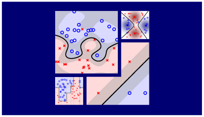
Machine Learning Techniques
(機器學習技法)



Lecture 6: Support Vector Regression

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Roadmap

1. Embedding Numerous Features: Kernel Models

Lecture 5: Kernel Logistic Regression

two-level learning for **SVM-like sparse model** for soft classification, or using **representer theorem** with **regularized logistic error** for dense model

Lecture 6: Support Vector Regression

- Kernel Ridge Regression
- Support Vector Regression Primal
- Support Vector Regression Dual
- Summary of Kernel Models

2. Combining Predictive Features: Aggregation Models
3. Distilling Implicit Features: Extraction Models

Recall: Representer Theorem

for any L2-regularized linear model

$$\min_{\mathbf{w}} \qquad \frac{\lambda}{N}\mathbf{w}^T\mathbf{w} + \frac{1}{N}\sum_{n=1}^{N} \mathrm{err}(y_n, \mathbf{w}^T\mathbf{z}_n)$$

optimal $\mathbf{w}_* = \sum_{n=1}^{N} \beta_n \mathbf{z}_n$.

—any L2-regularized linear model can be **kernelized**!

regression with squared error

$$\mathrm{err}(y, \mathbf{w}^T\mathbf{z}) = (y - \mathbf{w}^T\mathbf{z})^2$$

—analytic solution for linear/ridge regression

analytic solution for **kernel** ridge regression?

Kernel Ridge Regression Problem

solving ridge regression $\min_{\mathbf{w}} \frac{\lambda}{N}\mathbf{w}^T\mathbf{w} + \frac{1}{N}\sum_{n=1}^{N}(y_n - \mathbf{w}^T\mathbf{z}_n)^2$

yields optimal solution $\mathbf{w}_* = \sum_{n=1}^{N} \beta_n \mathbf{z}_n$

with out loss of generality, can solve for optimal $\boldsymbol{\beta}$ instead of $\mathbf{w}$

$$\min_{\boldsymbol{\beta}} \quad \frac{\lambda}{N} \underbrace{\sum_{n=1}^{N}\sum_{m=1}^{N}\beta_n\beta_m K(\mathbf{x}_n,\mathbf{x}_m)}_{\text{regularization of } \boldsymbol{\beta} \text{ on } K\text{-based regularizer}} + \underbrace{\frac{1}{N}\sum_{n=1}^{N}\left(y_n - \sum_{m=1}^{N}\beta_m K(\mathbf{x}_n,\mathbf{x}_m)\right)^2}_{\text{linear regression of } \boldsymbol{\beta} \text{ on } K\text{-based features}}$$

$$= \quad \frac{\lambda}{N}\boldsymbol{\beta}^T\mathrm{K}\boldsymbol{\beta} + \frac{1}{N}\left(\boldsymbol{\beta}^T\mathrm{K}^T\mathrm{K}\boldsymbol{\beta} - 2\boldsymbol{\beta}^T\mathrm{K}^T\mathbf{y} + \mathbf{y}^T\mathbf{y}\right)$$

kernel ridge regression:
use **representer theorem** for kernel trick on **ridge regression**

Solving Kernel Ridge Regression

$$\begin{aligned} E_{\text{aug}}(\boldsymbol{\beta}) &= \frac{\lambda}{N}\boldsymbol{\beta}^T \mathrm{K}\boldsymbol{\beta} + \frac{1}{N}\left(\boldsymbol{\beta}^T \mathrm{K}^T \mathrm{K}\boldsymbol{\beta} - 2\boldsymbol{\beta}^T \mathrm{K}^T \mathbf{y} + \mathbf{y}^T \mathbf{y}\right) \\ \nabla E_{\text{aug}}(\boldsymbol{\beta}) &= \frac{2}{N}\left(\lambda \mathrm{K}^T \mathrm{I}\boldsymbol{\beta} + \mathrm{K}^T \mathrm{K}\boldsymbol{\beta} - \mathrm{K}^T \mathbf{y}\right) = \frac{2}{N}\mathrm{K}^T\left((\lambda \mathrm{I} + \mathrm{K})\boldsymbol{\beta} - \mathbf{y}\right) \end{aligned}$$

want $\nabla E_{\text{aug}}(\boldsymbol{\beta}) = \mathbf{0}$: one analytic solution

$$\boldsymbol{\beta} = (\lambda \mathrm{I} + \mathrm{K})^{-1}\mathbf{y}$$

- $(\cdot)^{-1}$ always exists for $\lambda > 0$, because K positive semi-definite (**Mercer's condition, remember? :-)**)
- time complexity: $O(N^3)$ with simple **dense** matrix inversion

can now do **non-linear regression** 'easily'

Linear versus Kernel Ridge Regression

linear ridge regression

$$\mathbf{w} = (\lambda \mathrm{I} + \mathrm{X}^T\mathrm{X})^{-1}\mathrm{X}^T\mathbf{y}$$

- more restricted
- $O(d^3 + d^2N)$ training;
  $O(d)$ prediction
  —**efficient when $N \gg d$**

kernel ridge regression

$$\boldsymbol{\beta} = (\lambda \mathrm{I} + \mathrm{K})^{-1}\mathbf{y}$$

- **more flexible** with K
- $O(N^3)$ training;
  $O(N)$ prediction
  —hard for big data

linear versus **kernel**:
trade-off between **efficiency** and **flexibility**

Fun Time

After getting the optimal β from kernel ridge regression based on some kernel function K, what is the resulting $g(\mathbf{x})$?

① $\sum_{n=1}^{N} \beta_n K(\mathbf{x}_n, \mathbf{x})$
② $\sum_{n=1}^{N} y_n \beta_n K(\mathbf{x}_n, \mathbf{x})$
③ $\sum_{n=1}^{N} \beta_n K(\mathbf{x}_n, \mathbf{x}) + \lambda$
④ $\sum_{n=1}^{N} y_n \beta_n K(\mathbf{x}_n, \mathbf{x}) + \lambda$

Fun Time

After getting the optimal β from kernel ridge regression based on some kernel function K, what is the resulting $g(\mathbf{x})$?

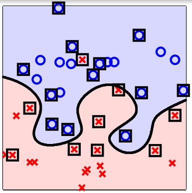
1. $\sum_{n=1}^{N} \beta_n K(\mathbf{x}_n, \mathbf{x})$
2. $\sum_{n=1}^{N} y_n \beta_n K(\mathbf{x}_n, \mathbf{x})$
3. $\sum_{n=1}^{N} \beta_n K(\mathbf{x}_n, \mathbf{x}) + \lambda$
4. $\sum_{n=1}^{N} y_n \beta_n K(\mathbf{x}_n, \mathbf{x}) + \lambda$

Reference Answer: ①

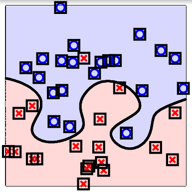
Recall that the optimal $\mathbf{w} = \sum_{n=1}^{N} \beta_n \mathbf{z}_n$ by representer theorem and $g(\mathbf{x}) = \mathbf{w}^T \mathbf{z}$. The answer comes from combining the two equations with the kernel trick.

Soft-Margin SVM versus Least-Squares SVM

least-squares SVM (LSSVM)
= **kernel ridge regression** for classification



soft-margin Gaussian SVM



Gaussian LSSVM

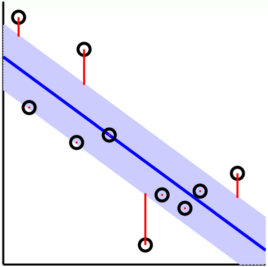
- LSSVM: similar boundary, **many more SVs**
  $\Longrightarrow$ slower prediction, **dense β (BIG g)**
- dense β: LSSVM, kernel LogReg;
  sparse α: standard SVM

want: **sparse** β like standard SVM

Tube Regression

will consider **tube regression**

- within a tube: **no error**
- outside a tube: **error** by distance to tube



error measure:

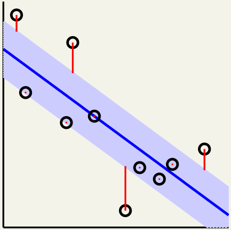
$$\mathrm{err}(y, s) = \max(0, |s - y| - \epsilon)$$

- $|s - y| \leq \epsilon$: 0
- $|s - y| > \epsilon$: $|s - y| - \epsilon$

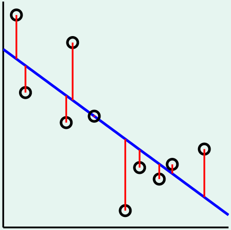
—usually called **ϵ-insensitive error** with $\epsilon > 0$

todo: L2-regularized **tube regression**
to get **sparse** β

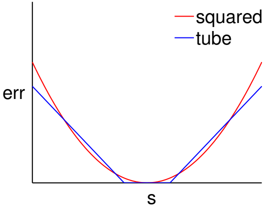
Tube versus Squared Regression



tube: $\mathrm{err}(y, s) = \max(0, |s - y| - \epsilon)$



squared: $\mathrm{err}(y, s) = (s - y)^2$



tube $\approx$ squared when $|s - y|$ small

& **less affected by outliers**

L2-Regularized Tube Regression

$$\min_{\mathbf{w}} \quad \frac{\lambda}{N}\mathbf{w}^T\mathbf{w} + \frac{1}{N}\sum_{n=1}^{N} \max\left(0, |\mathbf{w}^T\mathbf{z}_n - y| - \epsilon\right)$$

Regularized Tube Regr.

$\min \frac{\lambda}{N}\mathbf{w}^T\mathbf{w} + \frac{1}{N}\sum$ tube violation

- unconstrained,
  but max **not differentiable**
- 'representer' to kernelize,
  but **no obvious sparsity**

standard SVM

$\min \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum$ margin vio.

- not differentiable,
  but **QP**
- dual to kernelize,
  KKT conditions ⇒ **sparsity**

will mimic standard SVM derivation:

$$\min_{b,\mathbf{w}} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N} \max\left(0, |\mathbf{w}^T\mathbf{z}_n + b - y_n| - \epsilon\right)$$

Standard Support Vector Regression Primal

$$\min_{b,\mathbf{w}} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N} \max\left(0, |\mathbf{w}^T\mathbf{z}_n + b - y_n| - \epsilon\right)$$

mimicking standard SVM

$$\min_{b,\mathbf{w},\boldsymbol{\xi}} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N} \xi_n$$

$$\text{s.t.} \quad |\mathbf{w}^T\mathbf{z}_n + b - y_n| \le \epsilon + \xi_n$$

$$\xi_n \ge 0$$

making constraints linear

$$\frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N} \left(\xi_n^{\vee} + \xi_n^{\wedge}\right)$$

$$-\epsilon - \xi_n^{\vee} \le y_n - \mathbf{w}^T\mathbf{z}_n - b \le \epsilon + \xi_n^{\wedge}$$

$$\xi_n^{\vee} \ge 0, \xi_n^{\wedge} \ge 0$$

Support Vector Regression (SVR) primal:
minimize regularizer + (upper tube violations $\xi_n^{\wedge}$ & lower violations $\xi_n^{\vee}$)

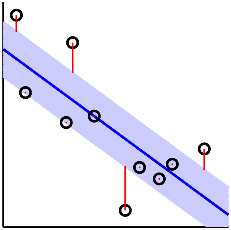
Quadratic Programming for SVR

$$\min_{b,\mathbf{w},\boldsymbol{\xi}^{\vee},\boldsymbol{\xi}^{\wedge}} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\left(\xi_n^{\vee} + \xi_n^{\wedge}\right)$$

$$\text{s.t.} \quad -\epsilon - \xi_n^{\vee} \le y_n - \mathbf{w}^T\mathbf{z}_n - b \le \epsilon + \xi_n^{\wedge}$$

$$\xi_n^{\vee} \ge 0, \xi_n^{\wedge} \ge 0$$

- parameter C: trade-off of regularization & tube violation
- parameter ϵ: vertical tube width —one more parameter to choose!
- QP of $\tilde{d} + 1 + 2N$ variables, $2N + 2N$ constraints



next: remove dependence on $\tilde{d}$ by SVR primal $\Rightarrow$ **dual**?

Fun Time

Consider solving support vector regression with $\epsilon = 0.05$. At the optimal solution, assume that $\mathbf{w}^T\mathbf{z}_1 + b = 1.234$ and $y_1 = 1.126$. What is $\xi_1^\vee$ and $\xi_1^\wedge$?

1. $\xi_1^\vee = 0.108, \xi_1^\wedge = 0.000$
2. $\xi_1^\vee = 0.000, \xi_1^\wedge = 0.108$
3. $\xi_1^\vee = 0.058, \xi_1^\wedge = 0.000$
4. $\xi_1^\vee = 0.000, \xi_1^\wedge = 0.058$

Fun Time

Consider solving support vector regression with $\epsilon = 0.05$. At the optimal solution, assume that $\mathbf{w}^T\mathbf{z}_1 + b = 1.234$ and $y_1 = 1.126$. What is $\xi_1^\vee$ and $\xi_1^\wedge$?

1. $\xi_1^\vee = 0.108, \xi_1^\wedge = 0.000$
2. $\xi_1^\vee = 0.000, \xi_1^\wedge = 0.108$
3. $\xi_1^\vee = 0.058, \xi_1^\wedge = 0.000$
4. $\xi_1^\vee = 0.000, \xi_1^\wedge = 0.058$

Reference Answer: ③

$y_1 - \mathbf{w}^T\mathbf{z}_1 - b = -0.108 < -0.05$, which means that there is a lower tube violation of amount 0.058. When there is a lower tube violation on some example, trivially there is no upper tube violation.

Lagrange Multipliers $\alpha^\wedge$ & $\alpha^\vee$

objective function $$\frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\left(\xi_n^\vee + \xi_n^\wedge\right)$$

Lagrange multiplier $\alpha_n^\wedge$ for $y_n - \mathbf{w}^T\mathbf{z}_n - b \le \epsilon + \xi_n^\wedge$

Lagrange multiplier $\alpha_n^\vee$ for $-\epsilon - \xi_n^\vee \le y_n - \mathbf{w}^T\mathbf{z}_n - b$

Some of the KKT Conditions

- $\frac{\partial \mathcal{L}}{\partial w_i} = 0$: $\mathbf{w} = \sum_{n=1}^{N} \underbrace{(\alpha_n^\wedge - \alpha_n^\vee)}_{\beta_n}\mathbf{z}_n$; $\frac{\partial \mathcal{L}}{\partial b} = 0$: $\sum_{n=1}^{N}(\alpha_n^\wedge - \alpha_n^\vee) = 0$
- complementary slackness: $$\begin{aligned} \alpha_n^\wedge(\epsilon + \xi_n^\wedge - y_n + \mathbf{w}^T\mathbf{z}_n + b) &= 0 \\ \alpha_n^\vee(\epsilon + \xi_n^\vee + y_n - \mathbf{w}^T\mathbf{z}_n - b) &= 0 \end{aligned}$$

standard dual can be derived
using the same steps as Lecture 4

SVM Dual and SVR Dual

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\xi_n \\ \text{s.t.} \quad & y_n(\mathbf{w}^T\mathbf{z}_n + b) \geq 1 - \xi_n \\ & \xi_n \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}(\xi_n^\wedge + \xi_n^\vee) \\ \text{s.t.} \quad & 1(y_n - \mathbf{w}^T\mathbf{z}_n - b) \leq \epsilon + \xi_n^\wedge \\ & 1(\mathbf{w}^T\mathbf{z}_n + b - y_n) \leq \epsilon + \xi_n^\vee \\ & \xi_n^\wedge \geq 0, \xi_n^\vee \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K(\mathbf{x}_n, \mathbf{x}_m) \\ & -\sum_{n=1}^{N} 1 \cdot \alpha_n \\ \text{s.t.} \quad & \sum_{n=1}^{N} y_n\alpha_n = 0 \\ & 0 \leq \alpha_n \leq C \end{aligned}$$

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}(\alpha_n^\wedge - \alpha_n^\vee)(\alpha_m^\wedge - \alpha_m^\vee)k_{n,m} \\ & +\sum_{n=1}^{N}\left((\epsilon - y_n)\cdot\alpha_n^\wedge + (\epsilon + y_n)\cdot\alpha_n^\vee\right) \\ \text{s.t.} \quad & \sum_{n=1}^{N} 1 \cdot (\alpha_n^\wedge - \alpha_n^\vee) = 0 \\ & 0 \leq \alpha_n^\wedge \leq C, 0 \leq \alpha_n^\vee \leq C \end{aligned}$$

similar QP, **solvable by similar solver**

Sparsity of SVR Solution

- $\mathbf{w} = \sum_{n=1}^{N} \underbrace{(\alpha_n^{\wedge} - \alpha_n^{\vee})}_{\beta_n} \mathbf{z}_n$
- complementary slackness:

$$\begin{aligned} \alpha_n^{\wedge}(\epsilon + \xi_n^{\wedge} - y_n + \mathbf{w}^T\mathbf{z}_n + b) &= 0 \\ \alpha_n^{\vee}(\epsilon + \xi_n^{\vee} + y_n - \mathbf{w}^T\mathbf{z}_n - b) &= 0 \end{aligned}$$

- strictly within tube $|\mathbf{w}^T\mathbf{z}_n + b - y_n| < \epsilon$
  $\Longrightarrow \xi_n^{\wedge} = 0$ and $\xi_n^{\vee} = 0$
  $\Longrightarrow (\epsilon + \xi_n^{\wedge} - y_n + \mathbf{w}^T\mathbf{z}_n + b) \neq 0$ and $(\epsilon + \xi_n^{\vee} + y_n - \mathbf{w}^T\mathbf{z}_n - b) \neq 0$
  $\Longrightarrow \alpha_n^{\wedge} = 0$ and $\alpha_n^{\vee} = 0$
  $\Longrightarrow \beta_n = 0$
- SVs ($\beta_n \neq 0$): **on or outside tube**

SVR: allows **sparse** β

Fun Time

What is the number of variables within the QP problem of SVR dual?

1. $\tilde{d} + 1$
2. $\tilde{d} + 1 + 2N$
3. N
4. $2N$

Fun Time

What is the number of variables within the QP problem of SVR dual?

1. $\tilde{d} + 1$
2. $\tilde{d} + 1 + 2N$
3. N
4. $2N$

Reference Answer: ④

There are N variables within $\boldsymbol{\alpha}^{\vee}$, and another N in $\boldsymbol{\alpha}^{\wedge}$.

Map of Linear Models

PLA/pocket

minimize $\text{err}_{0/1}$ specially

linear SVR

minimize regularized err_{TUBE} by QP

linear soft-margin SVM

minimize regularized $\widehat{\text{err}}_{\text{SVM}}$ by QP

linear ridge regression

minimize regularized err_{SQR} analytically

regularized logistic regression

minimize regularized err_{CE} by GD/SGD

second row: popular in **LIBLINEAR**

Map of Linear/Kernel Models

PLA/pocket	**linear SVR**	
linear soft-margin SVM	**linear ridge regression**	**regularized logistic regression**
	kernel ridge regression kernelized linear ridge regression	**kernel logistic regression** kernelized regularized logistic regression
SVM minimize SVM dual by QP	**SVR** minimize SVR dual by QP	**probabilistic SVM** run SVM-transformed logistic regression

fourth row: popular in **LIBSVM**

Map of Linear/Kernel Models

PLA/pocket	linear SVR	
linear soft-margin SVM	linear ridge regression	regularized logistic regression
	kernel ridge regression	kernel logistic regression
SVM	SVR	probabilistic SVM

first row: less used due to **worse performance**
third row: less used due to **dense** β

Kernel Models

possible kernels:

polynomial, Gaussian, . . ., your design (with Mercer's condition),

coupled with

kernel ridge regression

kernel logistic regression

SVM

SVR

probabilistic SVM

powerful extension of linear models
—*with great power comes great responsibility*
in **Spiderman, remember? :-)**

Fun Time

Which of the following model is less used in practice?

① pocket

② ridge regression

③ (linear or kernel) soft-margin SVM

④ regularized logistic regression

Fun Time

Which of the following model is less used in practice?

1. pocket
2. ridge regression
3. (linear or kernel) soft-margin SVM
4. regularized logistic regression

Reference Answer: ①

The pocket algorithm generally does not perform better than linear soft-margin SVM, and hence is less used in practice.

Summary

① Embedding Numerous Features: Kernel Models

Lecture 6: Support Vector Regression

- Kernel Ridge Regression
  representer theorem on ridge regression
- Support Vector Regression Primal
  minimize regularized tube errors
- Support Vector Regression Dual
  a QP similar to SVM dual
- Summary of Kernel Models
  with great power comes great responsibility

② Combining Predictive Features: Aggregation Models

- **next: making cocktail from learning models**

③ Distilling Implicit Features: Extraction Models