Sidework #1

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1.1 Hoeffding's Inequality

The proof that you will write below contains all the essential steps, but are not as rigorously written as the usual math texts.

(1) (Markov's Inequality) Prove that for any non-negative random variable ℓ and any positive constant a,

$$P(\ell \geq a) \leq \frac{E(\ell)}{a}.$$

(2) (Moment-Generating Function—Laplace Transform) Prove that for any finite random variable λ, any positive constant α, and any positive parameter s,

$$P(\lambda \geq \alpha) \leq e^{-s\alpha} E(e^{s\lambda}) \; ;$$
$$P(\lambda \leq \alpha) \leq e^{s\alpha} E(e^{-s\lambda}) \; .$$

(3) (Independence Decomposition) Let $z_1, z_2, \cdots, z_N$ be i.i.d. random variables and let $z = \frac{1}{N}\sum_{n=1}^{N} z_n$. For any positive constant α and any positive parameter s, prove that

$$P(z \geq \alpha) \leq (e^{-s\alpha} E(e^{sz_1}))^N \; ;$$
$$P(z \leq \alpha) \leq (e^{s\alpha} E(e^{-sz_1}))^N \; .$$

(4) (Bound Tightening) Let z_1 be a binary random variable with $P(z_1 = 0) = 1 - \theta$ and $P(z_1 = 1) = \theta$. Let

$$F(s) = e^{-s\alpha} E(e^{sz_1})$$

For any given α with $\theta < \alpha < 1$, prove that $F(s)$ is minimized on

$$s^* = \ln \frac{\alpha \cdot (1-\theta)}{(1-\alpha) \cdot \theta},$$

where s^* is positive.

(5) (Chernoff Bound) Use the fact that

$$P(z \geq \alpha) \leq (e^{-s^*\alpha} E(e^{s^* z_1}))^N$$

to prove

$$P(z \geq \alpha) \leq e^{-ND(\alpha||\theta)}$$

for $\theta < \alpha < 1$. Here $D(\alpha||\theta) = \alpha \log \frac{\alpha}{\theta} + (1-\alpha) \log \frac{1-\alpha}{1-\theta}$ is the KL-divergence between α and θ.

(6) (One-sided Hoeffding Inequality) Let $\alpha = \theta + \epsilon$ with $\epsilon > 0$, prove that

$$P(z - \theta \geq \epsilon) \leq e^{-2N\epsilon^2}$$

by showing that $D(\theta + \epsilon||\theta) \geq 2\epsilon^2$ for all $\epsilon > 0$.

(7) (Two-sided Hoeffding Inequality) Prove that

$$P(|z - \theta| \geq \epsilon) \leq 2e^{-2N\epsilon^2}.$$