Multi-Input Multi-Output Systems (MIMO)

- Channel Model for MIMO
- MIMO Decoding
- MIMO Gains
- Multi-User MIMO Systems
MIMO

- Each node has multiple antennas
  - Capable of transmitting (receiving) multiple streams concurrently
  - Exploit antenna diversity to increase the capacity

\[ H_{N \times M} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots \\ h_{21} & h_{22} & h_{23} & \cdots \\ h_{31} & h_2 & h_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]
Channel Model (2x2)

\[ \begin{align*}
  y_1 &= h_{11} x_1 + h_{21} x_2 + n_1 \\
  y_2 &= h_{12} x_1 + h_{22} x_2 + n_2 \\
  y &= H x + n
\end{align*} \]

- Can be extended to N x M systems
Antenna Space

M-antenna node receives in M-dimensional space

\[
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix} = \begin{bmatrix}
    h_{11} & h_{21} \\
    h_{12} & h_{22}
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + \begin{bmatrix}
    n_1 \\
    n_2
\end{bmatrix}
\]

\[\tilde{y} = \tilde{h}_1 x_1 + \tilde{h}_2 x_2 + \tilde{n}\]

\[\tilde{h}_2 = (h_{21}, h_{22})\]

\[\tilde{y} = (y_1, y_2)\]

\[\tilde{h}_1 = (h_{11}, h_{12})\]
MIMO Decoding (algebra)

\[
\begin{pmatrix}
  y_1 \\
  y_2 
\end{pmatrix} = \begin{pmatrix}
  h_{11} \\
  h_{12}
\end{pmatrix} x_1 + \begin{pmatrix}
  h_{21} \\
  h_{22}
\end{pmatrix} x_2 + \begin{pmatrix}
  n_1 \\
  n_2
\end{pmatrix} * h_{22} 
\]

\[
y_1 h_{22} - y_2 h_{21} = (h_{11} h_{22} - h_{12} h_{21}) x_1
\]

\[
x_1 = \frac{y_1 h_{22} - y_2 h_{21}}{h_{11} h_{22} - h_{12} h_{21}} \quad \text{Given } x_1, \text{ solve } x_2
\]

To guarantee the full rank of H, antenna spacing at the transmitter and receiver must exceed half of the wavelength.
MIMO Decoding (antenna space)

- Zero forcing

\[ \bar{h}_2 = (h_{21}, h_{22}) \]
\[ \bar{y} = (y_1, y_2) \]
\[ \| x' \|_1 < \| x_1 \| \]

- To decode \( x_1 \), decode vector \( y \) on the direction orthogonal to \( x_2 \)
- To improve the SNR, re-encode the first detected signal, subtract it from \( y \), and decode the second signal
Channel Estimation

• Estimate $N \times M$ matrix $H$

\[
\begin{align*}
    y_1 &= h_{11} x_1 + h_{21} x_2 + n_1 \\
    y_2 &= h_{12} x_1 + h_{22} x_2 + n_2
\end{align*}
\]

Two equations, but four unknowns

Antenna 1 at Tx

Access code 1

Stream 1

Antenna 2 at Tx

Access code 2

Stream 2

Estimate $h_{11}, h_{12}$

Estimate $h_{21}, h_{22}$
MIMO Gains

• Multiplex Gain
  ▶ Exploit antenna diversity to deliver multiple streams concurrently

• Diversity Gain
  ▶ Exploit antenna diversity to increase the SNR of a single stream
Diversity Gain

- 1 x 2 example

\[ y_1 = h_1 x + n_1 \]
\[ y_2 = h_2 x + n_2 \]

- Decode the SNR of \((y_1 + y_2)\)
- Uncorrelated whit Gaussian noise with zero mean
- Packet can be delivered through at least one of the many diver paths
Diversity Gain

• 1 x 2 example

\[ y_1 = h_1 x + n_1 \]
\[ y_2 = h_2 x + n_2 \]

\[ \text{SNR} = \frac{P(2X)}{P(n_1 + n_2)} \text{, where } P \text{ refers to the power} \]
\[ = \frac{E[(2X)^2]}{E[n_1^2 + n_2^2]} \]
\[ = \frac{4E[X^2]}{2\sigma^2} \text{, where } \sigma \text{ is the variance of AWGN} \]
\[ = 2 \times \text{SNR}_{\text{single antenna}} \]

• Increase SNR by 3dB
• Especially beneficial for the low SNR link
Diversity Gain

Multiply each $y$ with the conjugate of the channel

\[
\begin{align*}
  y_1 &= h_1 x + n_1 \\
  y_2 &= h_2 x + n_2
\end{align*}
\]

\[
\begin{align*}
  h_1^* y_1 &= \|h_1\|^2 x + h_1^* n_1 \\
  h_2^* y_2 &= \|h_2\|^2 x + h_2^* n_2
\end{align*}
\]

\[
\begin{align*}
  \text{SNR}_{\text{diversity}} &= \frac{\mathbb{E}[((\|h_1\|^2 + \|h_2\|^2)X)^2]}{\mathbb{E}[(h_1^* n_1 + h_2^* n_2)^2]} \\
  &= \frac{(\|h_1\|^2 + \|h_2\|^2)^2 \mathbb{E}(X^2)}{(\|h_1\|^2 + \|h_2\|^2)\sigma^2} \\
  &= \frac{(\|h_1\|^2 + \|h_2\|^2)\mathbb{E}(X^2)}{\sigma^2}
\end{align*}
\]

\[
\begin{align*}
  \text{SNR}_{\text{single}} &= \frac{\mathbb{E}[((\|h_1\|^2 + \|h_2\|^2)X)^2]}{\mathbb{E}[(h_1^* n_1 + h_2^* n_2)^2]} \\
  &= \frac{\|h_1\|^4 \mathbb{E}(X^2)}{(\|h_1\|^2)\sigma^2} \\
  &= \frac{\|h_1\|^2 \mathbb{E}(X^2)}{\sigma^2}
\end{align*}
\]

\[
\text{gain} = \frac{(\|h_1\|^2 + \|h_2\|^2)}{\|h_1\|^2}
\]
Trade off

• Between diversity gain and multiplex gain

• Say we have a N x N system
  ▶ Degree of freedom: N
  ▶ The transmitter can transmit k streams concurrently, where k <= N
  ▶ The optimal value of k is determined by the tradeoff between the diversity gain and multiplex gain
Degree of Freedom

• For $N \times M$ MIMO channel
  
  - Degree of Freedom (DoF): $\min \{N, M\}$
  - Maximum diversity: $NM$
Space-Time Code Examples: $2 \times 1$ Channel

**Repetition Scheme:**

$$X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

- **diversity:** 2
- **data rate:** $1/2$ sym/s/Hz

**Alamouti Scheme:**

$$X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

- **diversity:** 2
- **data rate:** 1 sym/s/Hz
Space-Time Code Examples: $2 \times 2$ Channel

Repetition Scheme:

$X = \begin{bmatrix}
  x_1 & 0 \\
  0 & x_1
\end{bmatrix}$

diversity: 4
data rate: $1/2$ sym/s/Hz

Alamouti Scheme:

$X = \begin{bmatrix}
  x_1 & -x_2 \\
  x_2 & x_1
\end{bmatrix}$

diversity: 4
data rate: 1 sym/s/Hz

But the $2 \times 2$ channel has 2 degrees of freedom!
Interference Nulling

- Signals cancel each other at Alice’s receiver
- Signals don’t cancel each other at Bob’s receiver
  - Because channels are different

\[
h_1 \alpha x + h_2 \beta x = 0
\]

\[\Rightarrow \text{Nulling: } h_1 \alpha = -h_2 \beta
\]

\[
(h_{1a} \alpha + h_{2a} \beta)x \neq 0
\]

\[
(h_{1b} \alpha + h_{2b} \beta)x \neq 0
\]
Homework

• Say there exist a 3x2 link, which has a channel

\[
H_{3\times2} = \begin{bmatrix}
    h_{11} & h_{12} \\
    h_{21} & h_{22} \\
    h_{31} & h_{32}
\end{bmatrix}
\]

How can a three-antenna transmitter transmit a signal $x$, but null its signal at two antennas of a two-antenna receiver?
Interference Alignment

2-antenna receiver

N-antenna node can only decode N signals

If $I_1$ and $I_2$ are aligned,
Interference Alignment

If \( I_1 \) and \( I_2 \) are aligned,

\( \rightarrow \) appear as one interferer

\( \rightarrow \) 2-antenna receiver can decode the wanted signal

N-antenna node can only decode N signals
Rotate Signal

1. Transmitter can rotate the received signal

To rotate received signal $y$ to $y' = Ry$, transmitter multiplies its transmitted signal by the same rotation matrix $R$. 

2-antenna receiver

$y' = Ry$
Rotate Signal

How to align the signal along the interference?

→ Find the direction of the interference and rotate the signal to that direction