Analysis Tools

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Asymptotic Notations

- goal: rough total rather than exact steps when input size large
- why rough total? constant not matter much

compare two complexity functions $f(n)$ and $g(n)$ when n large

growth of functions matters
—n^3 would eventually be bigger than $1000n$

- which is faster? n^2 or $10n$?
- which is faster? n or n^2 or 2^n?
- which is faster? $5n$ or n?

Asymptotic Notations: Symbols

- $f(n)$ grows slower than or similar to $g(n)$: $f(n) = O(g(n))$
  —like $f(n)[\leq]g(n)$
- $f(n)$ grows faster than or similar to $g(n)$: $f(n) = \Omega(g(n))$
  —like $f(n)[\geq]g(n)$
- $f(n)$ grows similar to $g(n)$: $f(n) = \Theta(g(n))$
  —like $f(n)[\approx]g(n)$

(note: $=$ in the asymptotic notations more like "$\in$")

Asymptotic Notations: Definitions

- $f(n)$ grows slower than or similar to $g(n)$:

  $f(n) = O(g(n))$, iff exist c, n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

- $f(n)$ grows faster than or similar to $g(n)$:

  $f(n) = \Omega(g(n))$, iff exist c, n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$

- $f(n)$ grows similar to $g(n)$:

  $f(n) = \Theta(g(n))$, iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Analysis of Sequential Search

Sequential Search

for $i \leftarrow 0$ to $n - 1$ **do**
  if $list[i] == searchnum$
    return i
  end if
end for
return -1

- best case (e.g. *searchnum* at 0): time $\Theta(1)$
- worst case (e.g. *searchnum* at last or not found): time $\Theta(n)$
- in general: time $\Omega(1)$ and $O(n)$

Analysis of Binary Search

Binary Search

$left \leftarrow 0$, $right \leftarrow n - 1$
while $left \leq right$ **do**
  $middle \leftarrow \text{floor}((left + right)/2)$
  if $list[middle] > searchnum$
    $left \leftarrow middle + 1$
  else if
  $list[middle] < searchnum$
    $right \leftarrow middle - 1$
  else
    return *middle*
  end if
end while
return -1

- best case (e.g. *searchnum* at *middle*): time $\Theta(1)$
- worst case (e.g. *searchnum* not found): because $(right - left)$ is halved in each WHILE iteration, needs time $\Theta(\log n)$ iterations if not found
- in general: time $\Omega(1)$ and $O(\log n)$

often care about the worst case (and thus see $O(\cdot)$ often)

Asymptotic Notation: Prove from Definition

Prove that $\log_2 n$ is $O(n)$.

Informal thoughts:

- i.e., show that there exists n_0 and c such that $\log_2 n \le cn$ for $n \ge n_0$.
- $\ln n \le n - 1 \le n$ for $n \ge 1$ because $g(n) = n - 1$ is the tangent line of $f(n) = \ln n$
- so $\ln 2 \log_2 n \le n$ for $n \ge 1$, where are n_0 and c?

Formal proof:

$$\text{Take } n_0 = \ldots, c = \ldots;$$
$$\text{for } n \ge n_0, \ldots \text{ so } \log_2 n \ge cn. \text{ That is, } \log_2 n = O(n).$$

Sequential and Binary Search

- Input: any integer array *list* with size *n*, an integer *searchnum*
- Output: if *searchnum* is not within *list*, -1; otherwise, othernum

DIRECT-SEQ-SEARCH (*list*, *n*, *searchnum*)

```
for i ← 0 to n − 1 do
    if list[i] == searchnum
        return i
    end if
end for
return −1
```

SORT-AND-BIN-SEARCH (*list*, *n*, *searchnum*)

```
SEL-SORT(list, n)
return BIN-SEARCH(list, n, searchnum)
```

- DIRECT-SEQ-SEARCH is $O(n)$ time
- SORT-AND-BIN-SEARCH is $O(n^2)$ time for SEL-SORT (Why?) and $O(\log n)$ time for BIN-SEARCH

want: show asymptotic complexity of SORT-AND-BIN-SEARCH as its bottleneck

Some Properties of Big-Oh

Theorem（封閉律）

if $f_1(n) = O(g_2(n))$, $f_2(n) = O(g_2(n))$ *then* $f_1(n) + f_2(n) = O(g_2(n))$

- When $n \geq n_1$, $f_1(n) \leq c_1 g_2(n)$
- When $n \geq n_2$, $f_2(n) \leq c_2 g_2(n)$
- So, when $n \geq \max(n_1, n_2)$, $f_1(n) + f_2(n) \leq (c_1 + c_2) g_2(n)$

Some Properties of Big-Oh

Theorem (遞移律)

if $f_1(n) = O(g_1(n))$, $g_1(n) = O(g_2(n))$ *then* $f_1(n) = O(g_2(n))$

- When $n \geq n_1$, $f_1(n) \leq c_1 g_1(n)$
- When $n \geq n_2$, $g_1(n) \leq c_2 g_2(n)$
- So, when $n \geq \max(n_1, n_2)$, $f_1(n) \leq c_1 c_2 g_2(n)$

Some Properties of Big-Oh

Theorem (併吞律)

if $f_1(n) = O(g_1(n))$, $f_2(n) = O(g_2(n))$ *and* $g_1(n) = O(g_2(n))$ *then* $f_1(n) + f_2(n) = O(g_2(n))$

Proof: use two theorems above.

Some Properties of Big-Oh

Theorem

If $f(n) = a_m n^m + \cdots + a_1 n + a_0$, then $f(n) = O(n^m)$

Proof: use the theorem above.

similar proof for Ω and Θ

Some More on Big-Oh

RECURSIVE-BIN-SEARCH is $O(\log n)$ time and $O(\log n)$ space

- by 遞移律, time also $O(n)$
- time also $O(n \log n)$
- time also $O(n^2)$
- also $O(2^n)$
- $\cdots$

prefer the tightest Big-Oh!

Practical Complexity

some input sizes are time-wise infeasible for some algorithms

when 1-billion-steps-per-second

n	n	$n \log_2 n$	n^2	n^3	n^4	n^{10}	2^n
10	$0.01\mu s$	$0.03\mu s$	$0.1\mu s$	$1\mu s$	$10\mu s$	$10s$	$1\mu s$
20	$0.02\mu s$	$0.09\mu s$	$0.4\mu s$	$8\mu s$	$160\mu s$	$2.84h$	$1ms$
30	$0.03\mu s$	$0.15\mu s$	$0.9\mu s$	$27\mu s$	$810\mu s$	$6.83d$	$1s$
40	$0.04\mu s$	$0.21\mu s$	$1.6\mu s$	$64\mu s$	$2.56ms$	$121d$	$18m$
50	$0.05\mu s$	$0.28\mu s$	$2.5\mu s$	$125\mu s$	$6.25ms$	$3.1y$	$13d$
100	$0.10\mu s$	$0.66\mu s$	$10\mu s$	$1ms$	$100ms$	$3171y$	$4 \cdot 10^{13}y$
10^3	$1\mu s$	$9.96\mu s$	$1ms$	$1s$	$16.67m$	$3 \cdot 10^{13}y$	$3 \cdot 10^{284}y$
10^4	$10\mu s$	$130\mu s$	$100ms$	$1000s$	$115.7d$	$3 \cdot 10^{23}y$	
10^5	$100\mu s$	$1.66ms$	$10s$	$11.57d$	$3171y$	$3 \cdot 10^{33}y$	
10^6	$1ms$	$19.92ms$	$16.67m$	$32y$	$3 \cdot 10^7y$	$3 \cdot 10^{43}y$	

note: similar for space complexity,
e.g. store an N by N double matrix when $N = 50000$?