Small-Scale Fading II
(and basics about random processes)

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2011/11/3

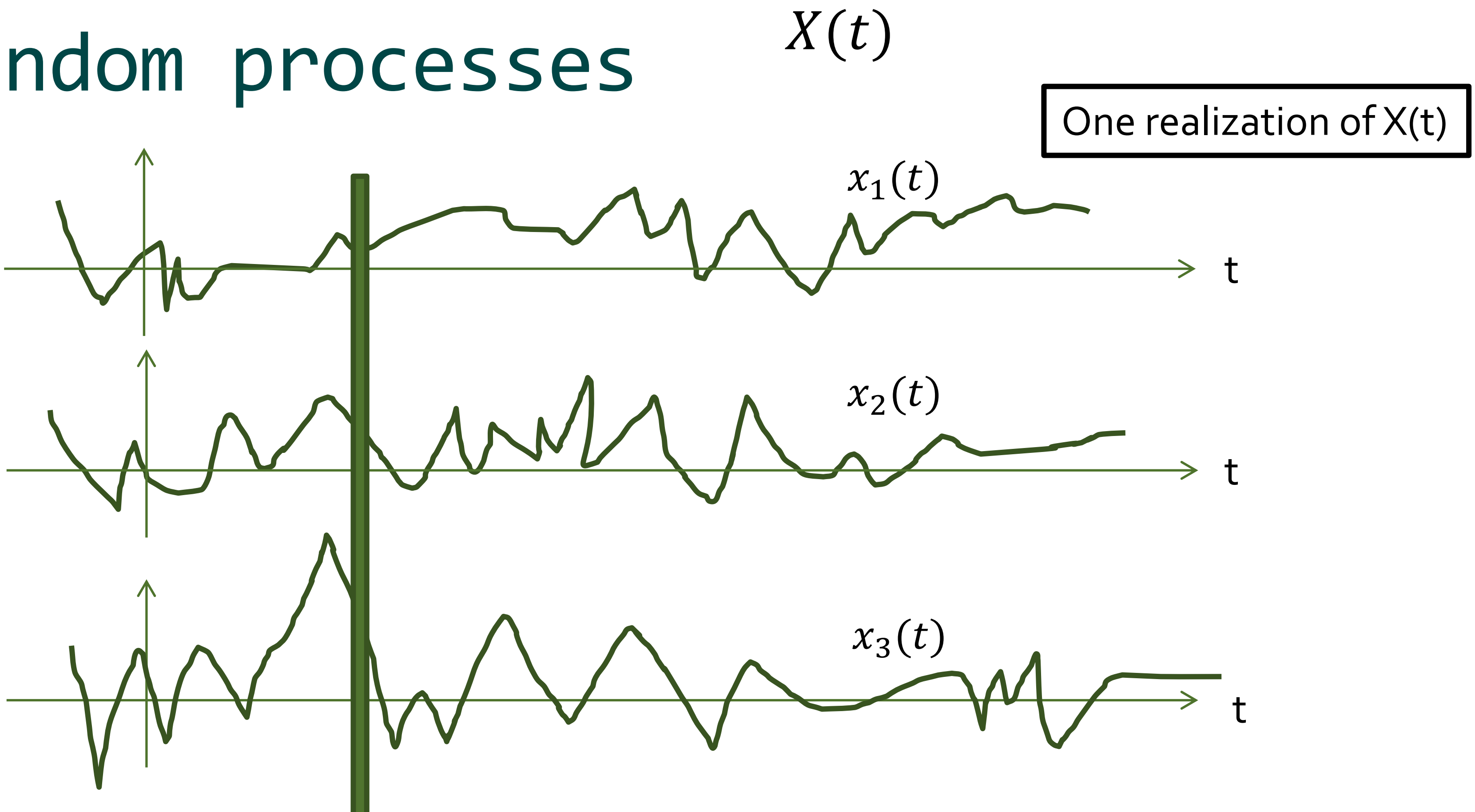
Logistics

- **11/10 HW1 is due and Lab3 will be announced.**
- **11/17 No class (I am out of town)**
- **Make-up class survey**
  - 11/28 (Monday) 6-9pm
  - 11/29 (Tuesday) 6-9pm
  - 11/30 (Wednesday) 6-9pm
- **Please take the survey here: http://ppt.cc/4EgU**
- **We will announce the time for the make-up class next week**
  - most likely the one with the least people who cannot come

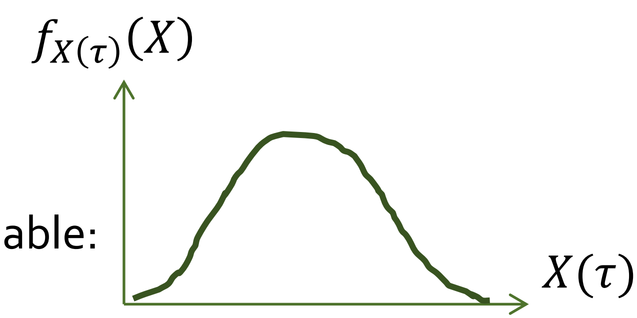
Logistics

- **11/24 & 12/1**
  Guest lectures by Dr. Kate Ching-Ju Lin (林靖茹)
  (see http://nms.citi.sinica.edu.tw/katelin/)
  Topics:
  - Modulation & rate adaption
  - OFDM encoding & decoding & recent researches
  - MIMO decoding & interference cancellation/alignment

Random processes



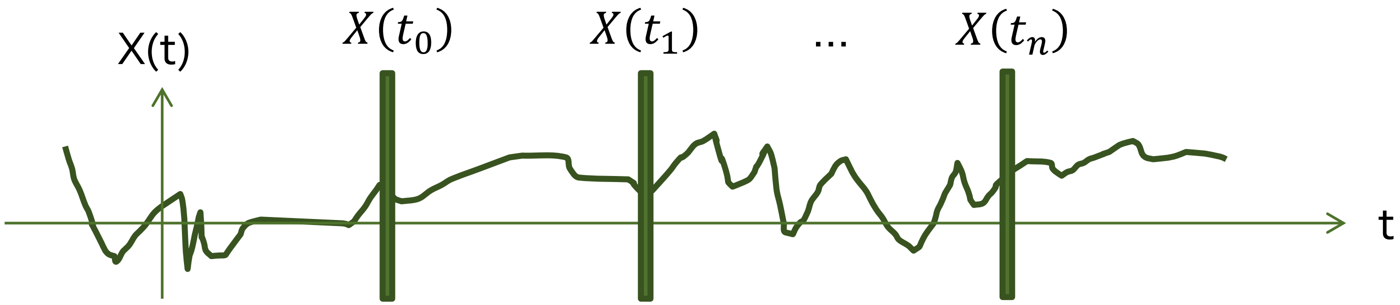
$X(\tau)$ is a random variable:



Joint CDF for a random process

- **If we sample X(t) at times $t_0, \ldots t_n$, we can have a joint cdf of samples at those times:**

$$P_{X(t_0),\ldots,X(t_n)}(x_0, \ldots, x_n) = p(X(t_0) \le x_0, X(t_1) \le x_1, \ldots, X(t_n) \le x_n)$$



Stationary Random Process (Strict-sense)

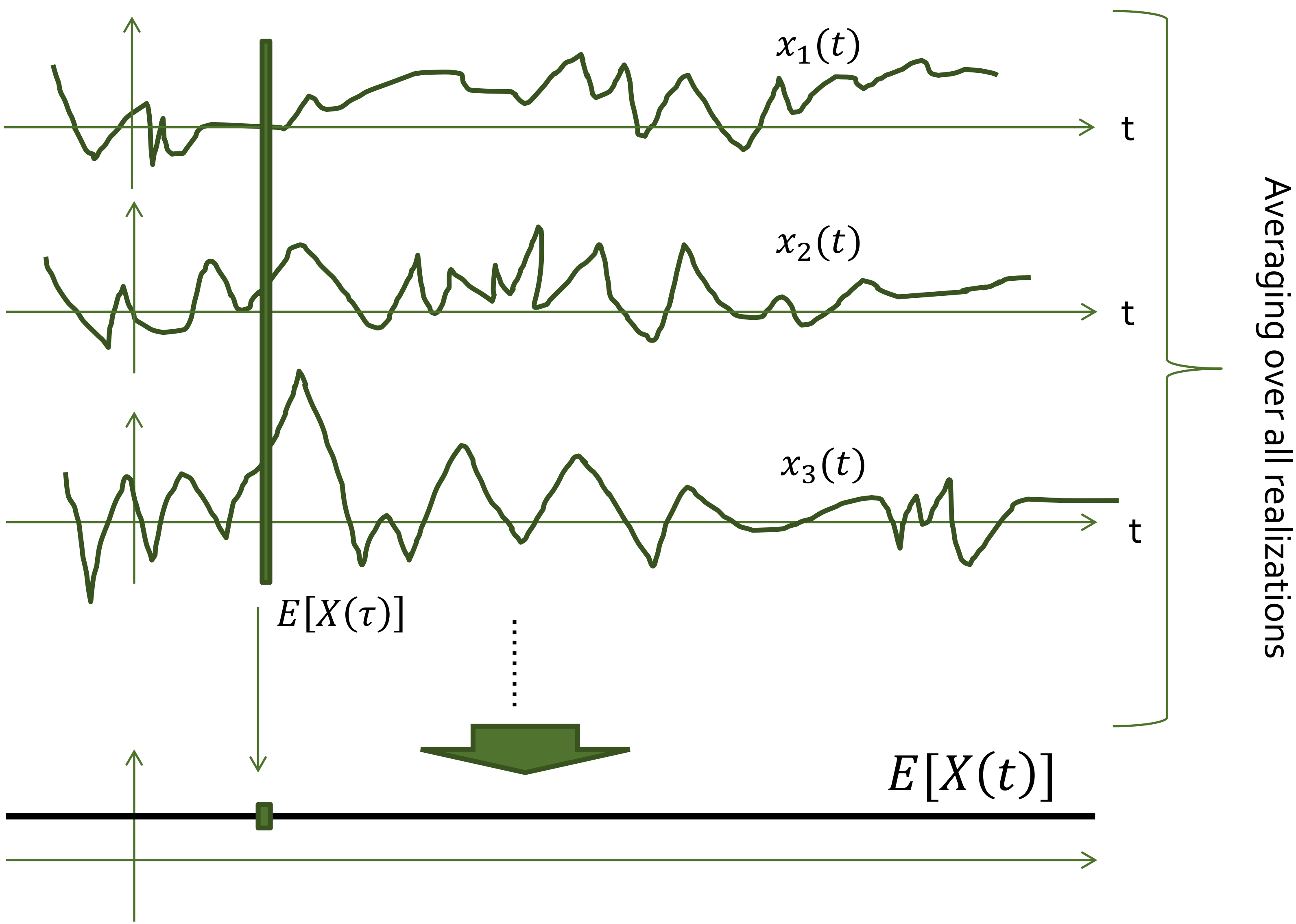
- **A random process X(t) is stationary if for all T, all n, and all sets of sample times $\{\boldsymbol{t_0}, \boldsymbol{t_1}, \dots, \boldsymbol{t_n}\}$ we have:**

$$p(X(t_0) \leq x_0, X(t_1) \leq x_1, \dots, X(t_n) \leq x_n) = p(X(t_0 + T) \leq x_0, X(t_1 + T) \leq x_1, \dots, X(t_n + T) < x_n)$$

If time shifts does not matter, then it is stationary

Mean (First Moment)

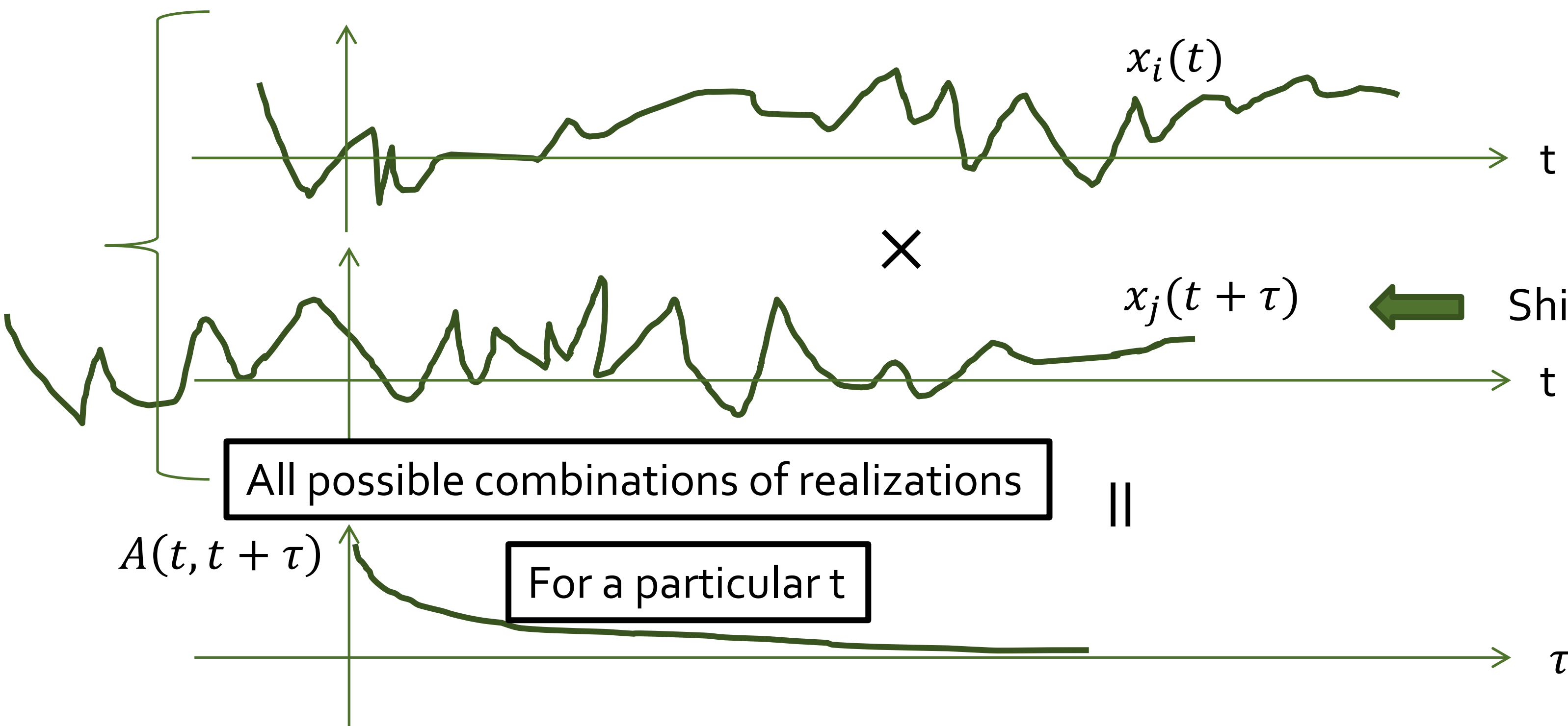
$E[X(t)]$



Autocorrelation (Second Moment)

- **"How similar a random process and a shifted version of itself is"**
- **Autocorrelation of a random process is defined as:**

$$A_X(t, t+\tau) \triangleq E[X(t)X(t+\tau)]$$



For stationary random processes…

- **Mean**

$$E[X(t)] = E[X(t-t)] = E[X(0)] = \mu_X$$

Constant. Does not change with t.

- **Autocorrelation**

$$A_X(t, t+\tau) = E[X(t-t)X(t+\tau-t)] = E[X(0)X(\tau)] \triangleq A_X(\tau)$$

Two random processes

- **Two random processes X(t) and Y(t) defined on the same underlying probability space have a joint cdf:**

$$p_{X(t_0),\dots,X(t_n)Y(t'_0),\dots,Y(t'_m)}(x_0,\dots,x_n,y_0,\dots,y_m) =$$
$$p(X(t_0)\le x_0,\dots,X(t_n)\le x_n, Y(t'_0)\le y_0,\dots,Y(t'_m)\le y_m)$$

for all possible sets of sample times $t_0, \dots, t_n$ and $t'_0, \dots, t'_m$.

Similar to how you can define a joint cdf for two random variables

- **Two random processes are <u>independent</u> if we have**

$$p_{X(t_0),\dots,X(t_n)Y(t'_0),\dots,Y(t'_m)}(x_0,\dots,x_n,y_0,\dots,y_m) =$$
$$p(X(t_0)\le x_0, X(t_1)\le x_1,\dots,X(t_n)\le x_n)p(Y(t'_0)\le y_0, Y(t'_1)\le y_1,\dots,Y(t'_n)\le y_n)$$

Cross-correlation

- **The cross-correlation between two random processes X(t) and Y(t) is defined as**

$$A_{XY}(t, t+\tau) \triangleq E[X(t)Y(t+\tau)]$$

- **Two random processes are uncorrelated if**

$$E[X(t)Y(t+\tau)] = E[X(t)]E[Y(t+\tau)]$$

for all t and τ.

- **If both random processes are stationary, we have**

$$A_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = E[X(0)Y(\tau)] = A_{\mathrm{XY}}(\tau)$$

Wide-Sense Stationary (WSS)

- **A process is wide-sense stationary if**

$$E[X(t)] = \mu_X$$

and

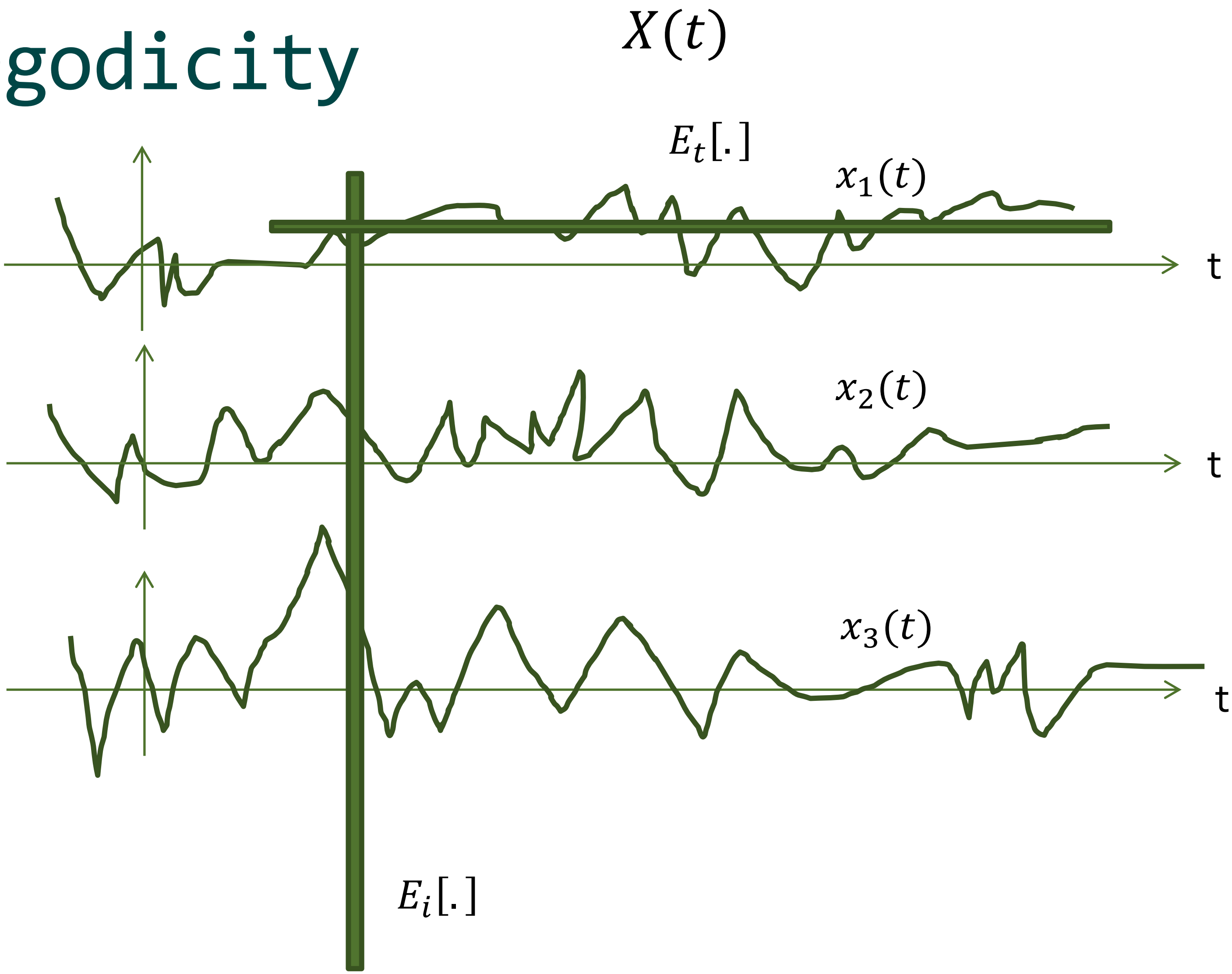
$$A_X(t, t+\tau) = E[X(t)X(t+\tau)] = A_X(\tau)$$

- $\boldsymbol{A_X(\tau)}$ **has its maximum value at** $\boldsymbol{\tau = 0}$**.**

$$|A_X(\tau)| \leq A_X(0) = E[X^2(t)]$$

A random process is always “the most similar” to the version of itself without shifting.

Ergodicity



Expectation value over time is the same as expectation over all possible realizations

Power Spectral Density

- **The power spectral density of a WSS process is defined as the Fourier transform of its autocorrelation function with respect to $\boldsymbol{\tau}$:**

$$S_X(f) = \int_{-\infty}^{\infty} A_X(\tau) \exp(-j2\pi f\tau)\, d\tau$$

- **PSD takes its name from the fact that the expected power of a random process X(t) is the integral of its PSD:**

$$E[X^2(t)] = A_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

Gaussian random processes

- **A random process X(t) is a Gaussian process if, for all values of T and all functions g(t), the random variable**

$$X_g = \int_0^T g(t)X(t)dt$$

Linear combination of samples

has a Gaussian distribution.

- **We usually use this to model the noise for a communication receiver.**
- **Mean & variance:**

$$E[X_g] = \int_0^T g(t)E[X(t)]dt$$

$$Var[X_g] = \int_0^T \int_0^T g(t)g(s)E[X(t)X(s)]dt\,ds - E^2[X_g]$$

$$X_g = \int_0^T g(t)X(t)dt$$

Gaussian random processes

- **Samples of a random process, $\boldsymbol{X(t_i), i = 0, \dots, n}$, are jointly Gaussian random variables, if we let $\boldsymbol{g(t) = \delta(t - t_i)}$.**

$$X_g = \int_0^T \delta(t - t_i)X(t)dt = X(t_i)$$

Example: white noise

- **White noise is a zero-mean WSS random process with a PSD that is constant over all frequencies.**

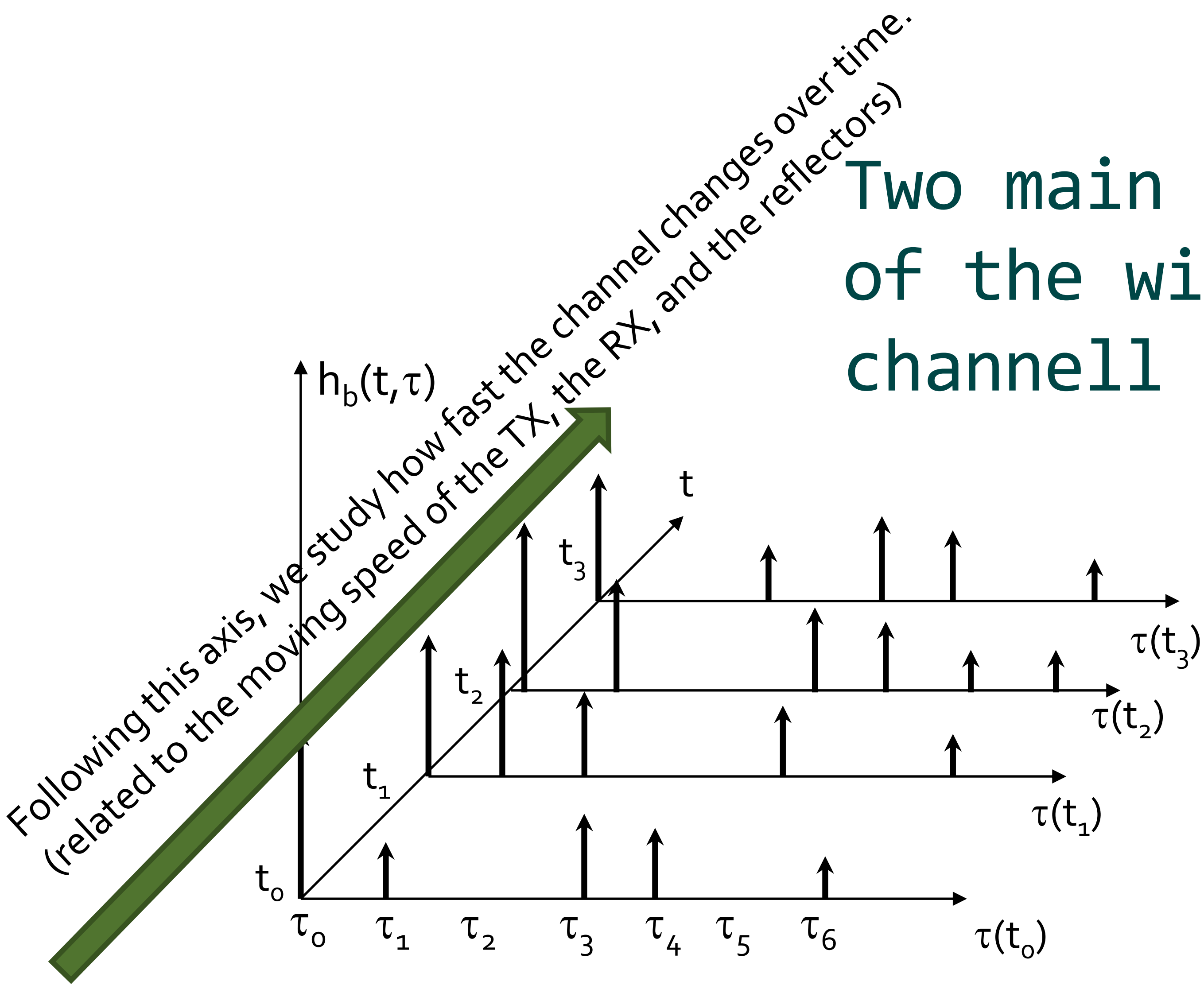
$E[X(t)] = 0$ $\qquad$ $S_X(f) = \frac{N_0}{2}$ for some constant N_0

- **N_0 is often called as one-sided white noise PSD.**
- **By inverse Fourier transform, the autocorrelation can be obtained:**

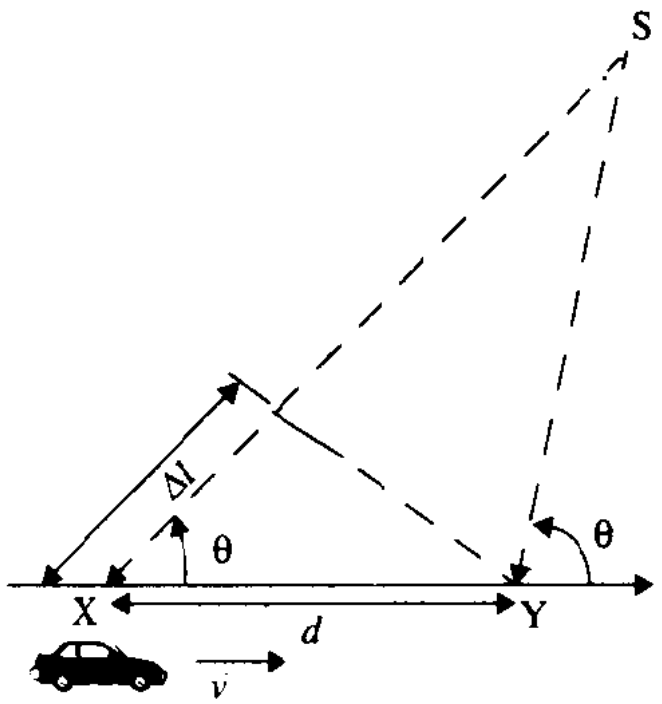
$$A_X(\tau) = \left(\frac{N_0}{2}\right)\delta(\tau)$$

White noise is not correlated with any shifted version of itself.
(Not similar at all after ANY time period)

Two main aspects of the wireless channell



Doppler Effect



- Difference in path lengths $\Delta l = d\,cos\theta = v\Delta t\,cos\theta$
- Phase change $\Delta\phi = \frac{2\pi\Delta l}{\lambda} = \frac{2\pi v\Delta t}{\lambda}cos\theta$
- Frequency change, or Doppler shift,

$$f_d = \frac{1}{2\pi}\,\frac{\Delta\phi}{\Delta t} = \frac{v}{\lambda}cos\theta$$

Example

$$f_d = \frac{1}{2\pi}\frac{\Delta\phi}{\Delta t} = \frac{v}{\lambda}\cos\theta$$

- **Consider a transmitter which radiates a sinusoidal carrier frequency of 1850 MHz. For a vehicle moving 60 mph, compute the received carrier frequency if the mobile is moving**

  1. directly toward the transmitter.
  2. directly away from the transmitter
  3. in a direction which is perpendicular to the direction of arrival of the transmitted signal.

- **Ans:**

  - Wavelength=$\lambda = \frac{c}{f_c} = \frac{3\times10^8}{1850\times10^6} = 0.162\ (m)$
  - Vehicle speed $v = 60mph = 26.82\ \left(\frac{m}{s}\right)$

  1. $f_d = \frac{26.82}{0.162}\cos(0) = 160\ (Hz)$
  2. $f_d = \frac{26.82}{0.162}\cos(\pi) = -160\ (Hz)$
  3. Since $\cos\left(\frac{\pi}{2}\right) = 0$, there is no Doppler shift!

Doppler Effect

- **If the car (mobile) is moving toward the direction of the arriving wave, the Doppler shift is positive**
- **Different Doppler shifts if different θ (incoming angle)**
- **Multi-path: all different angles**
- **Many Doppler shifts → Doppler spread**

Narrow-band Fading Model

- **Sending an unmodulated carrier wave with random phase offset $\boldsymbol{\phi_0}$:**

$$s(t) = Re\{\exp\big(j(2\pi f_c t + \phi_0)\big)\} = \cos(2\pi f_c t + \phi_0)$$

- **Received signal becomes**

$$r(t) = Re\left\{\underbrace{\left[\sum_{n=1}^{N(t)} \alpha_n(t)\exp\big(-j\phi_n(t)\big)\right]}_{\text{Sum of many MPC}} \underbrace{\exp(j2\pi f_c t)}_{\text{Carrier with frequency } f_c}\right\}$$

Sum of many MPC

Carrier with frequency f_c

$$= r_I(t)\cos(2\pi f_c t) - r_Q(t)\sin(2\pi f_c t)$$

$$r(t) = Re\left\{\left[\sum_{n=1}^{N(t)} \alpha_n(t)\exp\big(-j\phi_n(t)\big)\right]\exp(j2\pi f_c t)\right\}$$

$$= r_I(t)\cos(2\pi f_c t) - r_Q(t)\sin(2\pi f_c t)$$

$$r_I(t) = \sum_{n=1}^{N(t)} \alpha_n(t)\cos\phi_n(t) \qquad r_Q(t) = \sum_{n=1}^{N(t)} \alpha_n(t)\sin\phi_n(t)$$

$$\phi_n(t) = 2\pi f_c \tau_n(t) - \phi_{D_n} - \phi_0$$

Phase shift due to delay

Doppler Shift

Carrier phase shift (same for all MPC)

Since N(t) is large & we assume $\alpha_n(t)$ and $\phi_n(t)$ are independent for different MPC, we can approximate $r_I(t)$ and $r_Q(t)$ as **jointly Gaussian random processes**.

Some assumptions

- <u>**No dominant LOS component**</u>
- $\boldsymbol{\alpha_n(t), f_{D_n}(t), and\ \tau_n(t)}$ **change slowly over time**
- $\boldsymbol{2\pi f_c \tau_n}$ **changes rapidly relative to all other phase terms**

Very large

$$\phi_n(t) = 2\pi f_c \tau_n(t) - \phi_{D_n} - \phi_0$$

- $\boldsymbol{\phi_n(t)}$ **uniformly distributed on** $\boldsymbol{[-\pi, \pi]}$**.**
- $\boldsymbol{\alpha_n}$ **and** $\boldsymbol{\phi_n}$ **are independent of each other.**

Zero-mean

$$E[r_I(t)] = E\left[\sum_n \alpha_n \cos\phi_n(t)\right] = \sum_n E[\alpha_n] E[\cos\phi_n(t)] = 0$$

- **Similarly,**

$$E\left[r_Q(t)\right] = 0$$

- **So, E[r(t)]=0, and it is a <u>zero-mean</u> Gaussian process.**
- **If there is a dominant LOS component, then this is no longer true.**

Un-correlated

$$E\left[r_I(t)r_Q(t)\right] = E\left[\sum_n \alpha_n cos\phi_n(t) \sum_m \alpha_m \sin\alpha_m(t)\right]$$

α_n and ϕ_n are not correlated.

$$= \sum_n \sum_m E[\alpha_n \alpha_m]\, E[\cos\phi_n(t) \sin\phi_m(t)]$$

Different MPC's α_n and ϕ_n are independent

$$= \sum_{\substack{n,m \\ n \neq m}} E[\alpha_n]E[\alpha_m]\underbrace{E[\cos\phi_n(t)]}E[\sin\phi_m(t)] + \sum_n E[\alpha_n^2]E[\cos\phi_n(t)\sin\phi_n(t)]$$

Uniformly distributed over $[-\pi, \pi]$, so =0.

$$= \sum_n E[\alpha_n^2]E[\cos\phi_n(t)\sin\phi_n(t)] = \sum_n E[\alpha_n^2]E\overbrace{\left[\frac{\sin 2\phi_n(t)}{2}\right]} = 0$$

$r_I(t)$ and $r_Q(t)$ are uncorrelated, and they are Gaussian processes
→ they are independent.

$$\phi_n(t+\tau) = 2\pi f_c \tau_n - 2\pi f_{D_n}(t+\tau) - \phi_0$$

$$\phi_n(t) = 2\pi f_c \tau_n - 2\pi f_{D_n} t - \phi_0$$

Autocorrelation

$$A_{r_I}(t, t+\tau) = E[r_I(t) r_I(t+\tau)] = \sum_n E[\alpha_n^2] E[\cos\phi_n(t) \cos\phi_n(t+\tau)]$$

$$E[\cos\phi_n(t)\cos\phi_n(t+\tau)] =$$

$$= E[.5\cos(\phi_n(t+\tau) - \phi_n(t)) + .5\cos(\phi_n(t+\tau) + \phi_n(t))]$$

$$= .5E[\cos(2\pi f_{D_n}\tau)] + \underbrace{.5E\left[\cos\left(4\pi f_c\tau_n - 4\pi f_{D_n} t - 2\pi f_{D_n}\tau - 2\phi_0\right)\right]}_{0}$$

$4\pi f_c \tau_n$: Large and uniformly distributed over $[-\pi, \pi]$

$$A_{r_I}(t, t+\tau) = .5\sum_n E[\alpha_n^2] E[\cos(2\pi f_{D_n}\tau)] = .5 \sum_n E[\alpha_n^2]\, 2\pi v\tau \cos\left(\frac{\theta_n}{\lambda}\right)$$

Only depends on τ, so WSS!

Autocorrelation

$$A_{r_I,r_Q}(t,t+\tau) = A_{r_I,r_Q}(\tau) = E\left[r_I(t)r_Q(t+\tau)\right]$$

$$= -.5\sum_n E[\alpha_n^2]\sin\left(2\pi v\tau\cos\frac{\theta_n}{\lambda}\right) = -E\left[r_Q(t)r_I(t+\tau)\right]$$

- **Finally,**

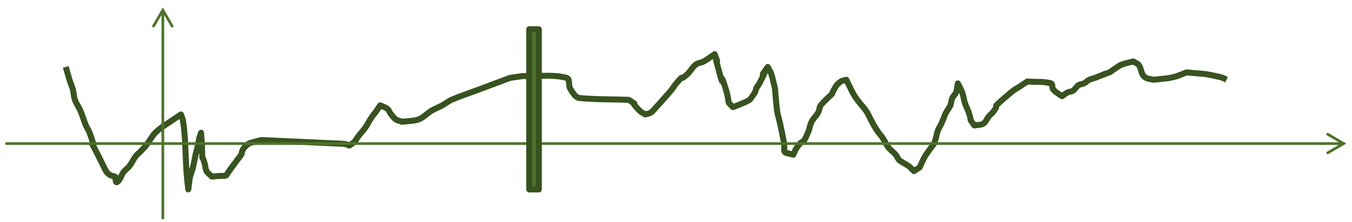
The received signal, representing how the channel changes over time

$$r(t) = r_I(t)\cos(2\pi f_c t) - r_Q(t)\sin(2\pi f_c t)$$

$$A_r(\tau) = E[r(t)r(t+\tau)] = A_{r_I}(\tau)\cos(2\pi f_c\tau) + A_{r_I,r_Q}(\tau)\sin(2\pi f_c\tau)$$

Also only depends on τ, WSS!

Amplitude distribution - Rayleigh



- $\boldsymbol{z(t) = |r(t)| = \sqrt{r_I^2(t) + r_Q^2(t)}}$
- $\boldsymbol{r_I(t)}$ **and** $\boldsymbol{r_Q(t)}$ **are both zero-mean Gaussian random process (so at a given time, two Gaussian random variables).**
- **z(t)'s distribution - the amplitude distribution of r(t):**

$$p_Z(z) = \frac{2z}{\overline{P}_r} \exp\left[-\frac{z^2}{\overline{P}_r}\right] = \frac{z}{\sigma^2} \exp\left[-\frac{z^2}{2\sigma^2}\right], z \geq 0$$

This is the famous Rayleigh distribution!

2-variable joint Gaussian distribution

- **PDF for 2-variable joint Gaussian distribution:**

$$f(X,Y) =$$

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(X-\mu_X)^2}{\sigma_X^2}+\frac{(Y-\mu_Y)^2}{\sigma_Y^2}-\frac{2\rho(X-\mu_X)(Y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

- $\boldsymbol{\rho}$: **X and Y's correlation (in our case, 0)**
- $\boldsymbol{\mu_X}$ **and** $\boldsymbol{\mu_Y}$**: X and Y's mean**
- $\boldsymbol{\sigma_X^2}$ **and** $\boldsymbol{\sigma_Y^2}$**: X and Y's variance (in our case, both are** $\boldsymbol{\sigma^2}$)

$$f(X,Y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2}\left[\frac{X^2+Y^2}{\sigma^2}\right]\right)$$

The rest of the derivation can be found here:
http://www.dsplog.com/2008/07/17/derive-pdf-rayleigh-random-variable/

Power distribution: Rayleigh

- **We can obtain the power distribution by making the change of variables $\boldsymbol{z^2(t) = |r(t)|^2}$ to obtain**

$$p_{Z^2}(x) = \frac{1}{\overline{\overline{P}}_r} \exp\left(-\frac{x}{\overline{\overline{P}}_r}\right) = \frac{1}{2\sigma^2} \exp\left(-\frac{x}{2\sigma^2}\right), x \geq 0$$

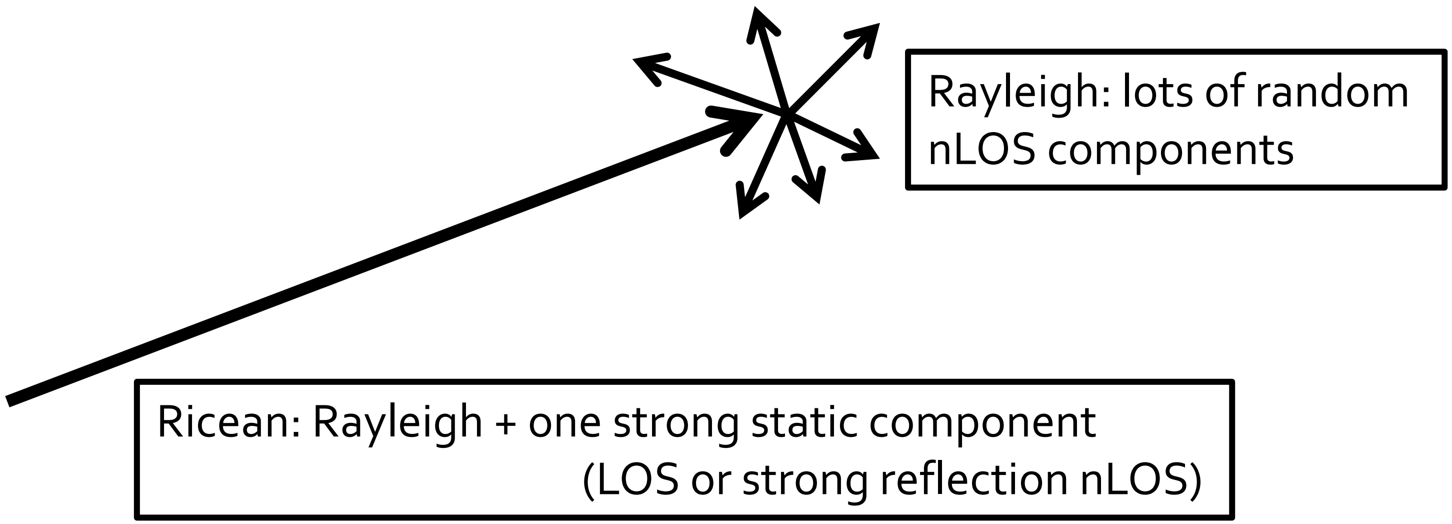
Example: Rayleigh fading

- **Consider a channel with Rayleigh fading (no LOS!) and average received power $\overline{P_r} = 20$ dBm. Find the probability that the received power is below 10 dBm.**
- **We want to find the probability that $Z^2 < 10\ dBm = 10\ mW$.**

$$p(Z^2 < 10) = \int_0^{10} \frac{1}{100} \exp\left(-\frac{x}{100}\right) dx = 0.095$$

With a LOS component – Ricean (or Rician)

- **If the channel has a fixed LOS component then $r_I(t)$ and $r_Q(t)$ are no longer zero-mean variables.**
- **The received signal becomes the superposition of a complex Gaussian component and a LOS component.**



Ricean distribution

- **Amplitude distribution:**

$$p_Z(z) = \frac{z}{\sigma^2} \exp\left[-\frac{z^2 + s^2}{2\sigma^2}\right] I_0\left(\frac{zs}{\sigma^2}\right), \qquad z \geq 0$$

- $2\sigma^2 = \sum_{n,n\neq 0} E[\alpha_n^2]$ **is the average power in the nLOS MPCs.**
- $s^2 = \alpha_0^2$ **is the power in the dominant strong component.**
- $I_0(x)$**: the modified Bessel function of zeroth order.**

Ricean distribution

- **The average power in the Ricean fading is**

$$\bar{P_r} = \int_0^\infty z^2 p_Z(z)dz = s^2 + 2\sigma^2$$

- **The Ricean distribution is often described in terms of a fading parameter K, defined by**

$$K = \frac{s^2}{2\sigma^2}$$

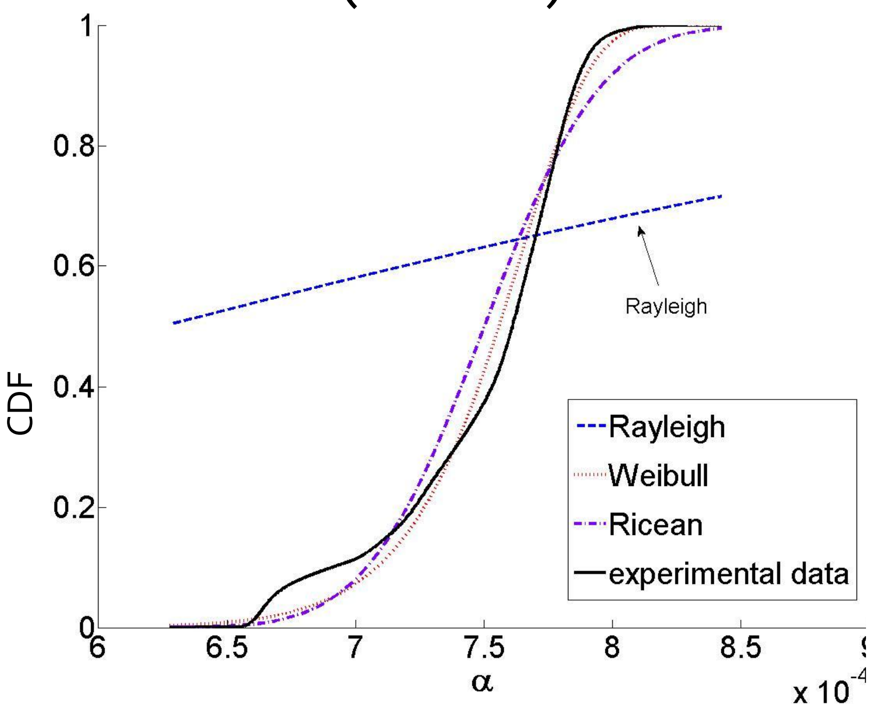
- **K is the ratio of the power in the dominant component to the power in the other random MPCs.**
  - K=0, then Ricean degenerates to Rayleigh
  - K=∞, then Ricean becomes a non-fading LOS channel.

Example: Intra-car Wireless Channel Measurements

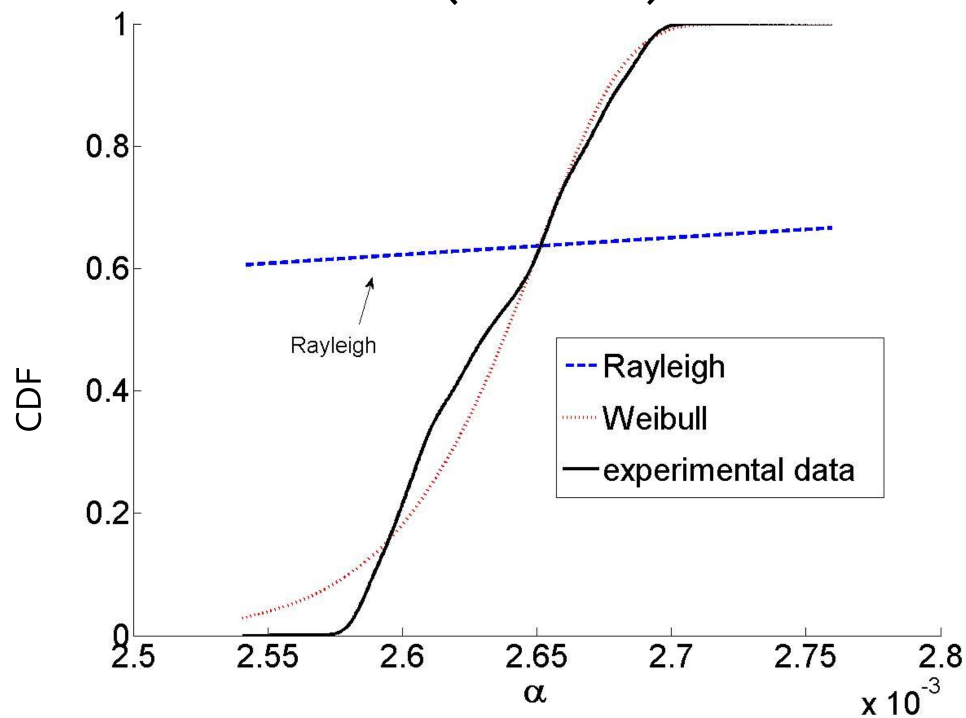
Which distribution fit the empirical amplitude distribution function the best?

Lognormal, Nakagami, Rician, Rayleigh, and Weibull

Engine compartment → Center (Parked)

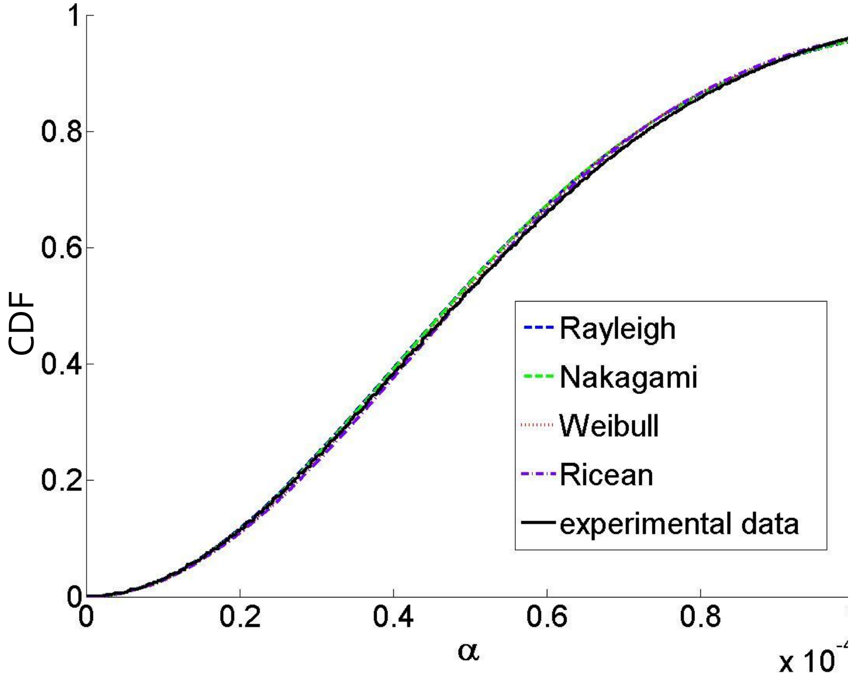


Engine compartment → Under the engine (Parked)

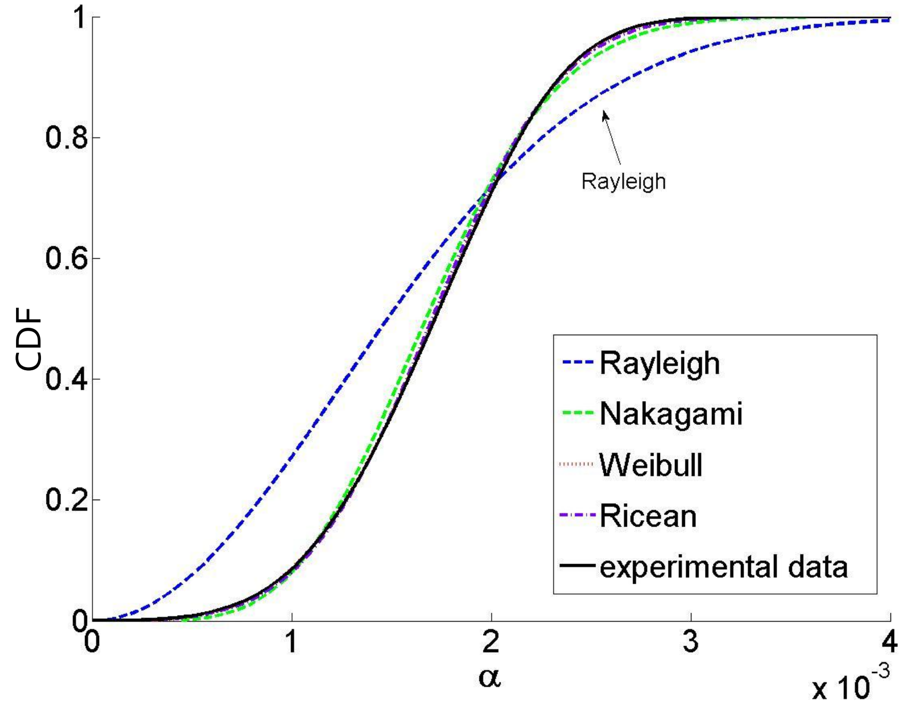


Parked: Weibull

Engine compartment → Trunk (Driving)



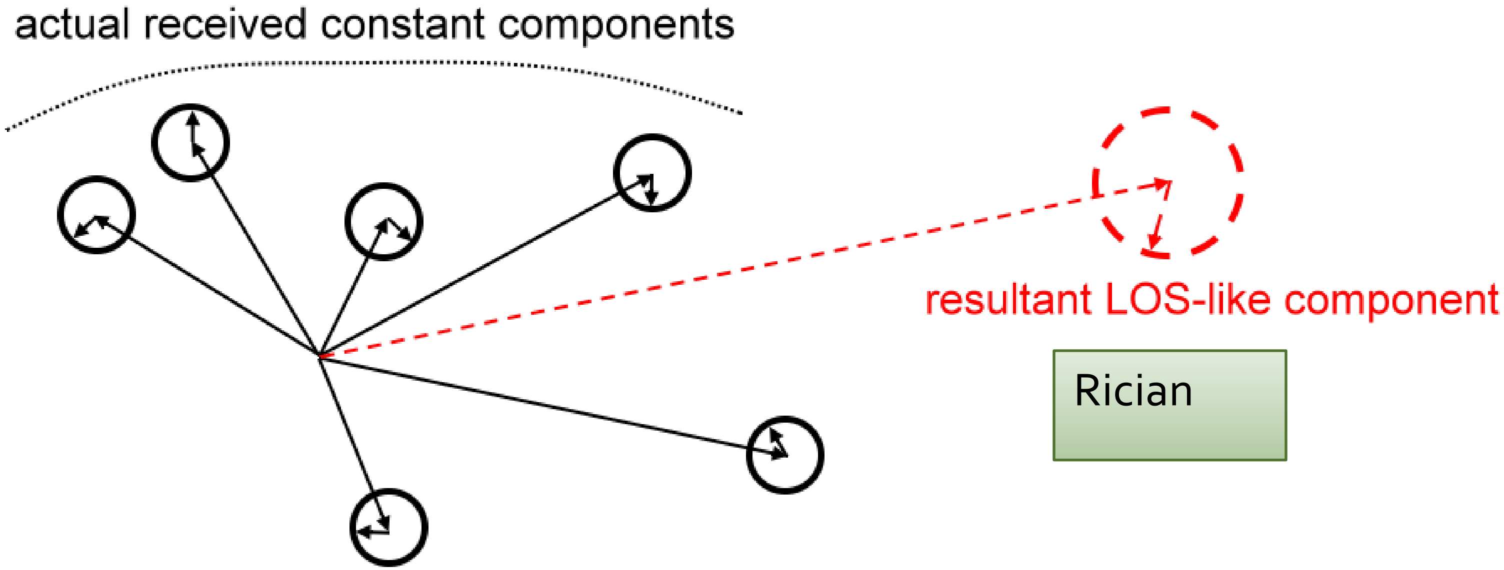
Under the engine → Engine Compartment (Driving)



Driving: Rician/Nakagami

Channel model

No Line-of-Sight component → Rayleigh



Rician

Coherence Time

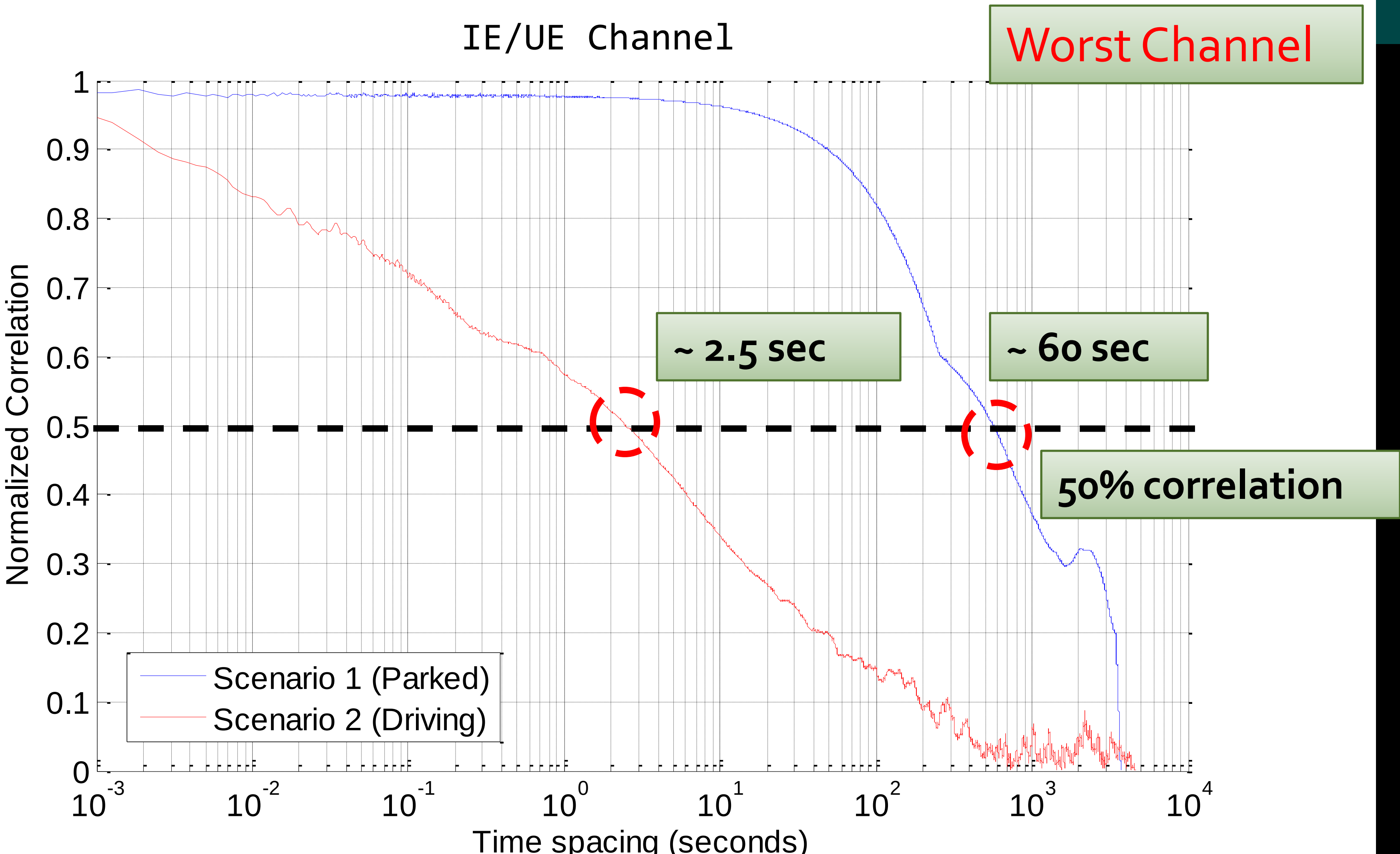
- **Coherence Time:**
  Coherence time is a statistical measure of the range of time over which the channel can be considered "static".
- **90% coherence time:**

$$T_{c,0.9} = argmin_{\tau}\left(\frac{A_r(\tau)}{A_r(0)} < 0.9\right)$$

The first time interval that normalized autocorrelation drops below the threshold.

- **We can define 50% coherence time in a similar way too.**

Coherence time is always on the order of **seconds**.



Fast and slow fading channel