Subject : Intro. to Info. Theory

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* minimum UD $\underbrace{\text{code length}}_{\text{avg. symbol}}$ is $H(X)$!

$$H(X) - \underbrace{\sum P_k n_k}_{\text{avg. code length}} = \sum P_k \log \frac{1}{P_k} + \sum P_k \log 2^{-n_k}$$

$$= \sum P_k \frac{\ln}{\ln 2} \left(\frac{1}{P_k} \cdot 2^{-n_k} \right)$$

$$\le \sum \frac{P_k}{\ln 2} \left(-1 + \frac{1}{P_k} 2^{-n_k} \right)$$

$$= \frac{1}{\ln 2} \left(-1 + \underbrace{\sum 2^{-n_k}}_{\le 1 \text{ by UD}} \right)$$

$$\le 0$$

equality if $\frac{1}{P_k} \cdot 2^{-n_k} = 1$ (and Kraft becomes "=")

$$\Downarrow$$

$$n_k = \log \frac{1}{P_k}$$

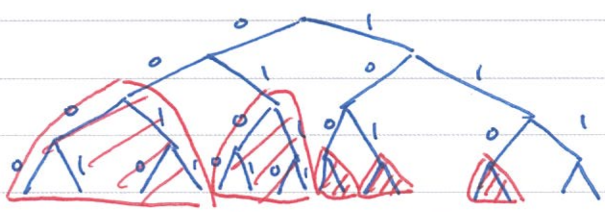
small P_k $\Longleftrightarrow$ large n_k achieves $H(X)$ (?)

without throwing those away like $H_\delta(X)$

* "achieves" ? does such UD code exist ?

construct a UD code w/ a full binary tree

wolg, assume $n_1 \le n_2 \le \cdots \le n_K$



$n_1 = 2$ $n_2 = 2$ $n_3 = 3$ $n_4 = 3$ $n_5 = 3$

for $i = 1, 2, \cdots, k$

pick 1st node @ level n_k

as $v(a_k)$

remove $\boxed{\text{sub-tree @ } v(a_k)}$

leaves eliminated

$$\sum_i 2^{n_K - n_i} \le \underbrace{2^{n_K}}_{\text{total leaves}}$$

(Kraft's for this special UD cod

decoding is just binary tree traversa

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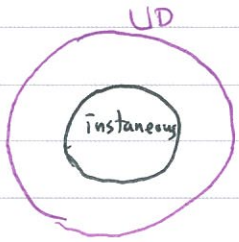
* decoding just binary tree traversal $\Leftrightarrow$ instaneous (can do "partial decode" per symbol

$\Updownarrow$

prefix-free (no codeword prefix of the other



	γ_1 (not instaneous)	γ_2 (instaneous)
a_1	0	0
a_2	01	10
a_3	011	110
a_4	0111	1110

usually preferred

* UD

Kraft's $\swarrow$ UD ; instaneous $\nwarrow$ special case UD

Kraft's $\xrightarrow{\text{construct}}$ instaneous

$\sum 2^{-n_k} \le 1$

* what if $\log \frac{1}{P_k}$ not integers ?

- naive idea (Shannon Coding)

$$n_k = \left\lceil \log \frac{1}{P_k} \right\rceil \Rightarrow \sum P_k n_k < \sum P_k \left(1 + \log \frac{1}{P_k}\right) = 1 + H(X)$$

- naive idea on a few blocks (M) : coding on X^M

$$\frac{1 + M H(X)}{M}$$

(can be arbitrarily close to $H(X)$, error-free)

by giving atypical sequence super-long codes

* Huffman Coding

$$\min_{n_k} \sum P_k n_k$$

s.t. n_k integers

& $\sum 2^{-n_k} \le 1$ (Kraft's)

* Consider : $P_1 \geq P_2 \geq P_3$

claim : $n_1 \leq n_2 \leq n_3$ at optimal
(otherwise can swap and still satisfy Kraft's)

claim : $\underbrace{n_2 = n_3}_{\text{smallest-two prob. elements w/ same longest codewords}}$ at some optimal instaneous
(prefix-free so can decrease n_3 w/o violating anything)

* Huffman Coding ($p_1, p_2, \cdots, p_K$)

recursively
- pick two smallest prob. elements
- merge and construct $HC(P_1', P_2', \cdots, P_{K-1}')$
- give the two elements c"0", c"1"

① need to prove that there's no loss of optimality here

② need to prove that $HC(P_1, P_2) = \{$"0", "1"$\}$

*