3.1 (1) Let $f(n) = n^2+1$, $g(n) = n^2$

$d(n) = n^2$, $e(n) = n$

(a) $d(n) = O(f(n))$ by letting $c=1$ & $n_0 = 1$ because $n^2+1 \geq n \;\; \forall n \geq 1$

(b) $e(n) = O(g(n))$ by letting $c=1$ & $n_0=1$

$n \leq n^2 \;\; \forall n \geq 1$

because $(n-\frac{1}{2})^2 \geq \frac{1}{4} \;\; \forall n \geq 1$

(c) $d(n) - e(n) = n^2 - n \neq O(\underbrace{f(n) - g(n)}_{=1})$

prove by contradiction

if $n^2 - n = O(1)$, then $\exists$ c, n_0 s.t.

$n^2 - n \leq c \quad \forall n \geq n_0$

$\Downarrow$

$(n-\frac{1}{2})^2 \leq c + \frac{1}{4}$

$\Downarrow$

$n \leq \frac{1}{2} + \sqrt{c+\frac{1}{4}}$

contradiction

by choosing

$n = \max(n_0, \lfloor \frac{1}{2} + \sqrt{c+\frac{1}{4}} + 1 \rfloor)$

so $d(n) - e(n) \neq O(f(n) - g(n))$

(2) (a) to prove $O(\max\{f(n), g(n)\}) \subseteq O(f(n)+g(n))$

if $h(n) = O(\max\{f(n), g(n)\})$

then $\exists$ c, n_0 s.t.

$h(n) \leq c \cdot \max\{f(n), g(n)\} \quad \forall n \geq n_0$

$\leq c \cdot (f(n) + g(n)) \quad \forall n \geq n_0$ by $\begin{cases} f(n) \geq 0 \\ g(n) \geq 0 \end{cases}$

thus, $h(n) = O(f(n)+g(n))$

(b) to prove $O(f(n)+g(n)) \subseteq O(\max\{f(n), g(n)\})$

if $h(n) = O(f(n)+g(n))$

then $\exists$ c, n_0 s.t.

$h(n) \leq c \cdot (f(n)+g(n)) \quad \forall n \geq n_0$

$\leq 2 \cdot c \cdot (\max\{f(n), g(n)\}) \quad \forall n \geq n_0$

let $c' = 2c$, $n_0' = n_0$

$\Rightarrow h(n) = O(\max\{f(n), g(n)\})$

(c) by (a), (b) the equivalence is proved

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(3) let $f(n) = \begin{cases} 1 & \text{for odd } n \\ n^2 & \text{for even } n \end{cases}$

if $f(n) = O(n)$

$\exists$ c, n_0

s.t. $f(n) \leq c \cdot n$ for all $n \geq n_0$

let $n' = \max(c, n_0) + 1$

and $n = \begin{cases} n'+1 & \text{if } n' \text{ odd} \\ n' & \text{if } n' \text{ even} \end{cases}$

then $f(n) = n^2 > \max(c, n_0) \cdot n$

$n \geq n_0$ and $\geq c \cdot n$ contradiction!

so $f(n) \neq O(n)$

if $f(n) = \Omega(n)$

$\exists$ c, n_0

s.t. $f(n) \geq c \cdot n$ for all $n \geq n_0$

let $n' = \max(\frac{1}{c}, n_0) + 1$

and $n = \begin{cases} n'+1 & \text{if } n' \text{ even} \\ n' & \text{if } n' \text{ odd} \end{cases}$

then $n \geq n_0$ and

$\frac{f(n)}{n} = \frac{1}{n} < c$ contradiction!

so $f(n) \neq \Omega(n)$

(4) $\sum_{i=1}^{n} \lceil \log_2 i \rceil \leq \sum_{i=2}^{n} ((\log_2 i) + 1)$

$\leq \sum_{i=2}^{n} ((\log_2 n) + 1)$

$\leq \sum_{i=2}^{n} ((\log_2 n) + (\log_2 n))$ for all $n \geq 1$

$= 2(n-1)\log_2 n$

$\leq 2(n-1)\log_2 n + 2\log_2 n$

$= 2n\log_2 n$

let $c = 2$, $n_0 = 1$, we showed $\sum_{i=1}^{n} \lceil \log_2 i \rceil = O(n \log_2 n)$

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(5) if $f(n) = O(g(n))$, $\exists\ c > 0$ and $n_0 \geq 0$ that

$$f(n) \leq c \cdot g(n) \quad \forall\ n \geq n_0$$

because $h(n) \geq 0$ and $c > 0$

$$h(n) \leq c \cdot h(n)$$

$$\Rightarrow \quad f(n) + h(n) \leq c \cdot (g(n) + h(n)) \quad \forall\ n \geq n_0$$

i.e. $f(n) + h(n) = O(g(n) + h(n))$

(6) consider $an^2 + bn - n^2$

$$\begin{cases} \text{if } a < 1 & an^2 + bn - n^2 \leq 0 \text{ for } n \geq \frac{b}{1-a} \\ \text{if } a = 1 \text{ and } b \leq 0 & an^2 + bn - n^2 \leq 0 \text{ for } n \geq 1 \end{cases}$$

when $an^2 + bn - n^2 \leq 0$

$$2^{an^2+bn+c} \leq 2^{n^2+c} = 2^c \cdot 2^{n^2}$$

let $c' = 2^c$, and $n_0' =$ $\frac{b}{1-a}$ or 1 according to the case

then $2^{an^2+bn+c} \leq c' \cdot 2^{n^2}$

$$\forall\ n \geq n_0'$$

i.e. $2^{an^2+bn+c} = O(2^{n^2})$

(7) $8n \log n \leq 2n^2$

$$\Uparrow$$

$4 \log n \leq n$

$$\Uparrow$$

$n \geq 16$ (by plotting)

(8) $O(n)$ mul & add