Homework #1

RELEASE DATE: 10/01/2019

DUE DATE: EXTENDED TO 11/05/2019, BEFORE 17:00 on GRADESCOPE

QUESTIONS ABOUT HOMEWORK MATERIALS ARE WELCOMED ON THE FACEBOOK FORUM.

Unless granted by the instructor in advance, you must upload your solution to Gradescope as instructed by the TA.

Any form of cheating, lying, or plagiarism will not be tolerated. Students can get zero scores and/or fail the class and/or be kicked out of school and/or receive other punishments for those kinds of misconducts.

Discussions on course materials and homework solutions are encouraged. But you should write the final solutions alone and understand them fully. Books, notes, and Internet resources can be consulted, but not copied from.

Since everyone needs to write the final solutions alone, there is absolutely no need to lend your homework solutions and/or source codes to your classmates at any time. In order to maximize the level of fairness in this class, lending and borrowing homework solutions are both regarded as dishonest behaviors and will be punished according to the honesty policy.

You should write your solutions in English or Chinese with the common math notations introduced in class or in the problems. We do not accept solutions written in any other languages.

This homework set comes with 200 points and 20 bonus points. In general, every homework set would come with a full credit of 200 points, with some possible bonus points.

1. Consider the ball-balance game with 100 balls where the probability that the heavy ball is number i is given by $p(i) = 2^{-i}$ for $i \neq 100$ and $p(100) = 2^{-99}$. How many uses of the balance are needed to determine the heavy ball without error? Describe the corresponding procedure.

2. Following the previous question, how many uses of the balance are needed to determine the heavy ball with

$$\Pr(\text{error}) \leq 0.0001?$$

Describe the corresponding procedure.

3. Following the previous question, what is the smallest δ such that $H_\delta(X) = 5$, where each $i \in X$ represents the situation where the i-th ball is the heavy ball? Prove your answer.

4. In the randomized name-identification scheme based on the probability transition matrices $[\nu_{kj}]$ and $[\mu_{jk}]$, define what the 'probability of correct identification' is, and prove the corresponding (probabilistic) pigeonhole principle.

5. Prove or disprove that for any given finite ensemble X with all $p_k > 0$, $H_\delta(X)$ is non-increasing within $0 < \delta < 1$.

6. Let $X = \{a_1, a_2\}$ be an ensemble with $p_1 = 0.01$ and $p_2 = 0.99$. Consider the block coding of X^{100} where every $\mathbf{x} \in X^{100}$ containing 3 or fewer a_1's is assigned a distinct codeword that is not all 0, while the other $\mathbf{x}$'s are assigned an all-0 codeword. If all codewords are of the same length, derive the minimum length required to provide the above set with distinct codewords.

7. Following the previous question, consider Huffman coding for symbol coding on X^2. List the 4 codewords for each element in X^2. What is the expected codeword length when using the Huffman coding on X^2 to encode a vector in X^{100} (i.e. apply the symbol coding 50 times)? Compare your result with the result of the previous question and describe your findings.

8. Following the previous question, consider Huffman coding for symbol coding on X^4. List the 16 codewords for each element in X^4. What is the expected codeword length when using the Huffman coding on X^4 to encode a vector in X^{100} (i.e. apply the symbol coding 25 times)? Compare your result with the result of the previous two questions and describe your findings.

9. Consider determining an object by asking yes-no questions. The object is drawn randomly from a finite set according to a certain distribution. If the optimal strategy needs 11.26 questions on the average to find the object. At least how many alternatives are within the finite set? Illustrate your answer.

10. Consider a finite ensemble X such that all $a_k \in X$ are integers. Define an ensemble Z that represents the sum of two independent outcomes from X. That is, Z contains all possible $a_i + a_j$ for $a_i \in X$ and $a_j \in X$, and $p(b_t) = \sum_{a_i+a_j=b_t} p(a_i)p(a_j)$ for all $b_t \in Z$. Prove or disprove that $H(Z) \leq H(X^2)$.

Bonus: Zero-error Coding towards Entropy Bitrate

11. (Bonus 20 points) In class, we proved that for any given finite ensemble X and parameters $0 < \delta < 1$ and $\epsilon > 0$, there exists a positive integer N_0 such that any integer $N \geq N_0$ implies

$$\frac{1}{N} H_\delta(X^N) < H(X) + \epsilon.$$

That is, there always exists a δ-error-tolerating coding scheme with 'bitrate' close to $H(X)$.

Now, prove or disprove that there exists a 0-error coding scheme with expected codeword length close to $H(X)$. That is, for any given $\epsilon > 0$, there exists a positive integer N_1 such that any integer $N \geq N_1$ implies that there exists a one-to-one deterministic function $\nu \colon X^N \to \{0,1\}^+$ such that

$$\frac{1}{N} \mathbb{E}(|\nu(X^N)|) < H(X) + \epsilon,$$

where the expectation $\mathbb{E}$ is taken with respect to the probability distribution of the finite ensemble X.