Mathematical Foundation I: Fourier Transform, Bandwidth, and Band-pass Signal Representation

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2011/10/13

Fourier Transform

- $g(t)$: a non-periodic deterministic signal.
- Definition: the Fourier transform of the signal $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) exp(-j2\pi f t) dt$$

- $j = \sqrt{-1}$
- t: time (in second)
- f: frequency (in Hz)
- Definition: $exp(i\theta) = cos(\theta) + i\, sin(\theta)$

Fourier Transform

- $\boldsymbol{g(t)}$: **a non-periodic deterministic signal.**
- **Definition: the Fourier transform of the signal** $\boldsymbol{g(t)}$ **is**

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) exp(-j2\pi f t) dt}$$

- **or we can have** $\boldsymbol{\omega = 2\pi f}$
- **and it becomes**

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) exp(-j\omega t) dt}$$

Inverse Fourier Transform

- **The original signal can be recovered exactly using inverse Fourier Transform:**

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f) exp(j2\pi f t) df}$$

- **We call $\boldsymbol{g(t) \Leftrightarrow G(f)}$ a Fourier transform pair.**

- **We can write $\boldsymbol{G(f) = F[g(t)]}$ or $\boldsymbol{g(t) = F^{-1}[G(f)]}$**

The meaning of Fourier transform

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} \overbrace{G(f)exp(j2\pi ft)df}^{\text{Contribution of one frequency to g(t)}}}$$

Summing over
all frequencies

Weighted by G(f)

Sinusoids at
different
frequency f

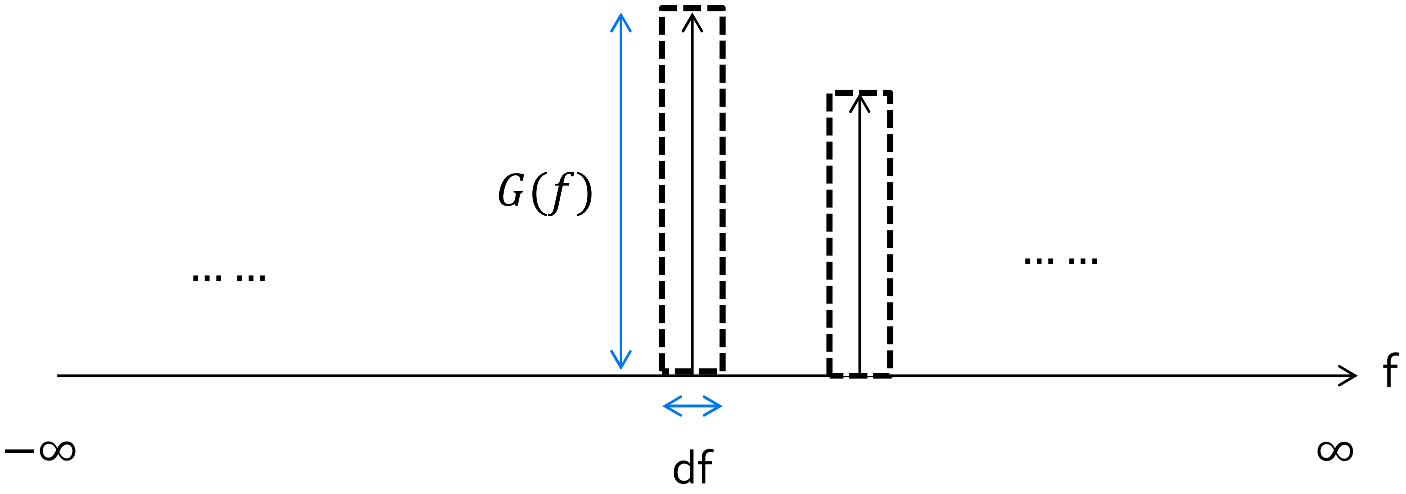
infinitesimally
small width

Decomposition of Rectangular Waves using sinusoids:
http://www.youtube.com/watch?v=y6crWlxKB_E

The meaning of Fourier transform

Fourier transform:

A tool to resolve a given signal g(t) into its complex exponential components occupying the entire frequency interval from $-\infty$ to ∞.



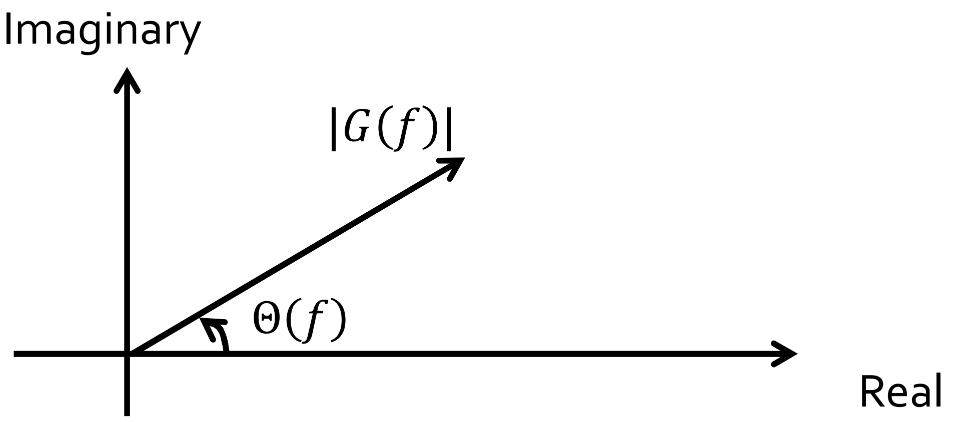
Continuous Spectrum

- **$G(f)$ is a complex function of frequency f, so it can be expressed as:**

$$\boldsymbol{G(f) = |G(f)|exp\left(j\theta(f)\right)}$$

Continuous amplitude spectrum

Continuous phase spectrum



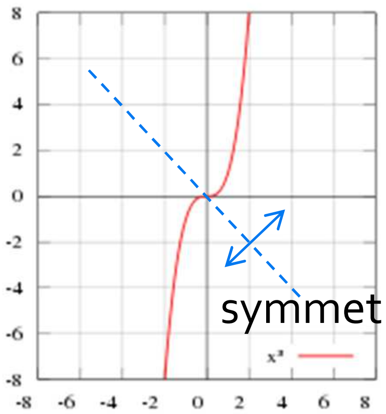
Real-value signal

- **Conjugation:**
  if $\boldsymbol{G(f) = a + bj}$, then $\boldsymbol{G^*(f) = a - bj}$

- **Odd function: f(-x)=-f(x)**
- **Even function: f(x)=f(-x)**

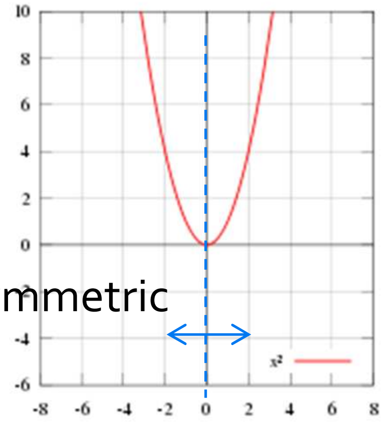
- **For a real g(t):** $G(-f) = G^*(f)$
- **And**
  - $|G(-f)| = |G(f)|$ ←even function
  - $\theta(-f) = -\theta(f)$ ←odd function

odd function



symmetric

even function



symmetric

Properties: Linearity

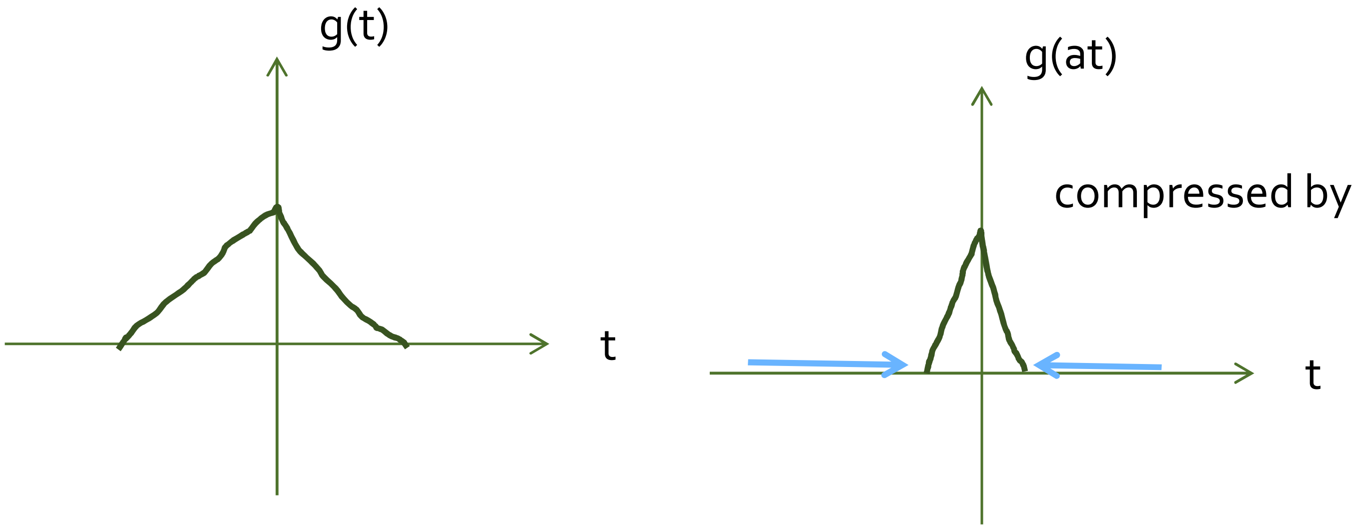
- If $g_1(t) \Leftrightarrow G_1(f)$ and $g_2(t) \Leftrightarrow G_2(f)$

- Then $c_1 g_1(t) + c_2 g_2(t) \Leftrightarrow c_1 G_1(f) + c_2 G_2(f)$

- The spectrum of a weighted-sum signal is the weighted-sum of individual signal’s spectrum.
- Nice!

Properties: Time scaling

- **g(t) → g(at)**



compressed by a factor of a

Properties: Time scaling

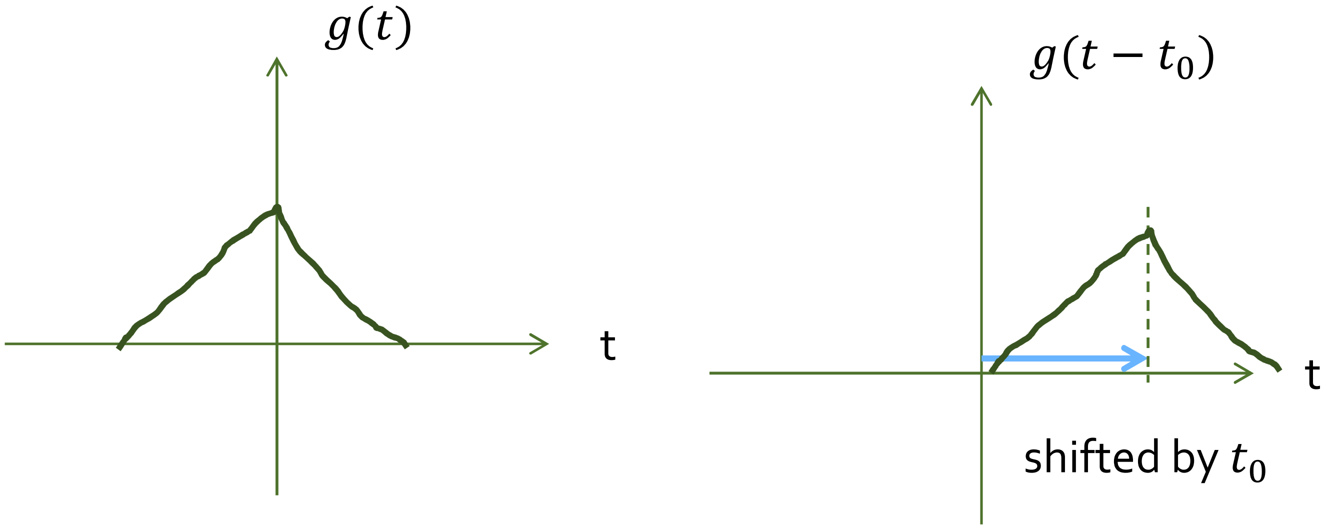
- if $\boldsymbol{F[g(t)] = G(f)}$,
- Then $\boldsymbol{F[g(at)] = \int_{-\infty}^{\infty} g(at) exp(-j2\pi f t) dt}$
- set $\boldsymbol{\tau = at \Rightarrow d\tau = adt}$
- $\boldsymbol{F[g(at)] = \frac{1}{|a|} \int_{-\infty}^{\infty} g(\tau) exp\left(-j2\pi \frac{f}{a} t\right) d\tau}$

$$\boldsymbol{= \frac{1}{|a|} G\left(\frac{f}{a}\right)}$$

Expanded in frequency by a factor of a!
("Faster signal" → occupies larger bandwidth)

Properties: Time shifting

- $\boldsymbol{g(t) \to g(t - t_0)}$



shifted by t_0

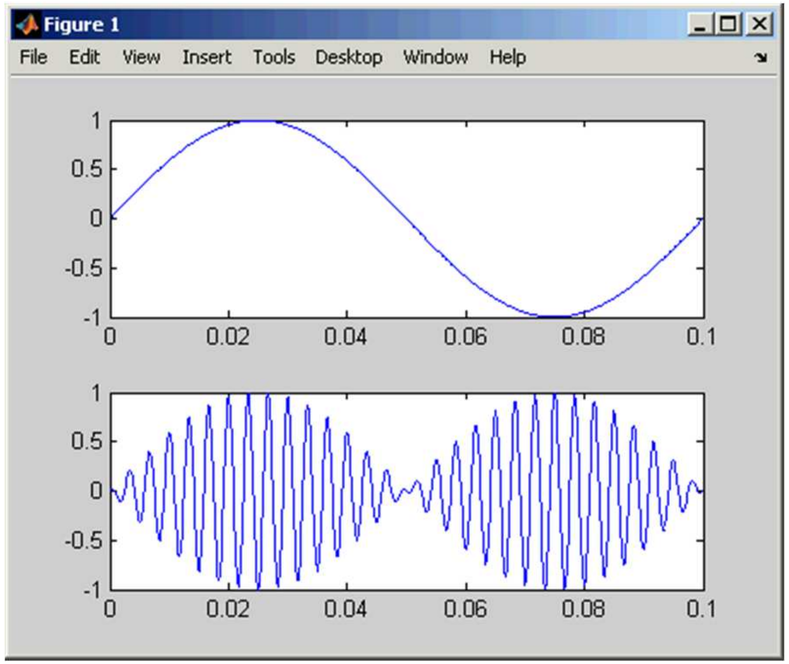
Properties: Time shifting

- **if $F[g(t)] = G(f)$,**
- **then $F[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)exp(-j2\pi ft)dt$**
- **Set $\tau = t - t_0$, then $d\tau = dt$**
- $F[g(t-t_0)] = \int_{-\infty}^{\infty} g(\tau)exp(-j2\pi f\tau)exp(-j2\pi ft_0)d\tau$
$$= exp(-j2\pi ft_0)G(f)$$
- **When the original signal is shifted in time:**
  - The amplitude of the Fourier transform is unchanged.
  - The phase is changed by the linear factor of $-j2\pi ft_0$.

Properties: Frequency shifting

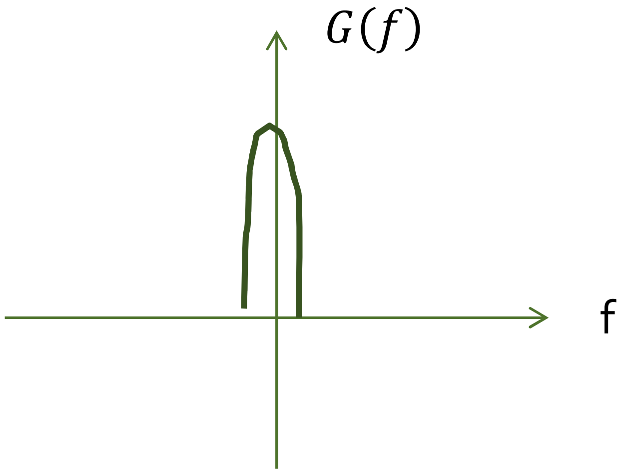
- if $F[g(t)] = G(f)$,
- $F[exp(j2\pi f_c t)g(t)] = \int_{-\infty}^{\infty} g(t)exp(j2\pi f_c t)exp(-j2\pi f t)dt$
$$= \int_{-\infty}^{\infty} g(t)exp(-j2\pi(f - f_c)t)dt$$
$$= G(f - f_c)$$
- This property is also called “modulation theorem”.
- Multiplication of a function by the factor $exp(j2\pi f_c t)$ is equivalent to shifting its Fourier transform $G(f)$ in the positive direction by f_c.

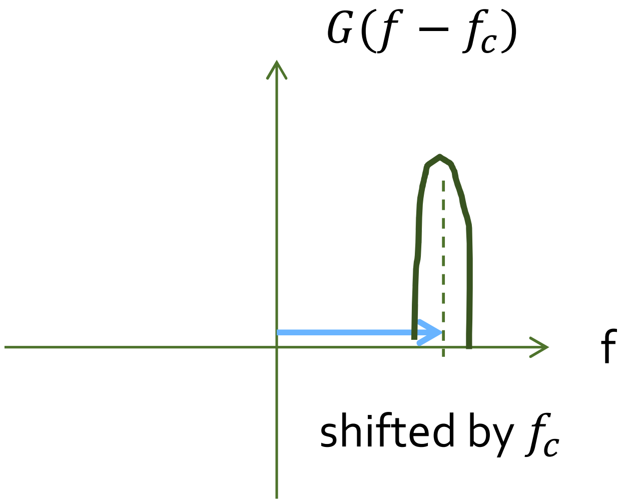
Properties: Frequency shifting



Original Signal: g(t)

After modulated: g(t)exp$(2\pi f_c t)$





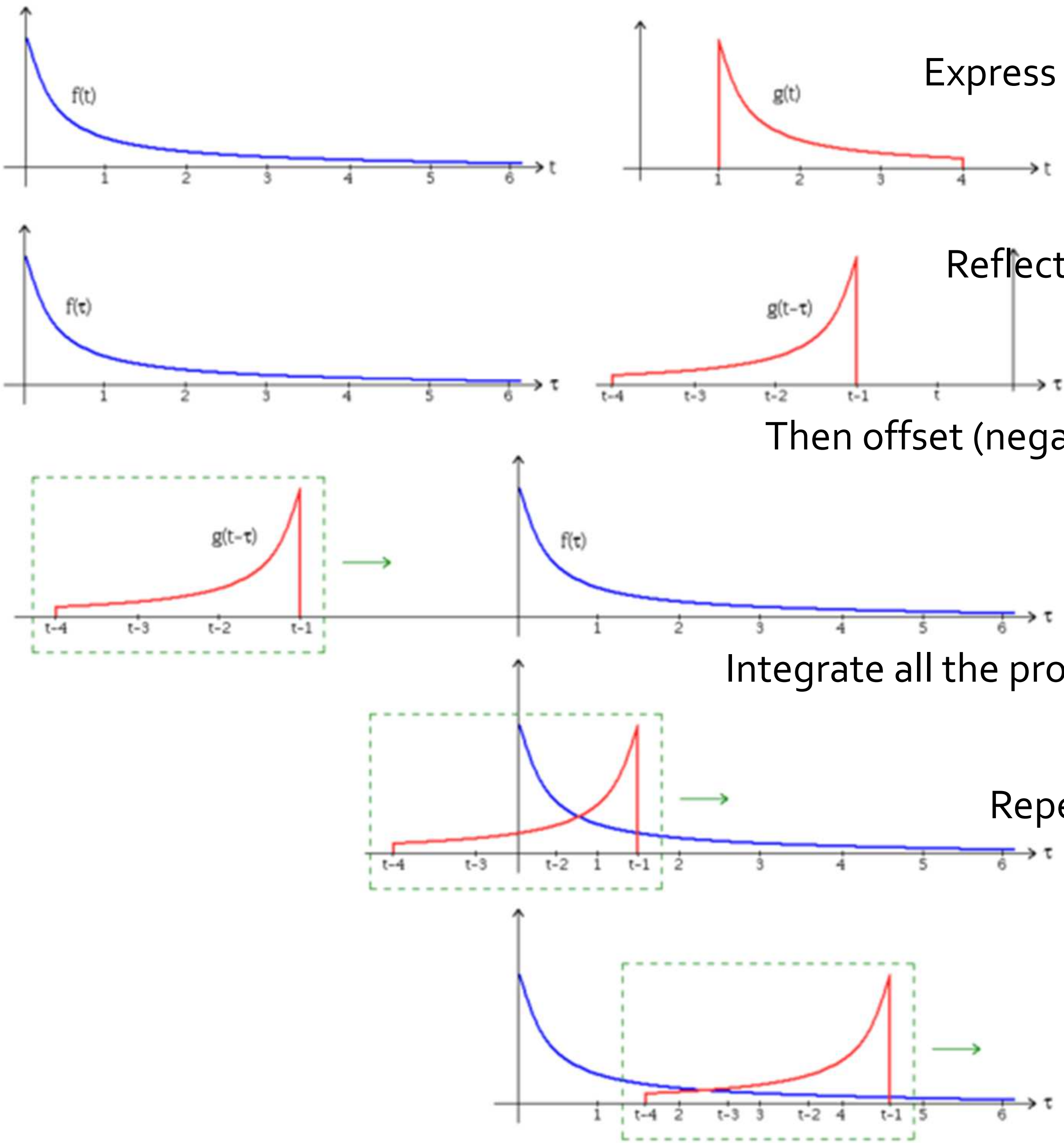
shifted by f_c

Convolution

- **The convolution of $\boldsymbol{f}$ and $\boldsymbol{g}$ is written as $\boldsymbol{f * g}$. It is a particular kind of integral transform:**

$$\boldsymbol{(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau}$$

$$\boldsymbol{= \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau}$$



Express each function in terms of τ

Reflect function g: $g(\tau) \rightarrow g(-\tau)$

Then offset (negatively) by t: $g(-\tau) \rightarrow g(t-\tau)$

Integrate all the product of all intersect parts.

Repeat for a different t.

Properties: Multiplication in time Domain

- if $\boldsymbol{F[g_1(t)] = G_1(f)}$ and $\boldsymbol{F[g_2(t)] = G_2(f)}$,
- Then $\boldsymbol{F[g_1(t)g_2(t)] = G_{12}(f) = (G_1 * G_2)(f)}$
- $\boldsymbol{G_{12}(f) = \int_{-\infty}^{\infty} g_1(t)g_2(t)exp(-j2\pi ft)dt}$
- $\boldsymbol{g_2(t) = \int_{-\infty}^{\infty} G_2(f')exp(j2\pi f't)df'}$
- $\boldsymbol{G_{12}(f) = \int_{-\infty}^{\infty} g_1(t) \int_{-\infty}^{\infty} G_2(f')exp(j2\pi f't)df' \, exp(-j2\pi ft)dt}$
- Set $\boldsymbol{\lambda = f - f', d\lambda = -df'}$
- $\boldsymbol{G_{12}(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(t)G_2(f - \lambda)exp(-j2\pi\lambda t)(-d\lambda)dt}$

$$\boldsymbol{= \int_{-\infty}^{\infty} G_1(f)G_2(f - \lambda)d\lambda}$$

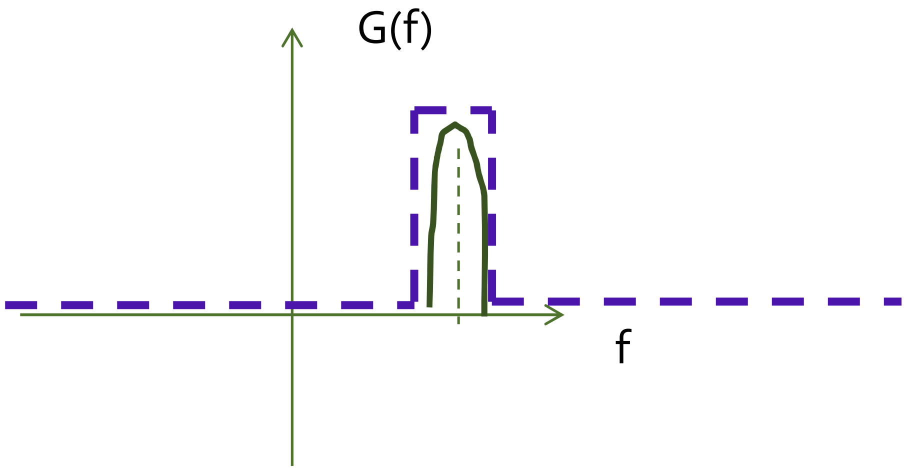
$$\boldsymbol{= (G_1 * G_2)(f)}$$

Properties: Convolution in time Domain

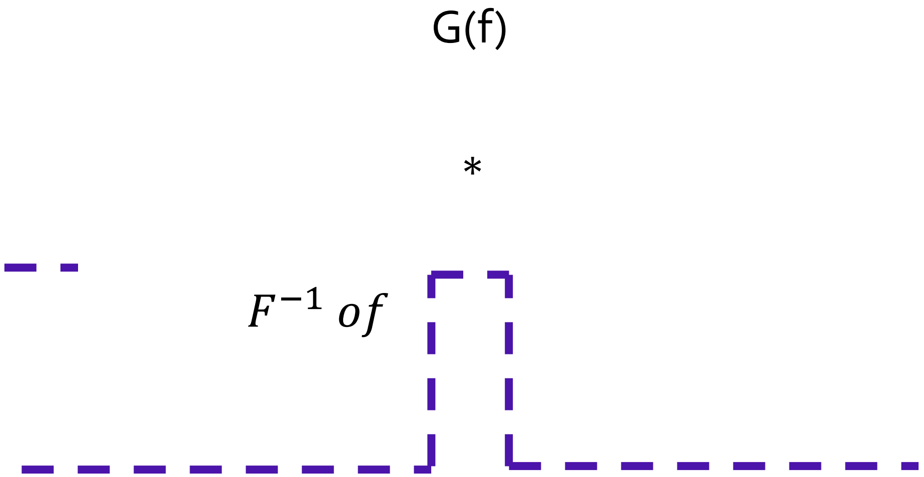
- **if $\boldsymbol{F[g_1(t)] = G_1(f)}$ and $\boldsymbol{F[g_2(t)] = G_2(f)}$,**
- **Then $\boldsymbol{F[(g_1 * g_2)(t)] = G_1(f)G_2(f)}$**

- **Filter:**

In frequency domain

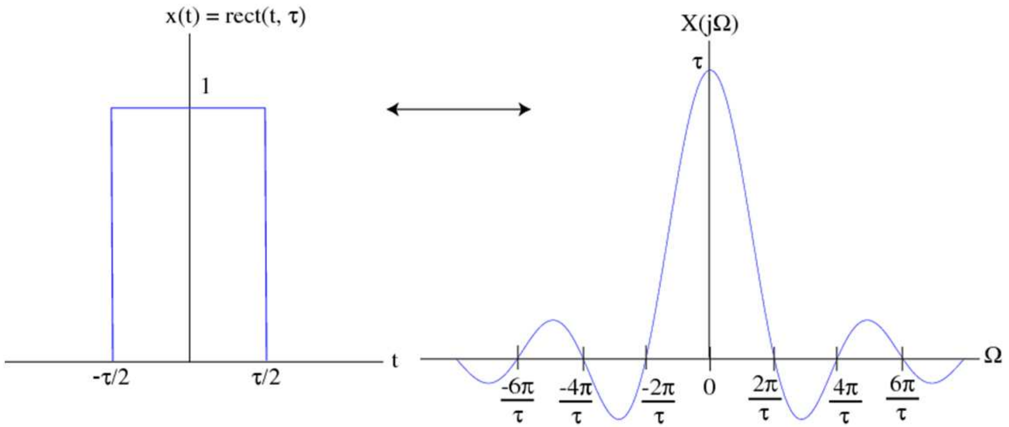


In time domain



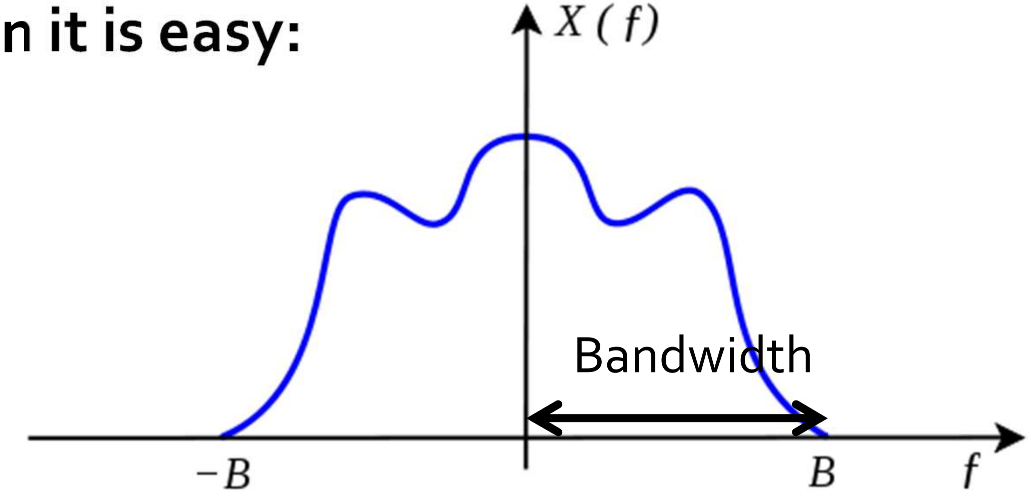
More properties

- **When the time domain signal is changed, the frequency domain signal is inversely changed.**
  - You can only specify one of the two, but not both at the same time.
- **X-limited: only a certain range is non-zero.**
  - Time-limited → The signal trails on indefinitely in the frequency domain
  - Band-limited → The signal trails on indefinitely in the time domain



Bandwidth

- The bandwidth of a signal provides a measure of the extent of significant spectral content of the signal for positive frequencies.
- But how "significant"?
- If band-limited, then it is easy:



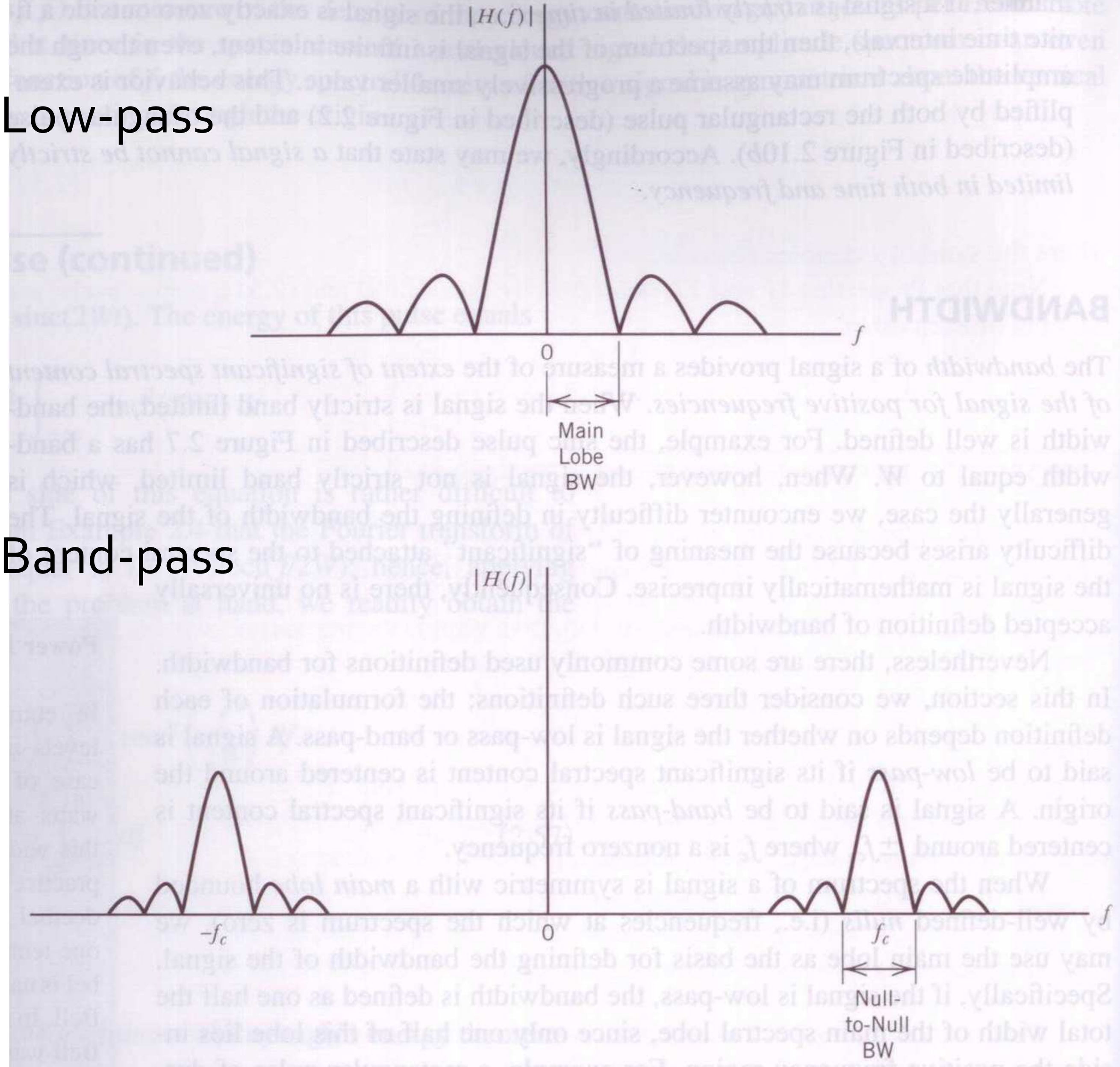
Bandwidth

- **Low-pass signal:**
  the signal's significant spectral content is centered around the origin.
- **Band-pass signal:**
  the signal's significant spectral content is centered around $\pm f_c$, $f_c \neq 0$.
- **Main-lobe:**
  Bounded by nulls (frequencies at which the spectrum = 0)

Main-lobe Bandwidth

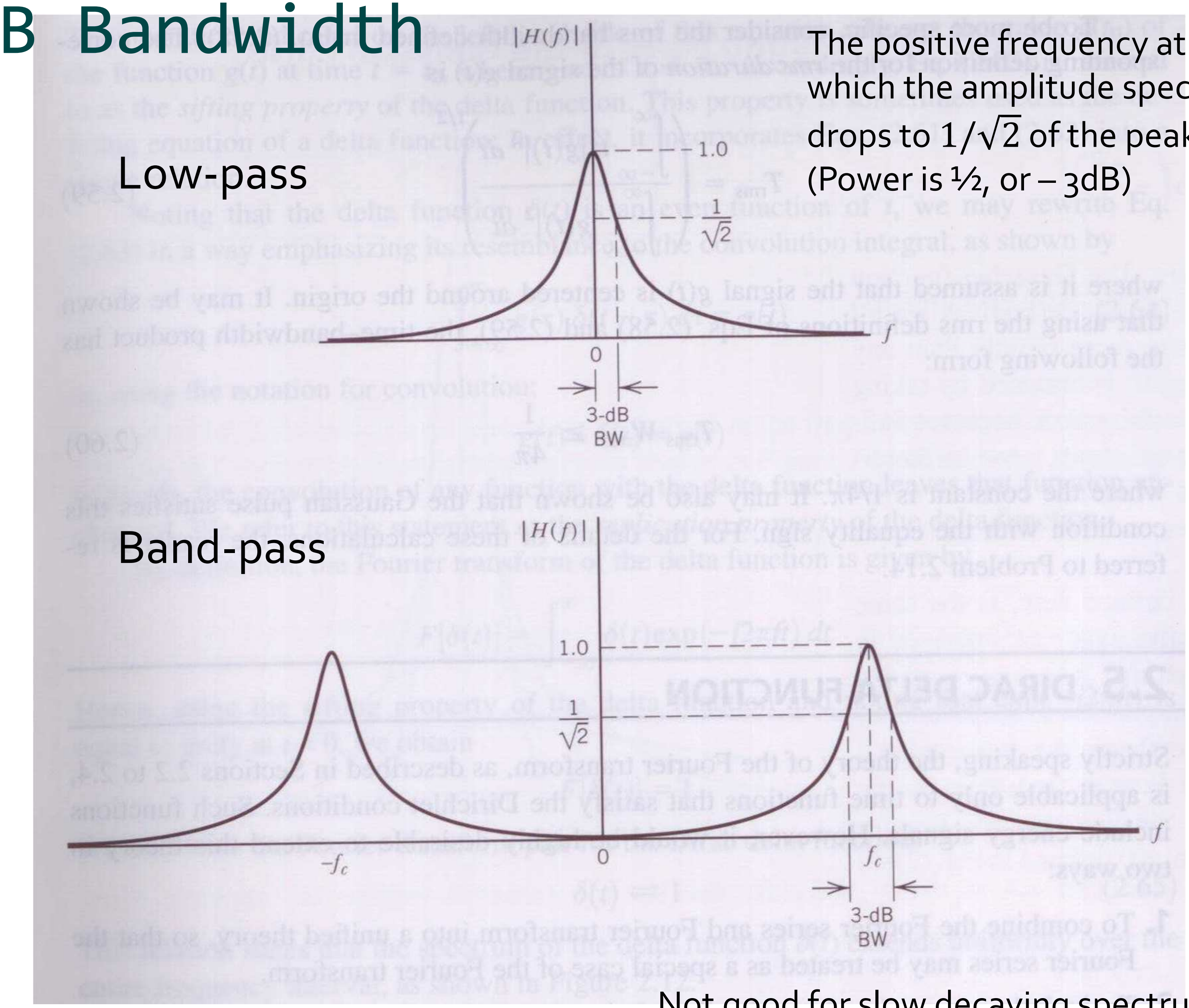
Low-pass

Band-pass



3-dB Bandwidth

The positive frequency at which the amplitude spectrum drops to $1/\sqrt{2}$ of the peak. (Power is ½, or – 3dB)

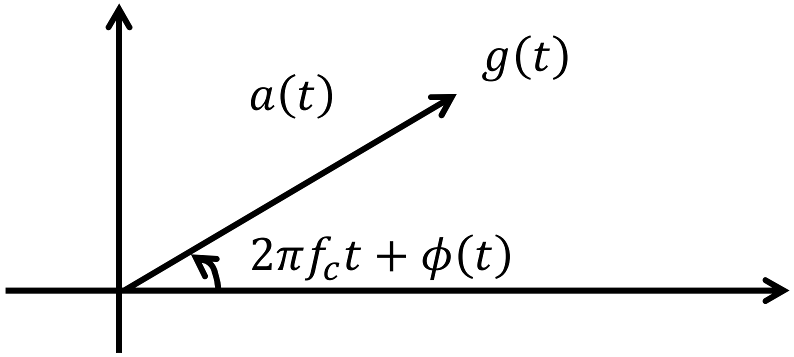


Low-pass

Band-pass

Not good for slow decaying spectrum.

Band-pass Signal Representation



- **General form:**

$$\boldsymbol{g(t) = a(t) cos\left(2\pi f_c t + \phi(t)\right)}$$

Envelope

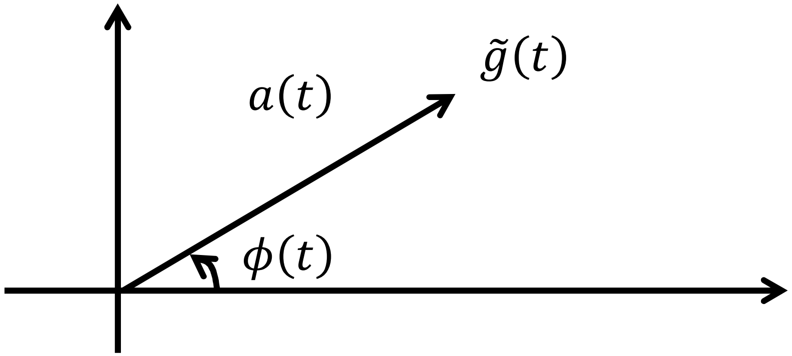
Phase

- **Envelope is always non-negative, or we can switch the phase by 180 degree**
- **This is called the canonical representation of a band-pass signal**

Band-pass Signal Representation

- $g(t) = a(t)cos\left(2\pi f_c t + \phi(t)\right)$ **can be re-arranged into**
- $g(t) = g_I(t)cos(2\pi f_c t) - g_Q(t)sin(2\pi f_c t)$
- $g_I(t) = a(t)cos\left(\phi(t)\right)$ **and** $g_Q(t) = a(t)sin\left(\phi(t)\right)$
- $g_I(t)$ **and** $g_Q(t)$ **are called <u>inphase</u> and <u>quadrature</u> components of the signal g(t), respectively**
- **Then** $a(t) = \sqrt{g_I^2(t) + g_Q^2(t)}$ **and** $\phi(t) = tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$

Band-pass Signal Representation



- We can also represent g(t) as

$$\boldsymbol{g(t) = Re[\tilde{g}(t)exp(j2\pi f_c t)]}$$

- $\boldsymbol{\tilde{g}(t) = g_I(t) + jg_Q(t)}$
- $\boldsymbol{\tilde{g}(t)}$ is called the complex envelope of the band-pass signal.
- This is to remove the annoying $\boldsymbol{exp(j2\pi f_c t)}$ in the analysis.