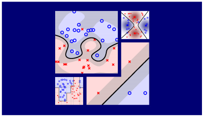
Machine Learning Techniques
(機器學習技法)



Lecture 9: Decision Tree

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Roadmap

1. Embedding Numerous Features: Kernel Models
2. Combining Predictive Features: Aggregation Models

Lecture 8: Adaptive Boosting

optimal re-weighting for diverse hypotheses and adaptive **linear aggregation** to **boost ‘weak’ algorithms**

Lecture 9: Decision Tree

- Decision Tree Hypothesis
- Decision Tree Algorithm
- Decision Tree Heuristics in C&RT
- Decision Tree in Action

3. Distilling Implicit Features: Extraction Models

What We Have Done

blending: aggregate after getting g_t;

aggregation type	blending	
uniform	voting/averaging	
non-uniform	linear	
conditional	stacking	

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conditional	stacking	**Decision Tree**

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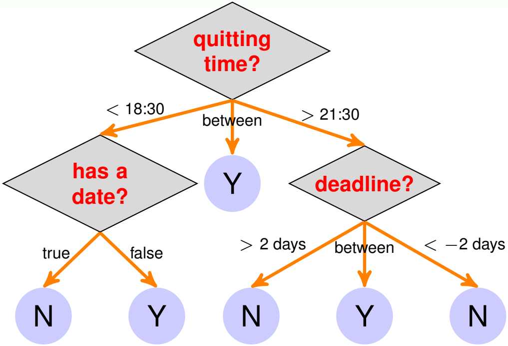
conditional	stacking	**Decision Tree**

decision tree: a traditional learning model that realizes **conditional aggregation**

Decision Tree for Watching MOOC Lectures



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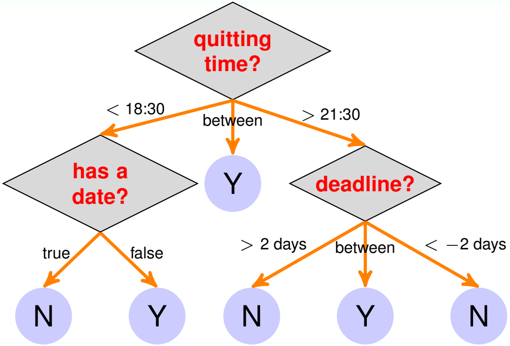

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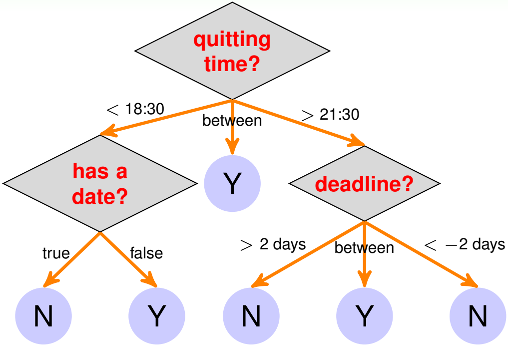

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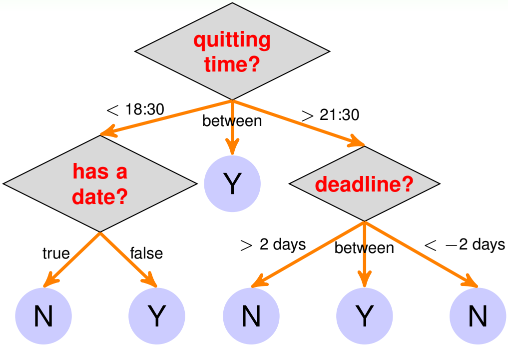

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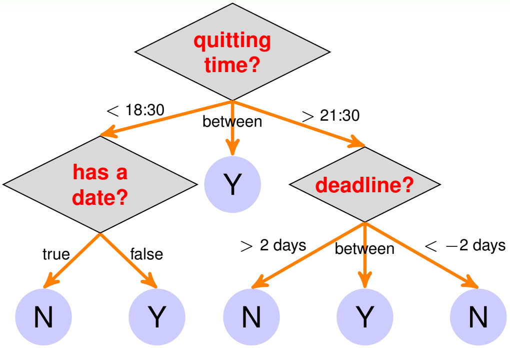

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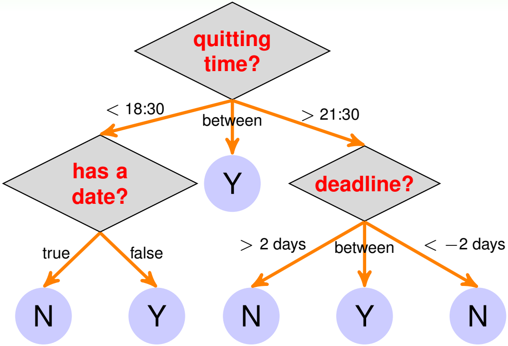
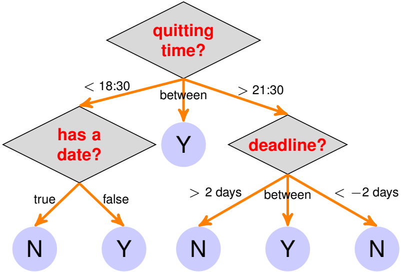


decision tree: arguably one of the most **human-mimicking models**

Recursive View of Decision Tree

Path View: $G(\mathbf{x}) = \sum_{t=1}^{T} [\![\mathbf{x} \text{ on path } t]\!] \cdot \text{leaf}_t(\mathbf{x})$



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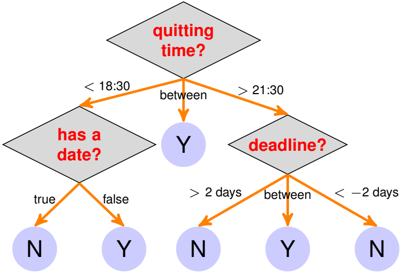

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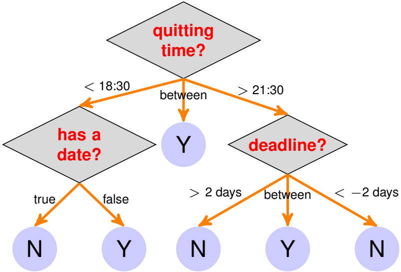

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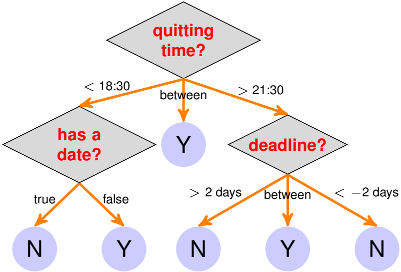

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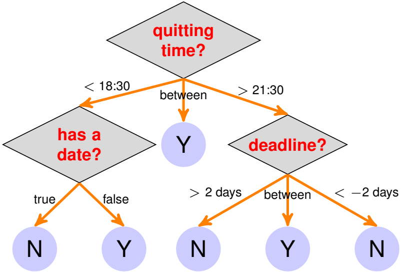

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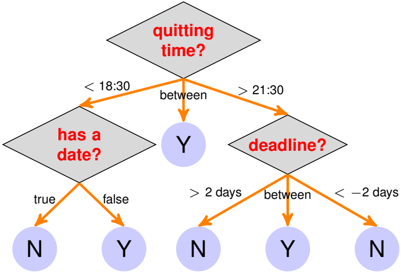

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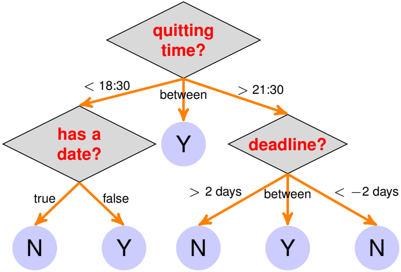

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tree = (root, sub-trees), just like what
your data structure instructor would say :-)

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Usefulness

- human-explainable: **widely used** in business/medical data analysis

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decision tree: mostly **heuristic** **but useful** on its own

Fun Time

The following C-like code can be viewed as a decision tree of three leaves.

```
if (income > 100000) return true;
else {
  if (debt > 50000) return false;
  else return true;
}
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What is the output of the tree for (`income`, `debt`) = (98765, 56789)?

1. true
2. false
3. 98765
4. 56789

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2. false
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Reference Answer: 2

You can simply trace the code. The tree expresses a complicated boolean condition $[\![\mathtt{income} > 100000 \text{ or } \mathtt{debt} \leq 50000]\!]$.

A Basic Decision Tree Algorithm

$$G(\mathbf{x}) = \sum_{c=1}^{C} [\![b(\mathbf{x}) = c]\!] \, G_c(\mathbf{x})$$

function DecisionTree$\left(\text{data } \mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{N}\right)$

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four choices: number of branches, branching criteria, termination criteria, & base hypothesis

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function DecisionTree(data $\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{N}$)
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disclaimer:
C&RT here is based on **selected components** of **CART**^{TM} **of California Statistical Software**

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C&RT: **bi-branching** by **purifying**

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by E_{in} of optimal constant

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$$1 - \max_{1 \le k \le K} \frac{\sum_{n=1}^{N}[\![y_n = k]\!]}{N}$$

—optimal $k = y^*$ only

Impurity Functions

by E_{in} of optimal constant

- regression error:

$$\text{impurity}(\mathcal{D}) = \frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y})^2$$

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$$1 - \sum_{k=1}^{K}\left(\frac{\sum_{n=1}^{N}[\![y_n = k]\!]}{N}\right)^2$$

—all k considered together

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popular choices: **Gini** for classification,
regression error for regression

Termination in C&RT

function DecisionTree(data $\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{N}$)
if termination criteria met
return base hypothesis $g_t(\mathbf{x})$ = E_{in}-optimal constant
else ...
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$$b(\mathbf{x}) = \underset{\text{decision stumps } h(\mathbf{x})}{\text{argmin}} \sum_{c=1}^{2} |\mathcal{D}_c \text{ with } h| \cdot \text{impurity}(\mathcal{D}_c \text{ with } h)$$

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C&RT: **fully-grown tree** with constant leaves
that come from **bi-branching** by **purifying**

Fun Time

For the Gini index, $1 - \sum_{k=1}^{K} \left(\frac{\sum_{n=1}^{N} [\![y_n = k]\!]}{N} \right)^2$. Consider $K = 2$, and let $\mu = \frac{N_1}{N}$, where N_1 is the number of examples with $y_n = 1$. Which of the following formula of μ equals the Gini index in this case?

1. $2\mu(1 - \mu)$
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Reference Answer: 1

Simplify $1 - (\mu^2 + (1 - \mu)^2)$ and the answer should pop up.

Basic C&RT Algorithm

function DecisionTree(data $\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{N}$)

if cannot branch anymore

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easily handle binary classification,
regression, & **multi-class classification**

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systematic choice of λ? **validation**

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numerical features

blood pressure:
130, 98, 115, 147, 120

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with $\theta \in \mathbb{R}$

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C&RT (& general decision trees):
handles **categorical features easily**

Missing Features by Surrogate Branch

possible $b(\mathbf{x}) = [\![\text{weight} \leq 50\text{kg}]\!]$

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C&RT: handles **missing features easily**

Fun Time

For a categorical branching criteria $b(\mathbf{x}) = [\![x_i \in S]\!] + 1$ with $S = \{1, 6\}$. Which of the following is the explanation of the criteria?

1. if i-th feature is of type 1 or type 6, branch to first sub-tree; else branch to second sub-tree
2. if i-th feature is of type 1 or type 6, branch to second sub-tree; else branch to first sub-tree
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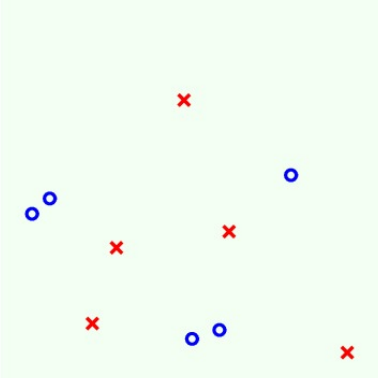
Reference Answer: 2

Note that '$\in S$' is an 'or'-style condition on the elements of S in human language.

A Simple Data Set



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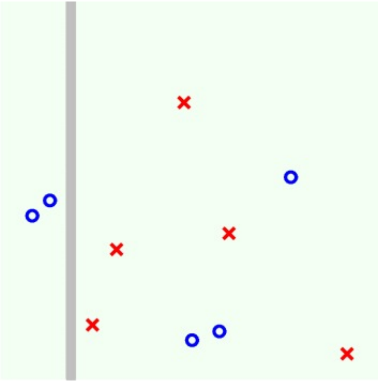

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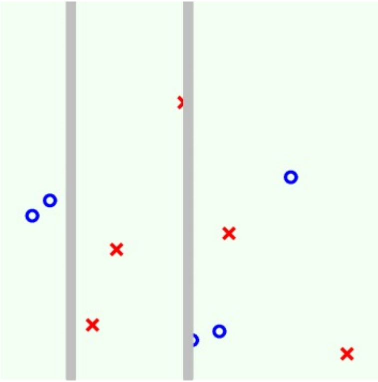

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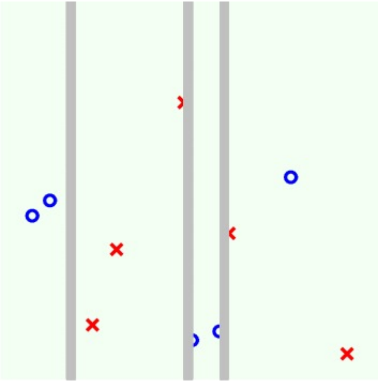

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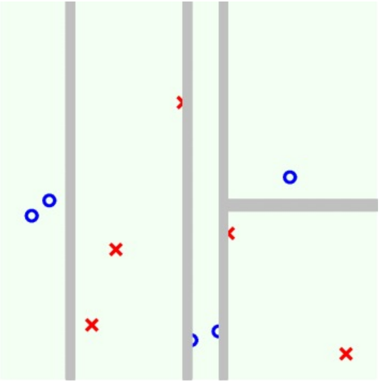

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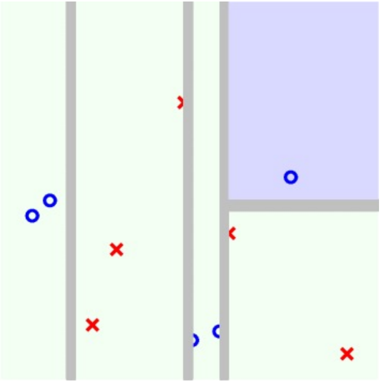

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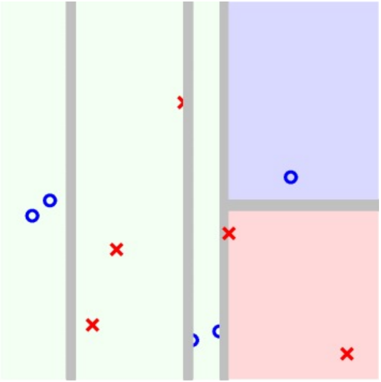

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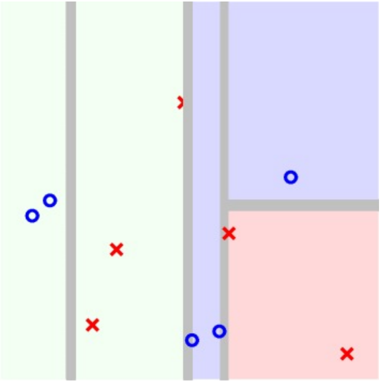

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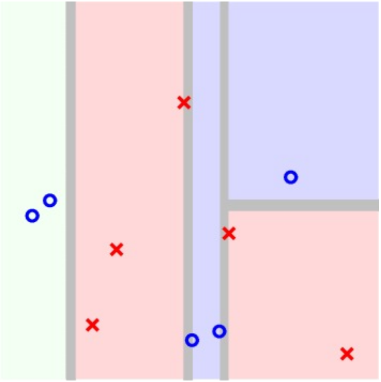

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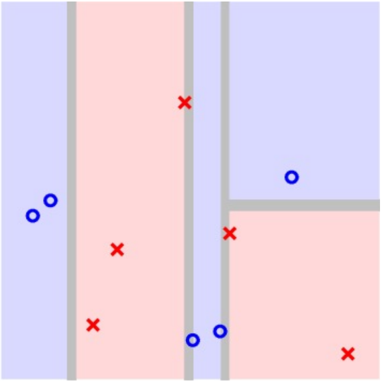

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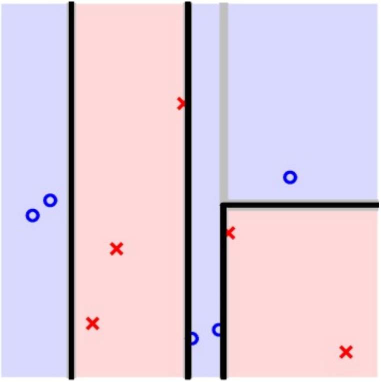
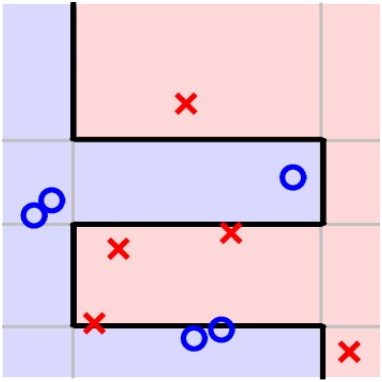


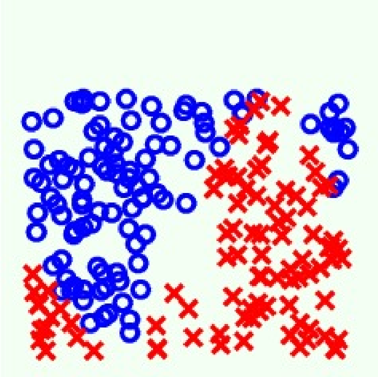
C&RT



AdaBoost-Stump

C&RT: 'divide-and-conquer'

A Complicated Data Set



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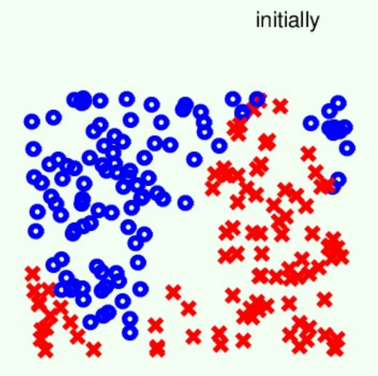

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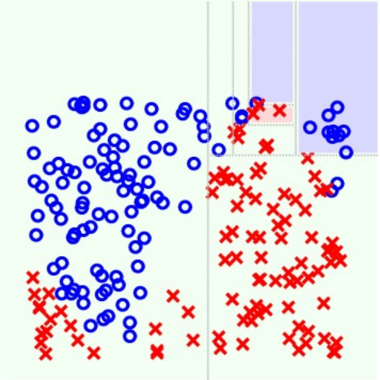

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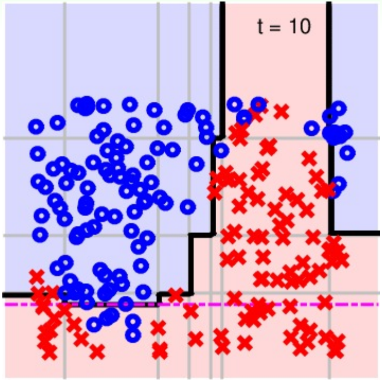

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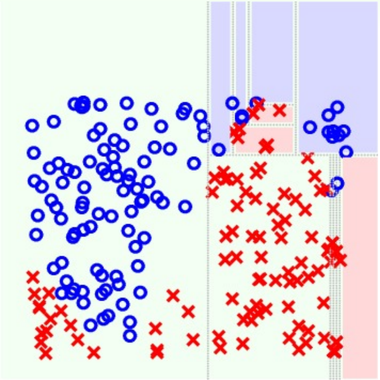

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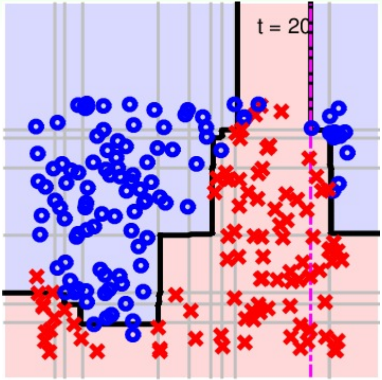

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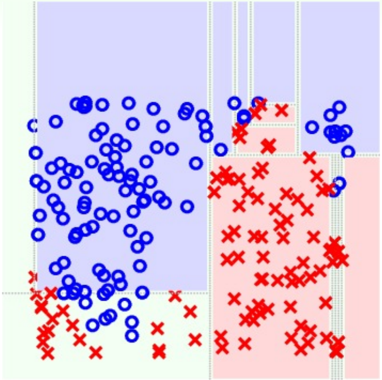

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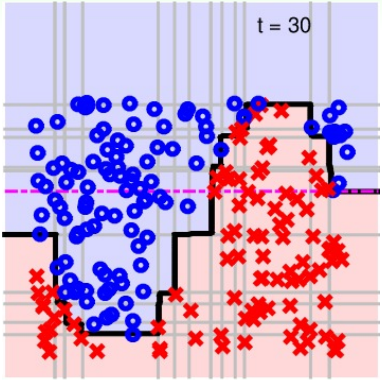

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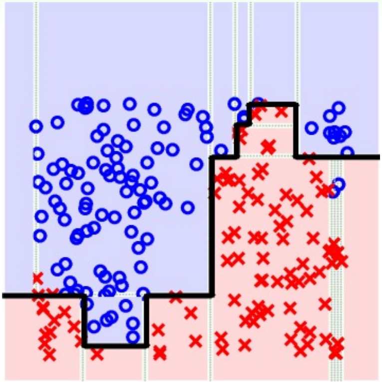

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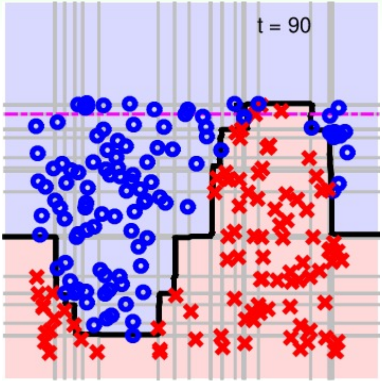

AdaBoost-Stump

C&RT: even more efficient than AdaBoost-Stump

Practical Specialties of C&RT

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another popular decision tree algorithm: **C4.5**, with different **choices of heuristics**

Fun Time

Which of the following is **not** a specialty of C&RT without pruning?

1. handles missing features easily
2. produces explainable hypotheses
3. achieves low E_{in}
4. achieves low E_{out}

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Reference Answer: 4

The first two choices are easy; the third comes from the fact that fully grown C&RT greedy minimizes E_{in} (almost always to 0). But as you may imagine, overfitting may happen and E_{out} may not always be low.

Summary

1. Embedding Numerous Features: Kernel Models
2. Combining Predictive Features: Aggregation Models

Lecture 9: Decision Tree

- Decision Tree Hypothesis
  express path-conditional aggregation
- Decision Tree Algorithm
  recursive branching until termination to base
- Decision Tree Heuristics in C&RT
  pruning, categorical branching, surrogate
- Decision Tree in Action
  explainable and efficient

- **next: aggregation of aggregation?!**

3. Distilling Implicit Features: Extraction Models