or $D_G(v, V)$. The *diameter* of a graph is the maximum of the eccentricity of any vertex in the graph. (The term "diameter" is overloaded. It is defined as the maximum eccentricity and also as the path of length equal to the maximum eccentricity.) In other words, the diameter is the longest distance between any two vertices in the graph. Recall that the distance between two vertices is the length of their shortest path in the graph. It should not be confused with the longest path in the graph.

The *radius* of a graph is the minimum eccentricity among all vertices in the graph, and a *center* of a graph is a vertex with eccentricity equal to the radius. For a general graph, there may be several centers and a center is not necessarily on a diameter. For example, in Figure 1(a), the shortest path between v_1 and v_4 is a diameter of length 6. Meanwhile v_2 and v_3 are endpoints of another diameter. The four vertices represented by white circles are centers of the graph, and the radius is 4. Note that the centers are not on any diameter. The diameter, radius and center of a graph can be found by computing the distances between all pairs of vertices. It takes $O(|V||E| + |V|^2 \log |V|)$ time for a general graph.

Any pair of vertices has a unique simple path in a tree. For this reason, the diameter, radius and centers of a tree are more related. For an unweighted tree $T = (V, E)$, it can be easily verified that

$$2 \times radius - 1 \le diameter \le 2 \times radius. \tag{1}$$

For positive weighted tree $T = (V, E, w)$, we can also have

$$2 \times radius - \max_e\{w(e)\} \le diameter \le 2 \times radius. \tag{2}$$

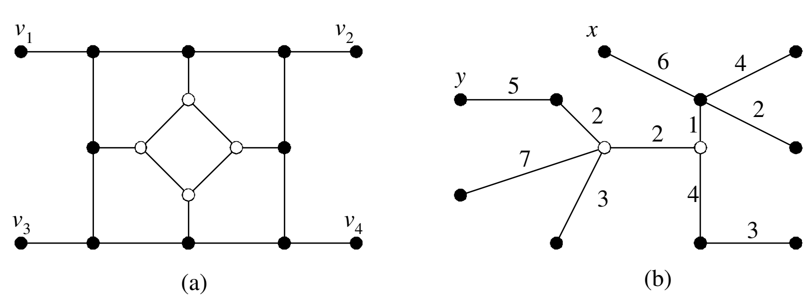

Figure 1: Diameters, centers and radii of (a) an unweighted graph; and (b) a weighted tree.

Figure 1(b) illustrates an example of the diameter, radius, and center of a tree. The two vertices represented by white circles are the centers of the tree. There are four diameters of length 16 in the tree. Vertices x and y are the endpoints of a diameter. The radius is 9, and the centers are on the diameters.

More efficient algorithms for computing diameter, radius, and centers are available if the graph is a tree. Let $T = (V, E, w)$ be a rooted tree. By T_r, we denote the subtree rooted at vertex $r \in V$, which is the subgraph induced on vertex r and all its descendants. Let $child(r)$ denote the set of children of v. The eccentricity of the root of a tree can be computed by the following recurrence relation.

$$D_{T_r}(r, V(T_r)) = \max_{s \in child(r)} \{D_{T_s}(s, V(T_s)) + w(r, s)\}. \tag{3}$$

By a recursive algorithm or an algorithm visiting the vertices in postorder, the eccentricity can be computed in linear time since each vertex is visited once. To make it clear, we give a recursive algorithm in the following.

Algorithm: ECCENT(T_r)
Input: A tree $T_r = (V, E, w)$ rooted at r.
Output: The eccentricity of r in T_r.
1: **if** r is a leaf **then**
  return 0;
2: **for** each child s of r **do**
  compute ECCENT(T_s) recursively;
3: **return** $\max_{s \in child(r)}\{$ECCENT$(s) + w(r, s)\}$.

To find the eccentricity of a vertex in an unrooted tree, we can root the tree at the vertex and employ the ECCENT algorithm. Therefore, we have the next lemma.

Lemma 1: The eccentricity of a vertex in a tree can be computed in linear time.

Let x, y and z be three vertices in a tree T. It can be easily verified that the three paths $SP_T(x, y)$, $SP_T(x, z)$ and $SP_T(y, z)$ intersect at a vertex. Define $c(x, y, z)$ to be the intersection vertex. We have

$$d_T(x, c(x, y, z)) = \frac{1}{2}\left(d_T(x, y) + d_T(x, z) - d_T(y, z)\right), \tag{4}$$

or equivalently

$$d_T(x, c(x, y, z)) = \frac{1}{2}\left(d_T(x, y) + d_T(x, z) + d_T(y, z)\right) - d_T(y, z). \tag{5}$$

We now derive some properties to help us find the diameter of a tree.

Fact 1: Suppose that $SP_T(v_1, v_2)$ is a diameter of T and r is a vertex on the diameter. For any vertex x, $d_T(x, r) \le \max\{d_T(r, v_1), d_T(r, v_2)\}$.

Proof: Otherwise $SP_T(x, v_1)$ or $SP_T(x, v_2)$ is a path longer than the diameter. □

The property can be easily extended to the case where r is not on the diameter. Let $u = c(r, v_1, v_2)$. Without loss of generality, let $d_T(u, v_1) \ge d_T(u, v_2)$. For any vertex x, we have $d_T(x, u) \le d_T(v_1, u)$ by Fact 1. Then

$$d_T(x, r) \le d_T(x, u) + d_T(u, r) \le d_T(v_1, u) + d_T(u, r) = d_T(v_1, r),$$

which implies that v_1 is the farthest vertex to r. We can conclude that, for any vertex, one of the endpoints of a diameter must be the farthest vertex. Furthermore the converse of the property is also true.

Let r be any vertex in a tree and v_3 be the vertex farthest to r. We shall show that v_3 must be an endpoint of a diameter. Suppose that $SP_T(v_1, v_2)$ is a diameter. By the above property, one of the endpoints, say v_1, is the farthest vertex to r, i.e.,

$$d_T(r, v_1) = d_T(r, v_3) \geq d_T(r, v_2).$$

Let $u_1 = c(r, v_1, v_2)$ and $u_2 = c(r, v_1, v_3)$. Then u_2 must be on the path $SP_T(v_1, u_1)$, for otherwise

$$\begin{aligned} d_T(v_1, v_3) &= d_T(v_1, u_2) + d_T(u_2, v_3) \\ &= 2d_T(v_1, u_2) > 2d_T(v_1, u_1) \\ &> d_T(v_1, v_2), \end{aligned}$$

a contradiction. As a result, $d_T(v_3, v_2) = d_T(v_1, v_2)$ and $SP_T(v_3, v_2)$ is also a diameter. We have the next lemma.

Lemma 2: Let r be any vertex in a tree T. If v is the farthest vertex to r, the eccentricity of v is the diameter of T.

The following algorithm uses the property to find the diameter of a tree.

Algorithm: TREEDIAMETER
Input: A tree $T = (V, E, w)$.
Output: The diameter of T.
1: Root T at an arbitrary vertex r.
2: Use ECCENT to find the farthest vertex v to r.
3: Root T at v.
4: Use ECCENT to find the eccentricity of v.
5: Output the eccentricity of v as the diameter of T.

It is obvious that the algorithm runs in linear time. The radius and the center can be obtained from a diameter. Suppose that $P = SP_T(v_1, v_2)$ is a diameter. Starting at v_1 and traveling along the path P, we compute the distance $d_T(u, v_1)$ for each vertex u on the path. Let u_1 be the last encountered vertex such that $d_T(v_1, u_1) \leq \frac{1}{2}w(P)$ and u_2 be the next vertex to u_1 (Figure 2(a)). By the definition of u_1, u_2 is the first encountered vertex such that $d_T(v_1, u_2) > \frac{1}{2}w(P)$. We claim that u_1 or u_2 is a center of the tree. Let $P_1 = SP_T(v_1, u_1)$ and $P_2 = SP_T(u_2, v_2)$. First, by Fact 1, the eccentricities of u_1 and u_2 must be $d_T(u_1, v_2)$ and $d_T(u_2, v_1)$ respectively. Otherwise P cannot be a diameter. For any vertex x connected to P at a vertex in P_1, we have $d_T(x, v_2) > d_T(u_1, v_2)$. Similarly, for any vertex x connected to P at a vertex in P_2, we have $d_T(x, v_1) > d_T(u_2, v_1)$. Consequently the eccentricity of any vertex is at least $\min\{d_T(u_1, v_2), d_T(u_2, v_1)\}$, and u_1 or u_2 must be a center. Therefore the center as well as the radius of a tree can be computed in linear time since the diameter can be found in linear time.

Theorem 3: The diameter, radius, and center of a tree can be computed in linear time.

Let us examine some more properties of the diameters of a tree. We assume that all edge lengths are positive.

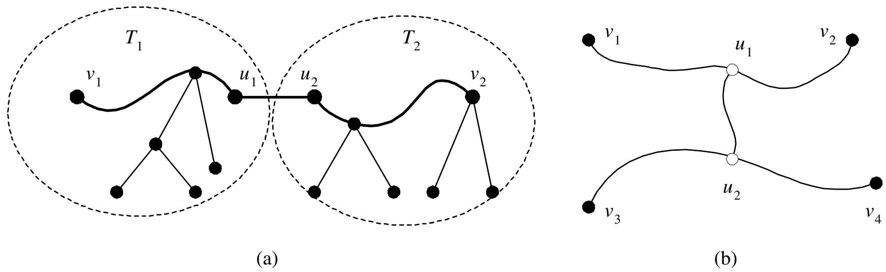

Figure 2: (a) Finding a center on a diameter. (b) Two diameters cannot be disjoint.

Fact 2: Two diameters of a tree cannot be disjoint.

Proof: Suppose that $SP_T(v_1, v_2)$ and $SP_T(v_3, v_4)$ are two disjoint diameters of a tree T. Let $u_1 = c(v_3, v_1, v_2)$ and $u_2 = c(v_1, v_3, v_4)$ (Figure 2(b)). We have

$$\begin{aligned} d_T(v_1, v_3) + d_T(v_2, v_4) &= d_T(v_1, v_2) + d_T(v_3, v_4) + 2d_T(u_1, u_2) \\ &> 2d_T(v_1, v_2) \end{aligned}$$

since $d_T(v_1, v_2) = d_T(v_3, v_4)$ is the diameter and $d_T(u_1, u_2) > 0$. It implies that the path from v_1 to v_3 or the path from v_2 to v_4 is longer than the diameter, a contradiction. $\square$

Let $\mathcal{P}$ be a set of more than two paths of a tree and the paths intersect each other. One can easily verify that all the paths in $\mathcal{P}$ share a common vertex. Otherwise there exists a cycle, which contradicts the definition of a tree. Therefore we can have the next property.

Fact 3: All diameters of a tree share at least one common vertex.