Homework #0

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1 Probability and Statistics

(1) (counting)
What is the probability of getting exactly 6 heads when flipping 10 fair coins?

What is the probability of getting a full house (XXXYY) when randomly drawing 5 cards out of a deck of 52 cards?

(2) (conditional probability)
If your friend flipped a fair coin twice, and tell you that one of the tosses resulted in head, what is the probability that both tosses resulted in heads?

(3) (Bayes theorem)
A program selects a random integer X like this: a random bit is first generated uniformly. If the bit is 0, X is drawn uniformly from $\{0, 1, \ldots, 7\}$; otherwise, X is drawn uniformly from $\{0, -1, -2, -3\}$. If we get an X from the program with $|X| = 1$, what is the probability that X is negative?

(4) (union/intersection)
If $P(A) = 0.3$ and $P(B) = 0.4$,
what is the maximum possible value of $P(A \cap B)$?
what is the minimum possible value of $P(A \cap B)$?
what is the maximum possible value of $P(A \cup B)$?
what is the minimum possible value of $P(A \cup B)$?

(5) (mean/variance)
Let mean $\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$ and variance $\sigma_X^2 = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \overline{X})^2$. Prove that

$$\sigma_X^2 = \frac{N}{N-1}\left(\frac{1}{N}\sum_{n=1}^{N} X_n^2 - \overline{X}^2\right).$$

(6) (Gaussian distribution)
If X_1 and X_2 are independent random variables, where $p(X_1)$ is Gaussian with mean 2 and variance 1, $p(X_2)$ is Gaussian with mean -3 and variance 4. Let $Z = X_1 + X_2$. Prove $p(Z)$ is Gaussian, and determine its mean and variance.

2 Linear Algebra

(1) (rank)

What is the rank of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}$?

(2) (inverse)

What is the inverse of $\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$?

(3) (eigenvalues/eigenvectors)
What are the eigenvalues and eigenvectors of $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$?

(4) (PD/PSD)
A symmetric real matrix A is positive definite (PD) *iff* $x^T A x > 0$ for all $x \neq 0$, and positive semi-definite (PSD) if "$>$" is changed to "$\geq$". Prove:

(a) For any real matrix Z, ZZ^T is PSD.

(b) A is PD *iff* all eigenvalues of A are strictly positive.

(5) (inner product)
Consider $x \in R^d$ and some $u \in R^d$ with $\|u\| = 1$.
What is the maximum value of $u^T x$?
What is the minimum value of $u^T x$?
What is the minimum value of $|u^T x|$?

(6) (distance)
Consider two parallel hyperplanes in R^d:

$$H_1 : w^T x = +3,$$
$$H_2 : w^T x = -2,$$

where w is the norm vector. What is the distance between H_1 and H_2?

3 Calculus

(1) (differential)
Let $f(x) = \ln(1 + e^{-2x})$. What is $\dfrac{df(x)}{dx}$?

(2) (partial differential)
Let $f(x, y) = e^x + e^{2y} + e^{3xy^2}$. What is $\dfrac{\partial f(x, y)}{\partial y}$?

(3) (chain rule)
Let $f(x, y) = xy$, $x(u, v) = \cos(u + v)$, $y(u, v) = \sin(u - v)$. What is $\dfrac{\partial f}{\partial v}$?

(4) (integral)
What is $\displaystyle\int_5^{10} \frac{2}{x - 3} dx$?

(5) (gradient and Hessian)
Let $E(u, v) = (ue^v - 2ve^{-u})^2$. Calculate the gradient ∇E and the Hessian $\nabla^2 E$ at $u = 1$ and $v = 1$.

(6) (optimization)
For some given $A > 0, B > 0$, solve

$$\min_{\alpha} Ae^{\alpha} + Be^{-2\alpha}.$$