Homework #0

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QUESTIONS ARE WELCOMED ON DISCORD.

Please use Gradescope to upload your choices. For homework 0, you do not need to upload your scanned/printed solutions.

Any form of cheating, lying, or plagiarism will not be tolerated. Students can get zero scores and/or fail the class and/or be kicked out of school and/or receive other punishments for those kinds of misconducts.

Discussions on course materials and homework solutions are encouraged. But you should write the final solutions alone and understand them fully. Books, notes, and Internet resources can be consulted, but not copied from.

Since everyone needs to write the final solutions alone, there is absolutely no need to lend your homework solutions and/or source codes to your classmates at any time. In order to maximize the level of fairness in this class, lending and borrowing homework solutions are both regarded as dishonest behaviors and will be punished according to the honesty policy.

This homework set is of 40 points, which is much smaller than that of a usual homework set. For each problem, there is one correct choice. If you choose the correct answer, you get 2 points; if you choose an incorrect answer, you get 0 points.

Combinatorics and Probability

1. Let $H(N, 0) = 1$ for all integers $N \geq 1$, and $H(1, K) = 1$ for all integers $K \geq 0$. For $N \geq 2$ and $K \geq 1$, let $H(N, K) = H(N-1, K) + H(N, K-1)$. What is the closed-form equation of $H(N, K)$ for integers $N \geq 1$ and $K \geq 0$?

[a] $H(N, K) = \frac{(N+K-1)!}{K!(N-1)!}$

[b] $H(N, K) = \frac{N!}{K!(N-K)!}$

[c] $H(N, K) = \frac{(N+K)!}{K!N!}$

[d] $H(N, K) = \frac{(N+K-1)!}{(K-1)!N!}$

[e] none of the other choices

2. What is the probability of getting exactly 4 heads when flipping 8 fair coins? Choose the closest number.

[a] 0.1

[b] 0.2

[c] 0.3

[d] 0.4

[e] 0.5

3. If your friend flipped a fair coin three times, and then tells you that at least two of the tosses resulted in heads, what is the probability that all three tosses resulted in heads?

[a] 1/8

[b] 1/4

[c] 3/8

[d] 1/2

[e] 1/3

4. A program selects a random integer y like this: a random bit is first generated uniformly. If the bit is 0, y is drawn uniformly from $\{1, 2, 3, 4, 5\}$; otherwise, y is drawn uniformly from $\{-1, -2, -3\}$. If we get a y from the program with $|y| \le 2$, what is the probability that y is positive?

[**a**] $1/4$

[**b**] $3/8$

[**c**] $1/3$

[**d**] $5/8$

[**e**] $2/5$

5. For N pairs of random variables $(x_1, y_1), \ldots, (x_N, y_N)$, let their sample covariance be $q_{xy} = \frac{1}{N-1}\sum_{n=1}^{N}(x_n - \bar{x})(y_n - \bar{y})$, where $\bar{x}$ and $\bar{y}$ are the sample means. Which of the following is provably the same as q_{xy}?

[**a**] $\frac{1}{N}\sum_{n=1}^{N}(x_n y_n - \bar{x}\bar{y})$

[**b**] $\frac{1}{N-1}\sum_{n=1}^{N}(x_n y_n - \bar{x}\bar{y})$

[**c**] $\frac{1}{N-1}\sum_{n=1}^{N}(\bar{x}\bar{y} - x_n y_n)$

[**d**] $\frac{N}{N-1}(\bar{x}\bar{y})$

[**e**] none of the other choices

6. For two events A and B, if their probability $P(A) = 0.6$ and $P(B) = 0.8$, what is the tightest possible range of $P(A \cap B)$?

[**a**] $[0.4, 0.8]$

[**b**] $[0, 0.6]$

[**c**] $[0.4, 0.6]$

[**d**] $[0.6, 1]$

[**e**] $[0.48, 0.6]$

Linear Algebra

7. Consider a plane $w_0 + w_1x_1 + w_2x_2 + w_3x_3 = 0$ in $\mathbb{R}^3$ with a non-zero w_3. Which of the following points is guaranteed to be on the plane?

[**a**] $(0, 0, -\frac{w_0}{w_3})$

[**b**] $(-\frac{w_0}{w_1}, 0, 0)$

[**c**] (w_1, w_2, w_3)

[**d**] $(0, -\frac{w_3}{w_0}, 0)$

[**e**] none of the other choices

8. What is the diagonal on the inverse of $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$?

[**a**] $[1/4, 1/8, 1/2]$

[**b**] $[1/2, 1/8, 1/4]$

[**c**] $[1/8, 1/4, 1/2]$

[**d**] $[1/2, 1/4, 1/8]$

[**e**] none of the other choices

9. What is the smallest eigenvalue of $\begin{pmatrix} 2027 & 1 & 1 \\ 2 & 2028 & 2 \\ 3 & 3 & 2029 \end{pmatrix}$?

[**a**] 2026

[**b**] 2027

[**c**] 2029

[**d**] 2032

[**e**] 2034

10. For a real matrix M, let $\mathrm{M} = \mathrm{U}\Sigma\mathrm{V}^T$ be its singular value decomposition, with U and V being unitary matrices. Define $\mathrm{M}^\dagger = \mathrm{V}\Sigma^\dagger\mathrm{U}^T$, where $\Sigma^\dagger[j][i] = \frac{1}{\Sigma[i][j]}$ when $\Sigma[i][j]$ is nonzero, and 0 otherwise. Which of the following is always the same as $\mathrm{M}^\dagger\mathrm{M}\mathrm{M}^\dagger$?

[**a**] M

[**b**] M^T

[**c**] $\mathrm{M}^\dagger$

[**d**] $\mathrm{U}^T\mathrm{M}^\dagger\mathrm{V}^T$

[**e**] the identity matrix I

11. For a real matrix M, let $\mathrm{M} = \mathrm{U}\Sigma\mathrm{V}^T$ be its singular value decomposition, with U and V being unitary matrices. Define $\mathrm{M}^\dagger = \mathrm{V}\Sigma^\dagger\mathrm{U}^T$, where $\Sigma^\dagger[j][i] = \frac{1}{\Sigma[i][j]}$ when $\Sigma[i][j]$ is nonzero, and 0 otherwise. Which of the following is always the same as $\mathrm{M}\mathrm{M}^\dagger\mathrm{M}$?

[**a**] $\mathrm{M}\mathrm{M}^T\mathrm{M}$

[**b**] $\mathrm{M}\mathrm{V}^T$

[**c**] $\mathrm{U}^T\mathrm{M}$

[**d**] $\mathrm{U}^T\mathrm{M}\mathrm{V}^T$

[**e**] M

12. Which of the following matrices is mathematically guaranteed to be positive semi-definite?

[**a**] A real symmetric matrix whose entries are all positive

[**b**] Any real matrix with all positive eigenvalues

[**c**] A real symmetric matrix with a strictly positive trace

[**d**] $\mathrm{X}\mathrm{X}^T$ for any real matrix X

[**e**] none of the other choices

13. Consider a fixed $\mathbf{x} \in \mathbb{R}^d$ and some varying $\mathbf{u} \in \mathbb{R}^d$ constrained to $\|\mathbf{u}\| = 1.126$. Which of the following is the largest possible value of $\mathbf{u}^T\mathbf{x}$?

[**a**] $\|\mathbf{x}\|/1.126$

[**b**] $1.126\|\mathbf{x}\|$

[**c**] $1.126\|\mathbf{x}\|^2$

[**d**] ∞

[**e**] none of the other choices

14. Consider two parallel hyperplanes in $\mathbb{R}^d$:

$$H_1 : 2\mathbf{w}^T\mathbf{x} = 4,$$
$$H_2 : \mathbf{w}^T\mathbf{x} = -3,$$

What is the Euclidean distance between H_1 and H_2?

[**a**] $1/\|\mathbf{w}\|$
[**b**] $5/\|\mathbf{w}\|$
[**c**] $7/\|\mathbf{w}\|$
[**d**] 5
[**e**] none of the other choices

Calculus and Optimization

15. Let $f(x,y) = xy$, $x(u,v) = \sin(u-v)$, $y(u,v) = \cos(u+v)$. What is $\dfrac{\partial f}{\partial u}$?

[**a**] $\cos(u+v)\cos(u-v) - \sin(u-v)\sin(u+v)$
[**b**] $-\cos(u+v)\cos(u-v) - \sin(u-v)\sin(u+v)$
[**c**] $\cos(u+v)\cos(u-v) + \sin(u-v)\sin(u+v)$
[**d**] $-\cos(u+v)\cos(u-v) + \sin(u-v)\sin(u+v)$
[**e**] none of the other choices

16. Let $E(u,v) = (u^2e^{-v} + ve^u)^2$. Calculate the gradient $\nabla E(u,v) = \begin{pmatrix} \frac{\partial E}{\partial u} \\ \frac{\partial E}{\partial v} \end{pmatrix}$ at $[u,v] = [1,1]$. Choose the closest vector.

[**a**] $[14.51, 21.32]$
[**b**] $[21.32, -14.51]$
[**c**] $[21.32, 14.51]$
[**d**] $[-21.32, 14.51]$
[**e**] $[1, 1]$

17. For some given $C > 0, D > 0$, what is the optimal β that solves

$$\min_{\beta} Ce^{2\beta} + De^{-\beta}?$$

[**a**] $\frac{1}{3}\ln(\frac{D}{2C})$
[**b**] $\frac{1}{3}\ln(\frac{2C}{D})$
[**c**] $\ln(\frac{D}{2C})$
[**d**] $\frac{1}{2}\ln(\frac{D}{C})$
[**e**] none of the other choices

18. Let $\mathbf{v}$ be a vector in $\mathbb{R}^d$ and $E(\mathbf{v}) = \mathbf{v}^T\mathrm{B}\mathbf{v} - \mathbf{c}^T\mathbf{v}$ for some symmetric matrix B and vector $\mathbf{c}$. What is the gradient $\nabla E(\mathbf{v})$?

[**a**] $\mathrm{B}\mathbf{v} - \mathbf{c}$
[**b**] $2\mathrm{B}\mathbf{v} - \mathbf{c}$
[**c**] $2\mathrm{B}\mathbf{v} + \mathbf{c}$
[**d**] $\mathrm{B}\mathbf{v} + \mathbf{c}$
[**e**] none of the other choices

19. Let $\mathbf{v}$ be a vector in $\mathbb{R}^d$ and $E(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T \mathrm{Q}\mathbf{v} - \mathbf{p}^T\mathbf{v}$ for some symmetric *and strictly positive definite* matrix Q and vector $\mathbf{p}$. What is the optimal $\mathbf{v}$ that minimizes $E(\mathbf{v})$?

[**a**] $+\mathrm{Q}^{-1}\mathbf{p}$

[**b**] $-\mathrm{Q}^{-1}\mathbf{p}$

[**c**] $+\mathrm{Q}^{-1}\mathbf{1} - \mathbf{p}$, where $\mathbf{1}$ is a vector of all 1's

[**d**] $\mathrm{Q}\mathbf{p}$

[**e**] none of the other choices

20. Solve

$$\begin{aligned} & \min_{w_1, w_2, w_3} \frac{1}{2}(2w_1^2 + w_2^2 + 4w_3^2) \\ \text{subject to} \quad & w_1 + w_2 + w_3 \geq 14, \\ & w_1 - w_3 \leq 5. \end{aligned}$$

What is the optimal (w_1, w_2, w_3)? (Hint: you can consider using "Lagrange multipliers" to solve this.)

[**a**] $(8, 4, 2)$

[**b**] $(2, 8, 4)$

[**c**] $(4, 8, 2)$

[**d**] $(4, 2, 8)$

[**e**] $(8, 2, 4)$