Mathematical Foundation II: Signals & Systems basics

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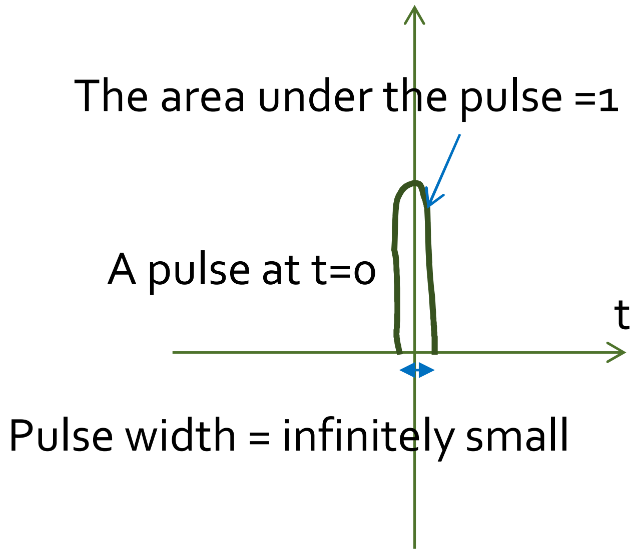
Dirac Delta Function Definition

- **Dirac Delta Function:**

$$\delta(t) = 0, \qquad t \neq 0$$

- **and**

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



- **, or by *sifting property:***

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$$

Two properties of the delta function

- **After rewriting the equation:**

$$\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau = g(t) \Rightarrow g(t) * \delta(t) = g(t)$$

Replication property of the delta function

- **Fourier Transform:**

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) \exp(-j2\pi f t)\, dt$$
$$= \exp(0) = 1$$

The spectrum of the delta function extends uniformly ***over the entire frequency interval.***

Duality property

- **If $\boldsymbol{F[g(t)] = G(f), then\ F[G(t)] = g(-f)}$**

$$g(-t) = \int_{-\infty}^{\infty} G(f)\exp(-j2\pi f t)df$$

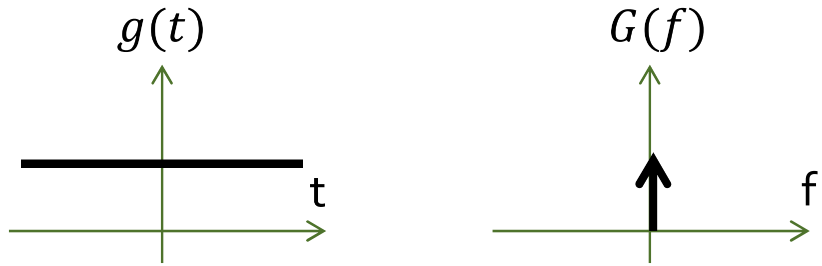
inverse Fourier transform definition

$$g(-f) = \int_{-\infty}^{\infty} G(t)\exp(-j2\pi f t)dt$$

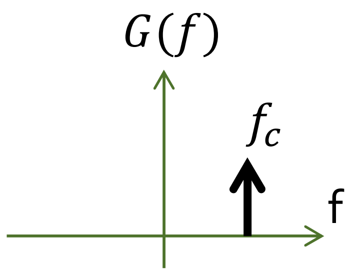
Interchanging f and t

Some example uses of the delta function

- **DC signal**
  - By duality property, we know $F[1] = \delta(f)$.

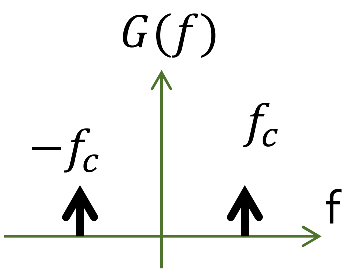


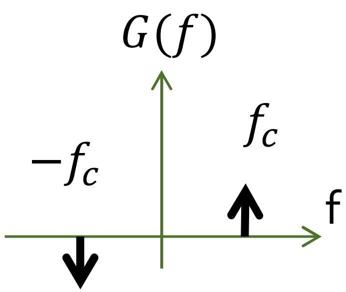
- **Complex exponential function**
  - By using the frequency-shifting property, we know $F[\exp(j2\pi f_c t)] = \delta(f - f_c)$.



Some example uses of the delta function

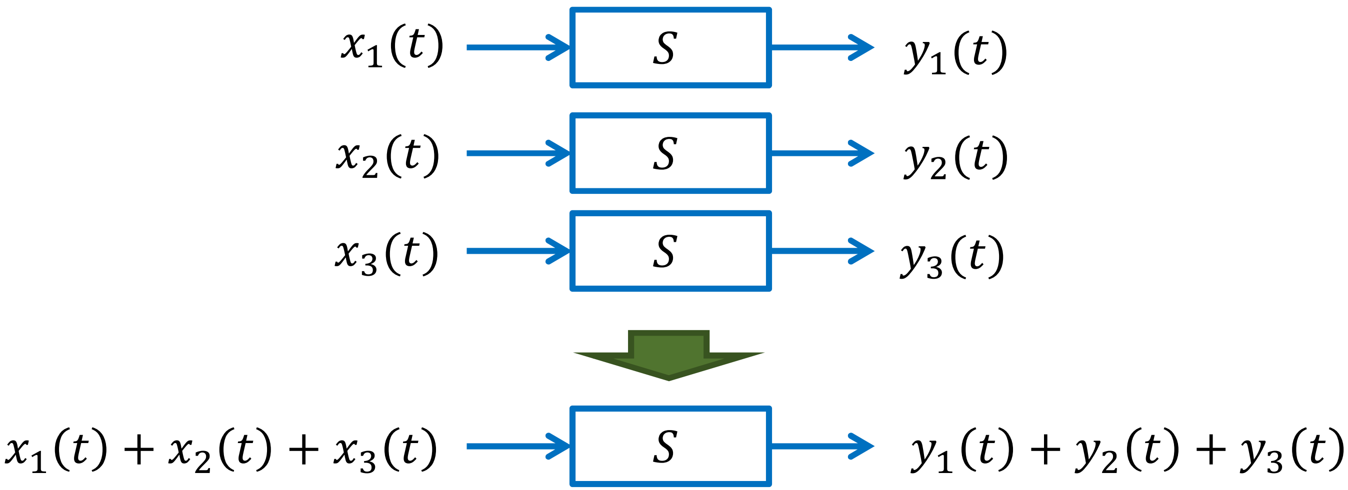
- **Sinusoidal functions:**
  - $\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$
    - $F[\cos(2\pi f_c t)] = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
  - $\sin(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$
    - $F[\sin(2\pi f_c t)] = \frac{1}{2}[\delta(f - f_c) - \delta(f + f_c)]$





Transmission of signals through linear systems

- **A system: any physical device that produces an output signal in response to an input signal**
- **Input signal = excitation**
- **Output signal=response**
- **A <u>linear</u> system: the principle of superposition holds.**

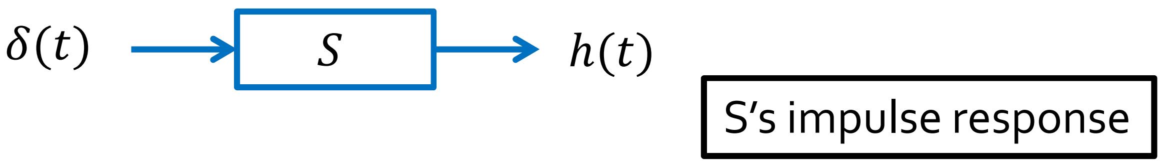


Filters and channels

- **Filters and channels →**
  important examples of linear systems
- **A filter: a frequency-selective device that is used to limit the spectrum of a signal to some band of frequencies.**
- **A channel: a transmission medium that connects the transmitter and receiver of a communication system.**

Time response

- **Defined in the time domain: described by its <u>impulse response</u>.**
- **Impulse response: the response of the system to delta function applied to the input of the system.**

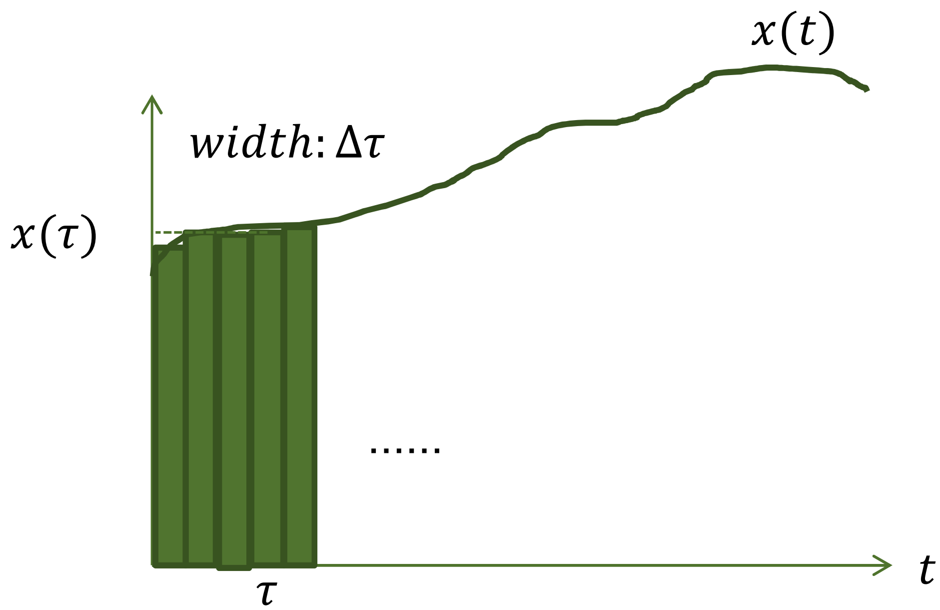


Approximating x(t) and its time response

- **Impulse response:** $h(t)$

- **Time response to** $\delta(t-\tau)$ **:**
  $h(t-\tau)$

  time shifted by $\boldsymbol{\tau}$



- **Time response to** $(x(\tau)\Delta\tau)\ \delta(t-\tau)$**:** $(x(\tau)\Delta\tau)\ h(t-\tau)$

  weighted by the area $x(\tau)\Delta\tau$

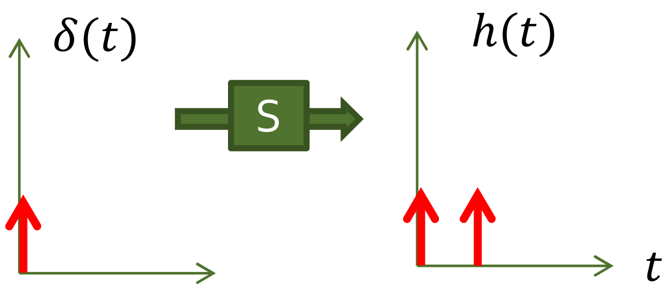
- **Finally, integrating over all** $\boldsymbol{\tau}$ **to obtain the output signal y(t):**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (x * h)(t)$$

Convolution!

The strength of the output signal y at time t is the integral of the responses whose total "delay" is t.

Simple example

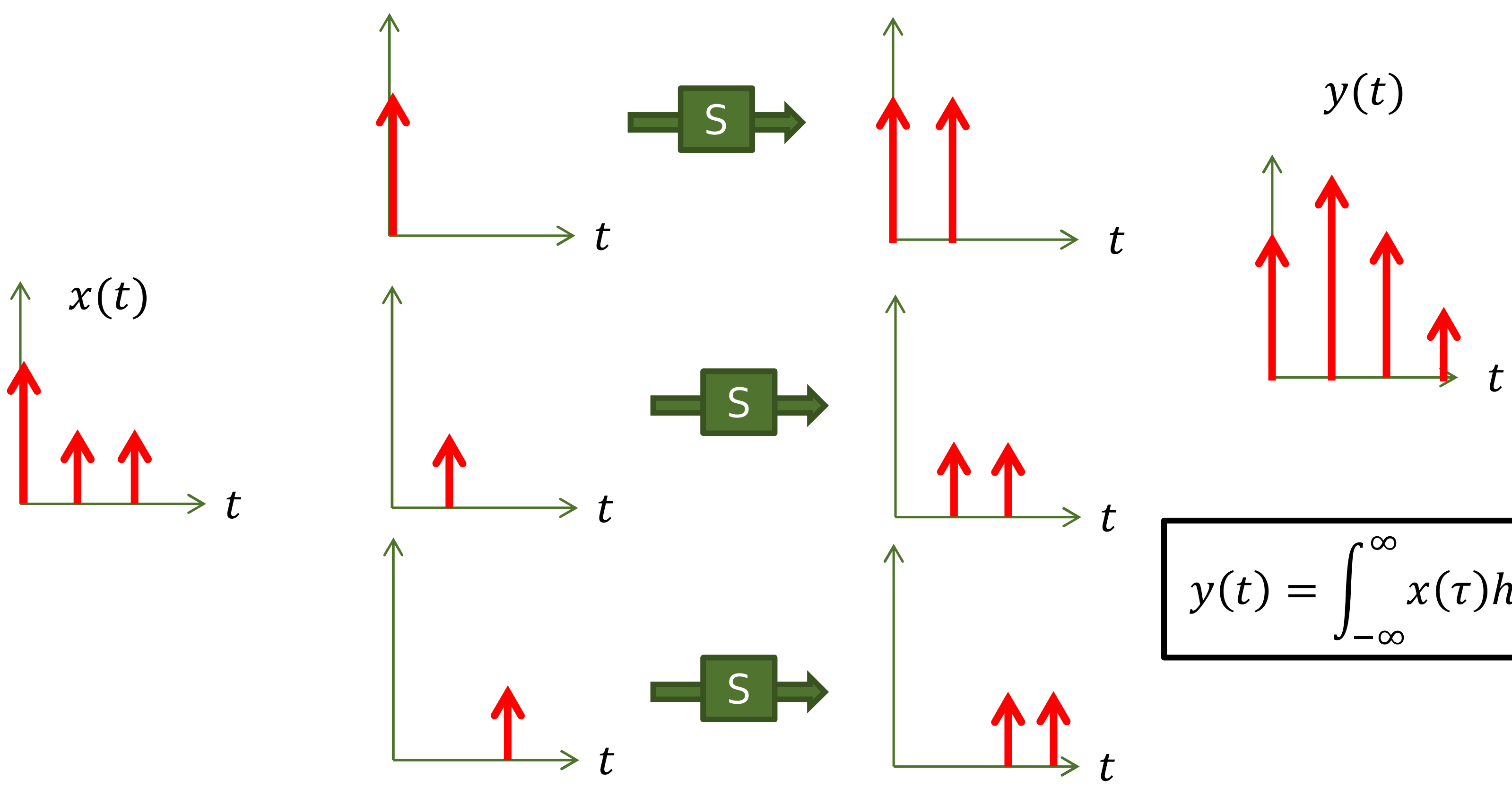


y(0)=x(0)h(0-0)=x(0)h(0)=2

y(1)=x(0)h(1-0)+x(1)h(1-1)=2+1=3

y(2)=x(1)h(2-1)+x(2)h(2-2)
=1+1=3

y(3)=x(2)h(3-2)=1



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

S's frequency response

Frequency Response

- **The system's impulse response is h(t).**
- **Input:**

$$x(t) = \exp(j2\pi f t)$$

- **What is the output?**

$$y(t) = (x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) \exp[j2\pi f (t - \tau)]\, d\tau$$

$$= \exp(j2\pi f t) \boxed{\int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi f \tau) d\tau}$$

Impulse response's Fourier transform

$$= \exp(j2\pi f t)\, H(f)$$

Transfer function

$\exp(j2\pi ft) \rightarrow \boxed{S} \rightarrow \exp(j2\pi ft)\,H(f)$

S's frequency response

Transfer Function

- **Start from x(t)'s inverse Fourier transform:**

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df$$

$$x(t) = \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} X(f)\exp(j2\pi ft)\,\Delta f$$

Sum of lots of exponentials at different frequencies

$$y(t) = \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} X(f)H(f)\exp(j2\pi ft)\Delta f$$

Sum of the frequency responses of lots of exponentials

$$= \int_{-\infty}^{\infty} H(f)X(f)\exp(j2\pi ft)\,df$$

$$y(t) = \int_{-\infty}^{\infty} Y(f)\exp(j2\pi ft)\,df \qquad \Rightarrow Y(f) = H(f)X(f)$$

A System's Description

- **A system (e.g., filter or channel) can be described by either**
- **In time-domain, the impulse response h(t), or**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- **in frequency-domain, the transfer function.**

$$Y(f) = H(f)X(f)$$

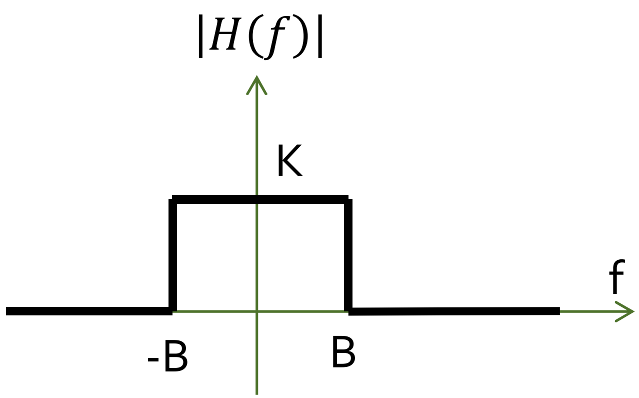
Either can be converted to the other description

$$Y(f) = H(f)X(f)$$

Example: filter

- **Filter:**
  - Frequencies inside the **passband** are transmitted wit little or no distortion.
  - Frequencies inside the **stopband** are rejected.
- **Types of filters: low-pass, high-pass, band-pass, band-stop**
- **Example: ideal low-pass filter**

$$H(f) = \begin{cases} \exp(-j2\pi f t_0), & -B \le f \le B \\ 0 & , |f| > B \end{cases}$$

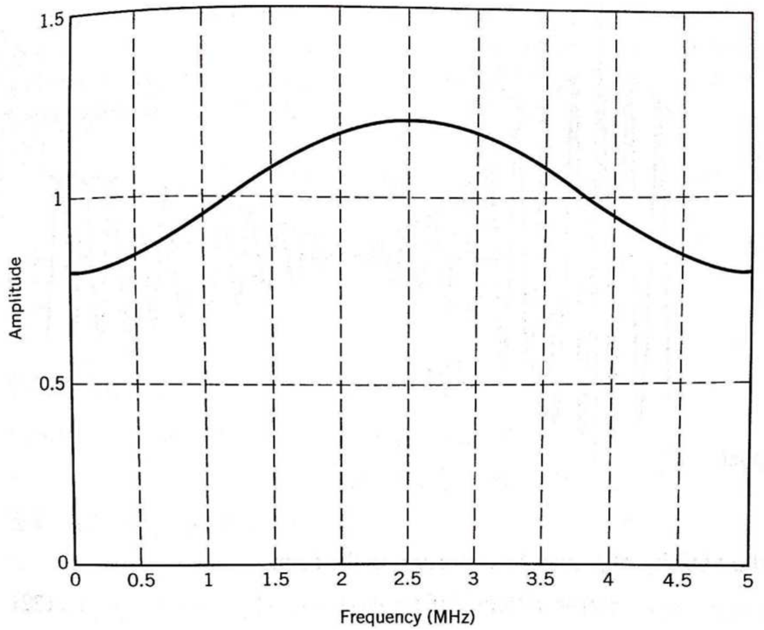


Communication channel as a filter

- **An example of a multipath channel:**

$$h(t) = \underbrace{\delta(t)}_{\text{Line-of-sight component}} + \underbrace{\alpha \exp(j\phi)\delta(t-\tau)}_{\text{reflected component}}$$

- **And its transfer function: (when $\boldsymbol{\alpha = 0.2, \phi = \pi, \tau = 0.2\ \mu s}$)**



$$Y(f) = H(f)X(f)$$

Signal with a bandwidth > 3 MHz will have significant distortion!