Digital Band-pass Modulation

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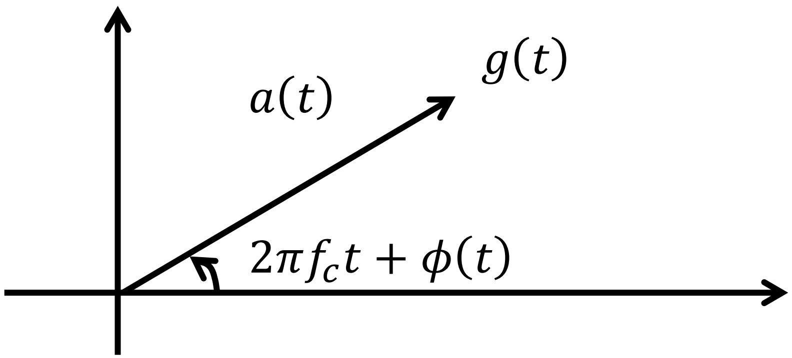
Band-pass Signal Representation



- **General form:**

$$g(t) = a(t)\cos\left(2\pi f_c t + \phi(t)\right)$$

Envelope

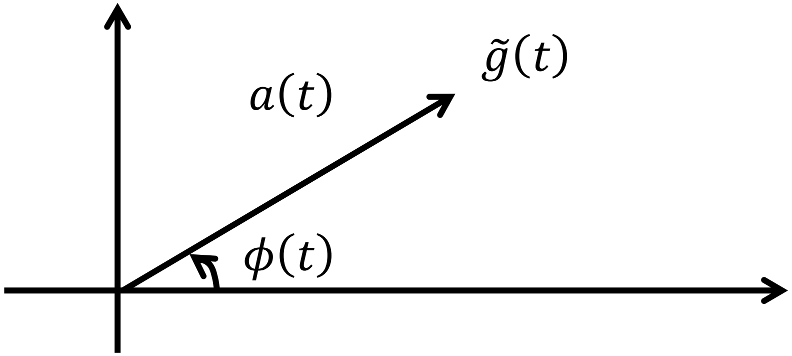
Phase

- **Envelope is always non-negative, or we can switch the phase by 180 degree**
- **This is called the canonical representation of a band-pass signal**

Band-pass Signal Representation

- $g(t) = a(t)cos\big(2\pi f_c t + \phi(t)\big)$ can be re-arranged into
- $g(t) = g_I(t)cos(2\pi f_c t) - g_Q(t)sin(2\pi f_c t)$
- $g_I(t) = a(t)cos\big(\phi(t)\big)$ and $g_Q(t) = a(t)sin\big(\phi(t)\big)$
- $g_I(t)$ and $g_Q(t)$ are called <u>inphase</u> and <u>quadrature</u> components of the signal g(t), respectively
- Then $a(t) = \sqrt{g_I^2(t) + g_Q^2(t)}$ and $\phi(t) = tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$

Band-pass Signal Representation



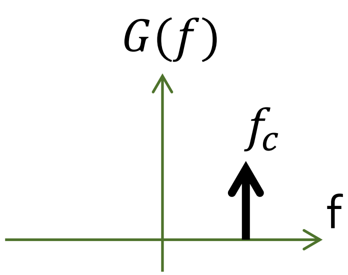
- **We can also represent g(t) as**

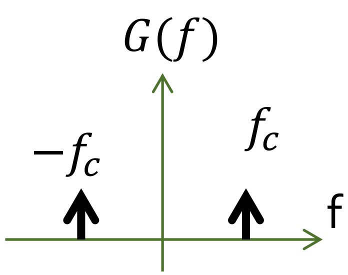
$$\boldsymbol{g(t) = Re[\tilde{g}(t) exp(j2\pi f_c t)]}$$

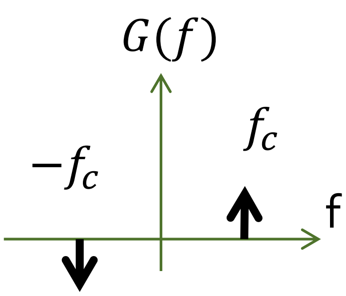
- $\boldsymbol{\tilde{g}(t) = g_I(t) + j g_Q(t)}$
- $\boldsymbol{\tilde{g}(t)}$ **is called the complex envelope of the band-pass signal.**
- **This is to remove the annoying** $\boldsymbol{exp(j2\pi f_c t)}$ **in the analysis.**

Sinusoidal Functions' Fourier Transform

- **Complex exponential function**
  - $F[\exp(j2\pi f_c t)] = \delta(f - f_c)$.
- **Sinusoidal functions:**
  - $\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$
    - $F[\cos(2\pi f_c t)] = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
  - $\sin(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$
    - $F[\sin(2\pi f_c t)] = \frac{1}{2}[\delta(f - f_c) - \delta(f + f_c)]$

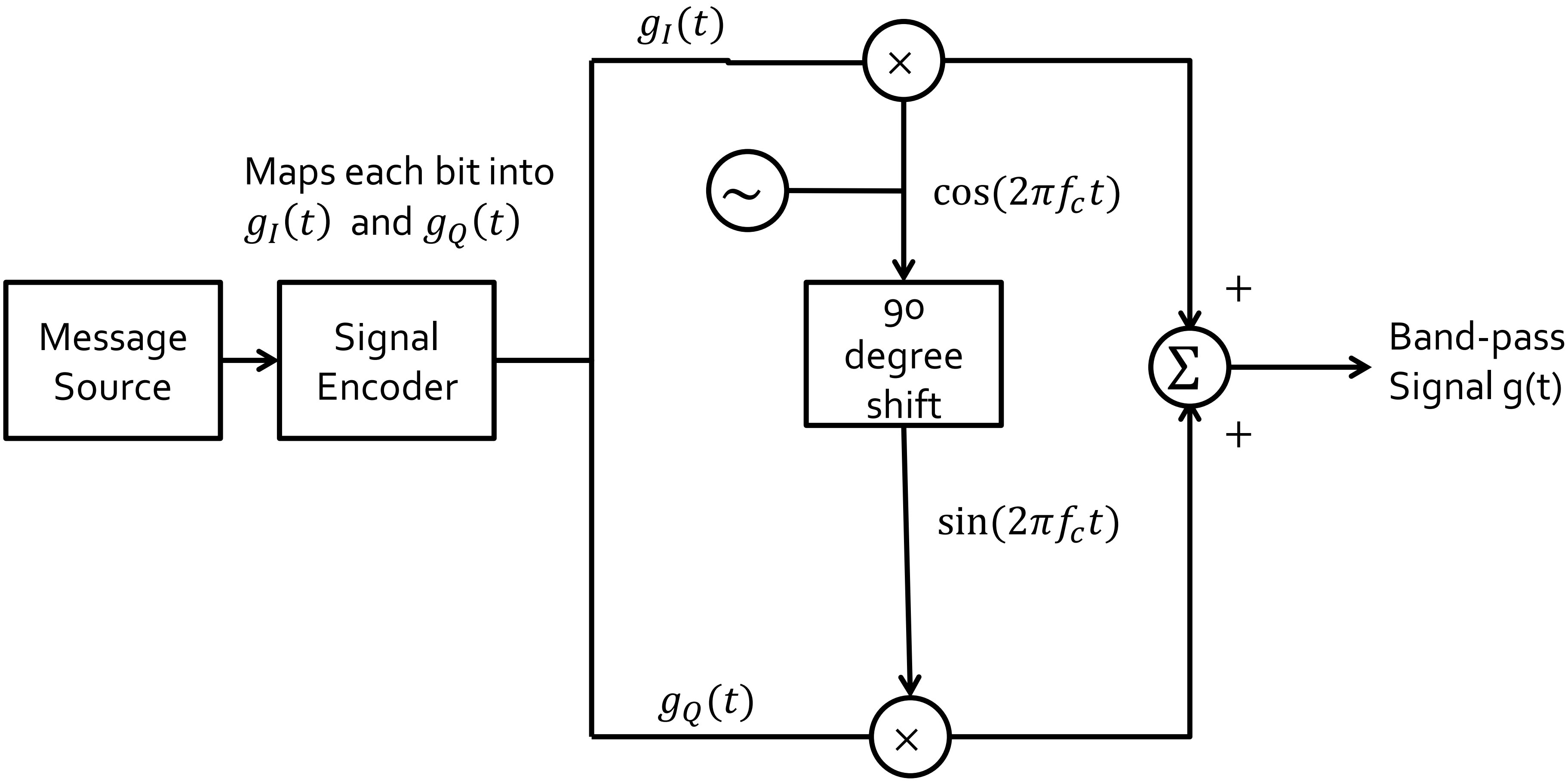






$$g(t) = g_I(t)\cos(2\pi f_c t) + g_Q(t)\sin(2\pi f_c t)$$

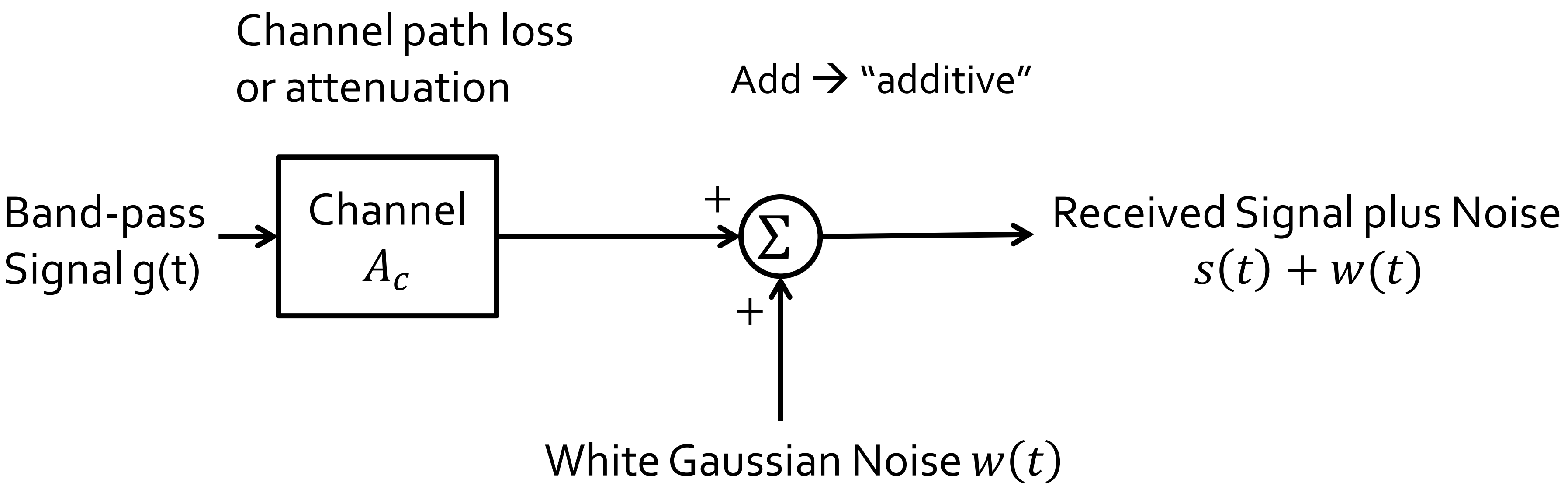
Band-pass Signal Transmitter



Assumption

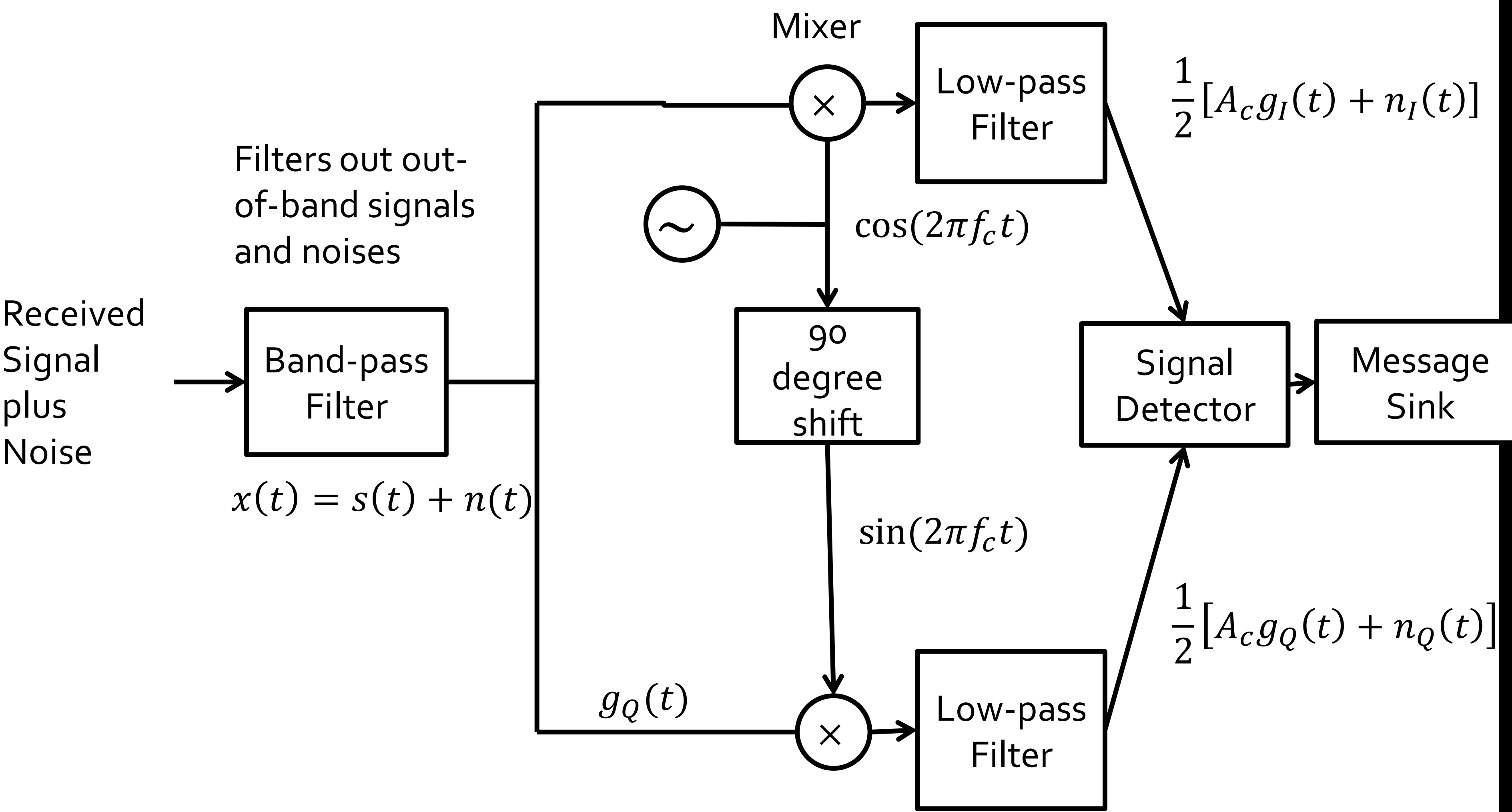
- **The channel is linear: flat-fading channel.**
  - $B_c > B_s$
  - Negligible distortion to $g(t)$
- **The received signal s(t) is perturbed by AWGN**
  - noise w(t) $\sim N\left(0, \frac{N_0}{2}\right)$
  - $\frac{N_0}{2}$ is the PSD of the noise and also its variance (since it's white)

AWGN Channel



$$\boxed{x(t) = s(t) + w(t) = A_c g(t) + w(t)}$$

Band-pass Signal Receiver

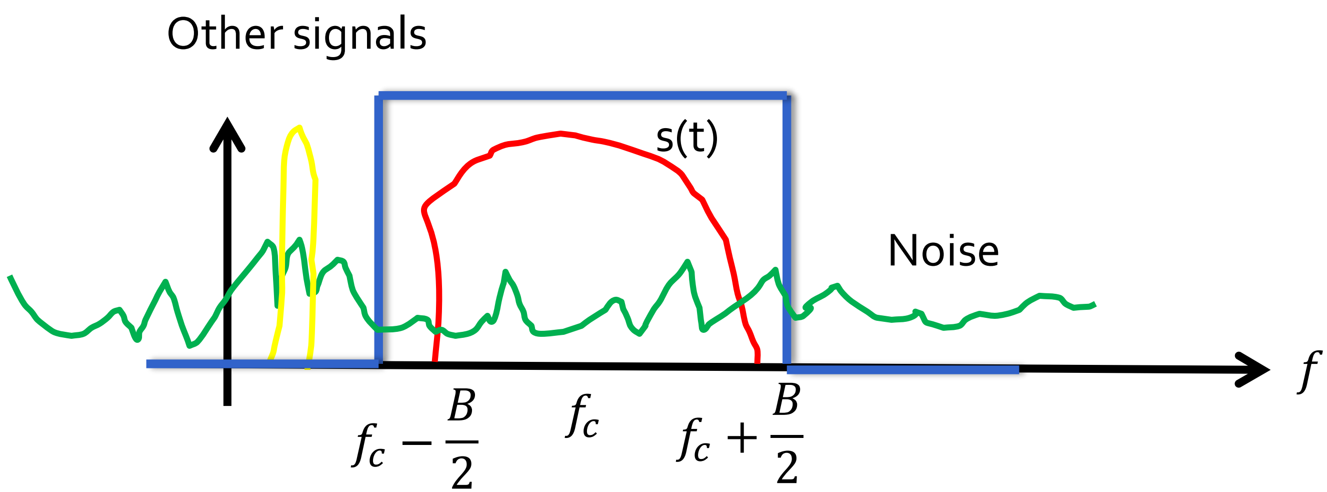


Band-pass Filter

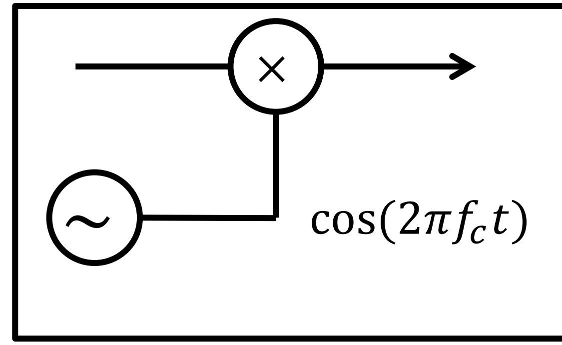
Band-pass Filter

- **The band-pass filter at the frontend filters out out-of-band signals and noises**

1. **Signal s(t) is within the band → not affected**
2. **White noise w(t) becomes narrowband noise n(t)**
    - Much smaller since now we only include noises within the band
    - Still "white over the bandwidth of the signal"
3. **Other signal (out-of-band) is filtered out**



Up-conversion (TX)



In time domain

$$A(t)\exp\big(j\theta(t)\big) \quad \otimes \quad \mathrm{A_c}\cos(2\pi f_c t)$$

In frequency domain

Convolution

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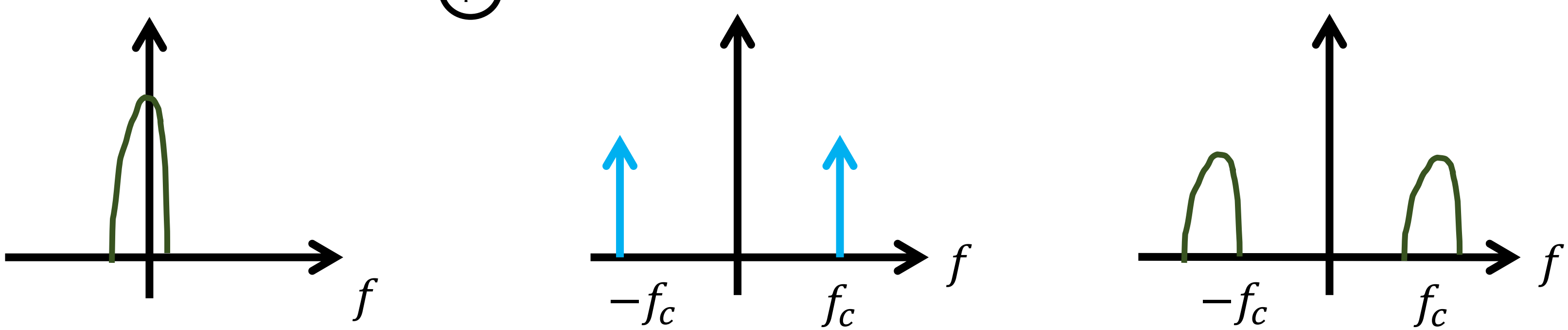
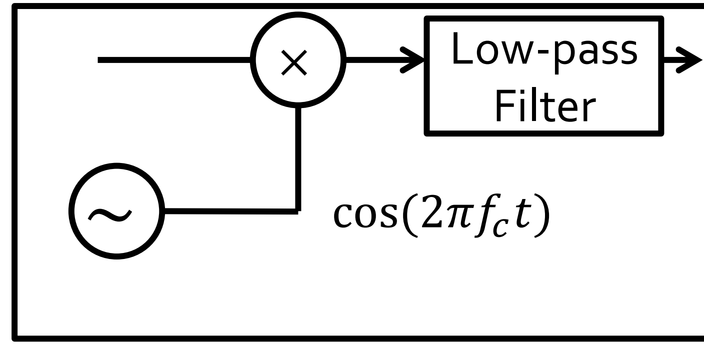


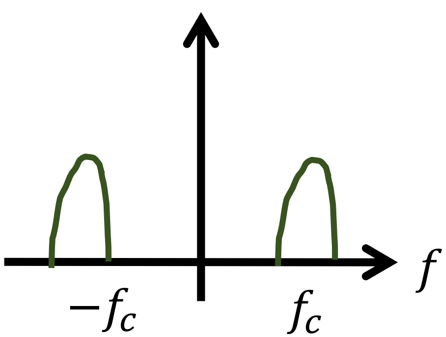
Down-conversion (RX)



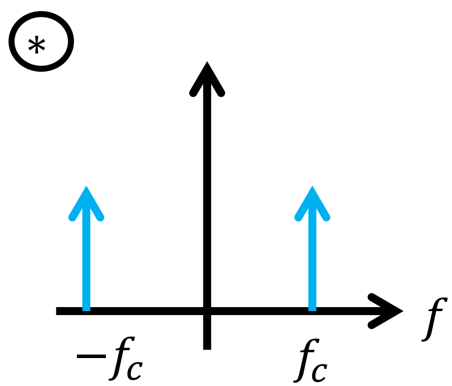
In time domain

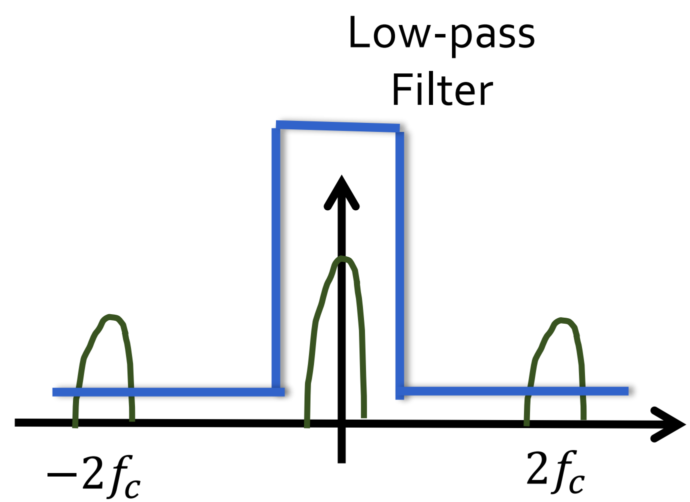
$$s(t)A'_c\cos(2\pi f_c t + \phi) \otimes A_c\cos(2\pi f_c t)$$

In frequency domain



Convolution





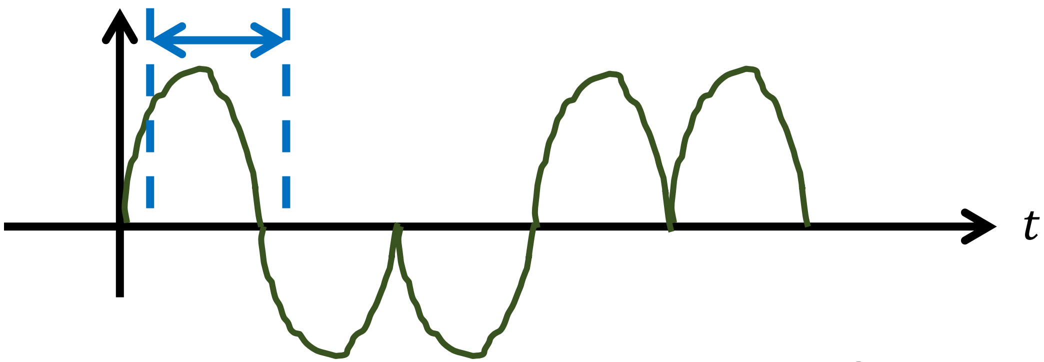
Signal Detector

- **The signal detector:**
  - Observes complex representation of the received signal, $[\boldsymbol{g_I(t)} + \boldsymbol{n_I(t)}] + \boldsymbol{j}[\boldsymbol{g_Q(t)} + \boldsymbol{n_Q(t)}]$,
  - For a duration of T seconds (symbol/bit period)
  - And the make its best estimate of the corresponding transmitted signal $\boldsymbol{g_I(t)} + \boldsymbol{j g_Q(t)}$
  - $\boldsymbol{g_I(t)} + \boldsymbol{j g_Q(t)}$ → bit stream

Time synchronization

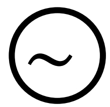
- **To simplify, we assume we have time synchronization between the TX and the RX**

Where does each symbol start and end?



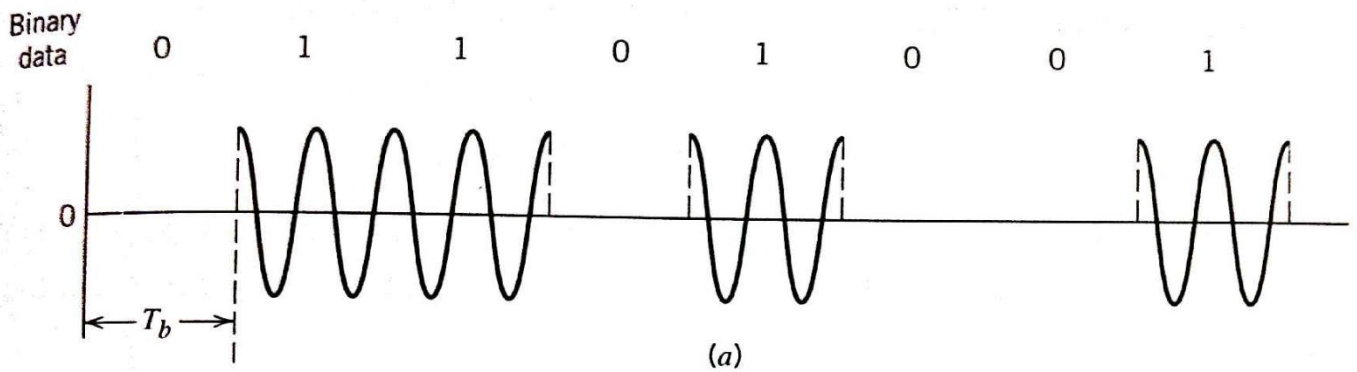
- **Symbol boundary needs to be same for TX and RX**
- **In practice, a timing recovery circuit is required**

Coherent & non-coherent

- **Sometimes, the receiver is phase-locked to the transmitter**
- **That means, the**  **in TX and in RX generate $\boldsymbol{cos(2\pi f_c t)}$ with no phase difference.**
  - RX looks at the received signal to lock onto TX's carrier
- **When that happens, we say**
  - The receiver is a coherent receiver, carrying out coherent detection
- **Otherwise, we say**
  - The receiver is a non-coherent receiver, carrying out non-coherent detection

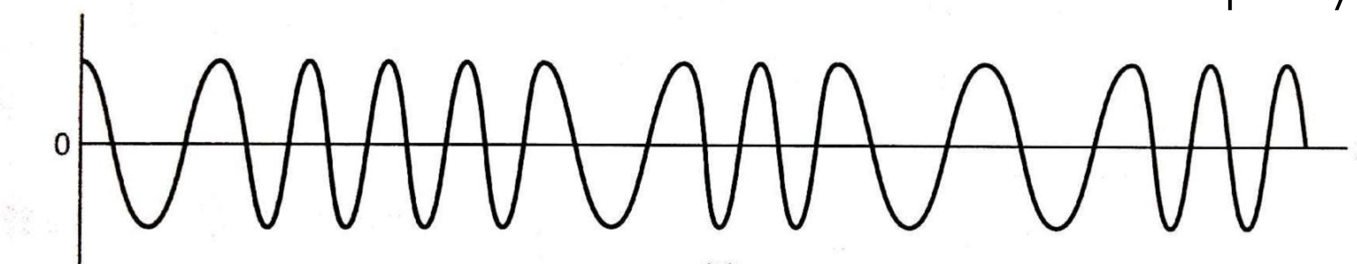
Basic forms of digital modulation

Amplitude Shift Keying



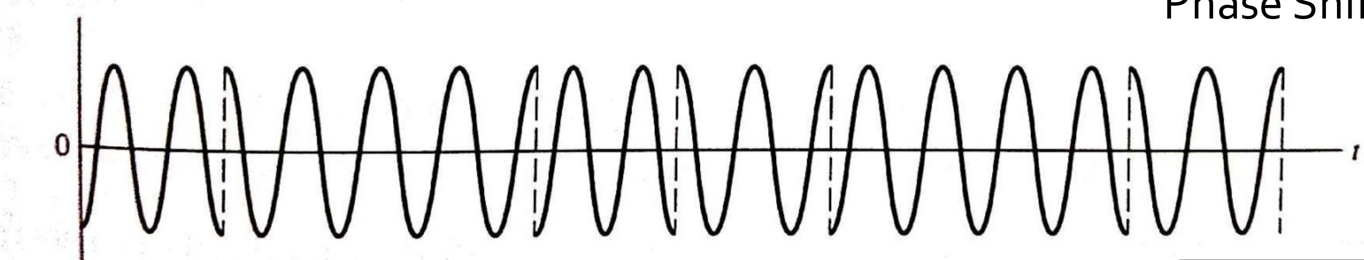
(a)

Frequency Shift Keying



(b)

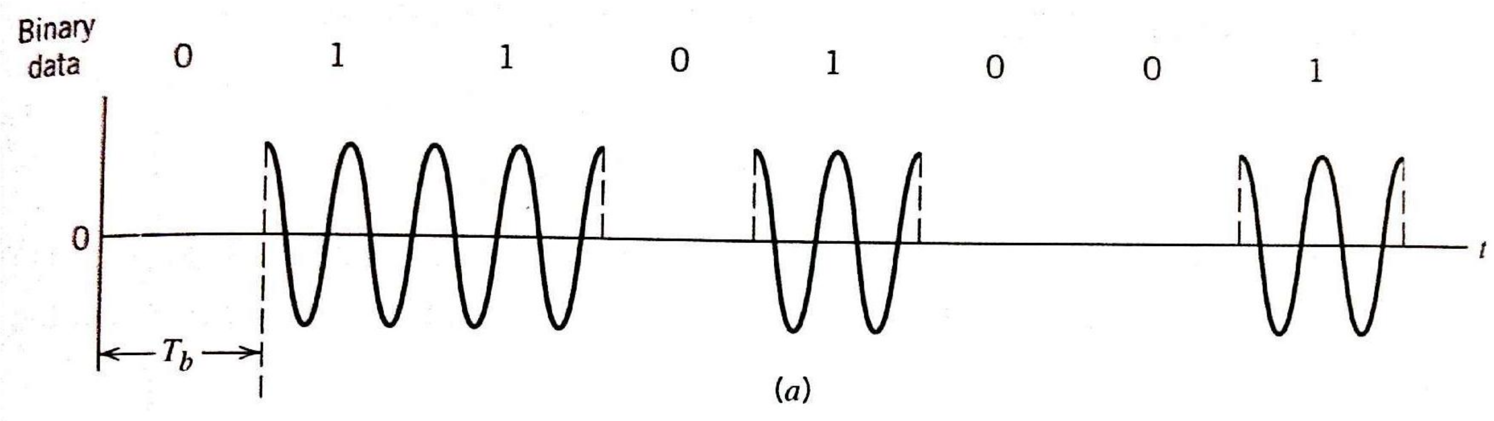
Phase Shift Keying



(c)

Keying == Switching

(Binary) Amplitude Shift Keying (BASK)

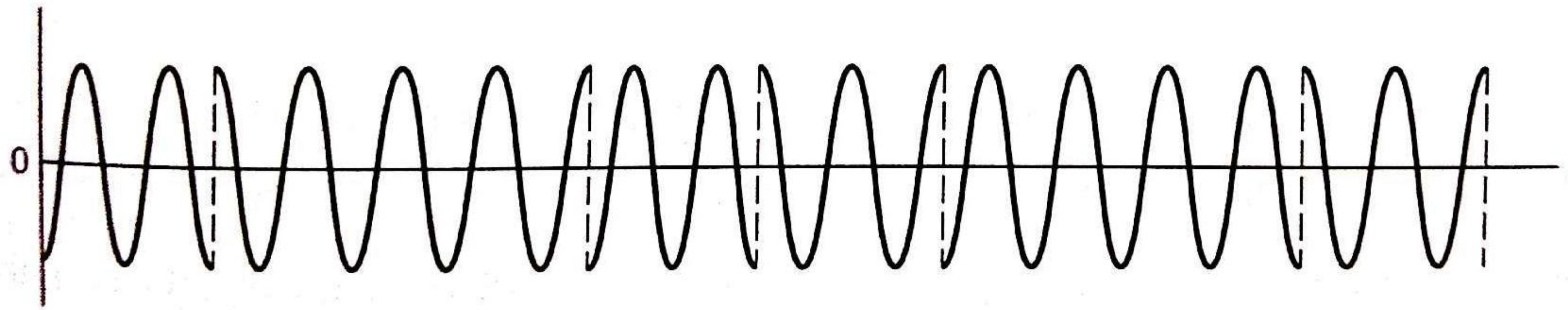


(a)

- **Fixed Amplitude/fixed frequency for a duration of T_b to represent "1"**
- **No transmission to represent "0"**
- **Or, more formally,**
  - $s_1(t) = A_c \cos(2\pi f_c t)$
  - $s_0(t) = 0$

$\left.\begin{array}{l} s_1(t) = A_c \cos(2\pi f_c t) \\ s_0(t) = 0 \end{array}\right\}$ for a duration of T_b

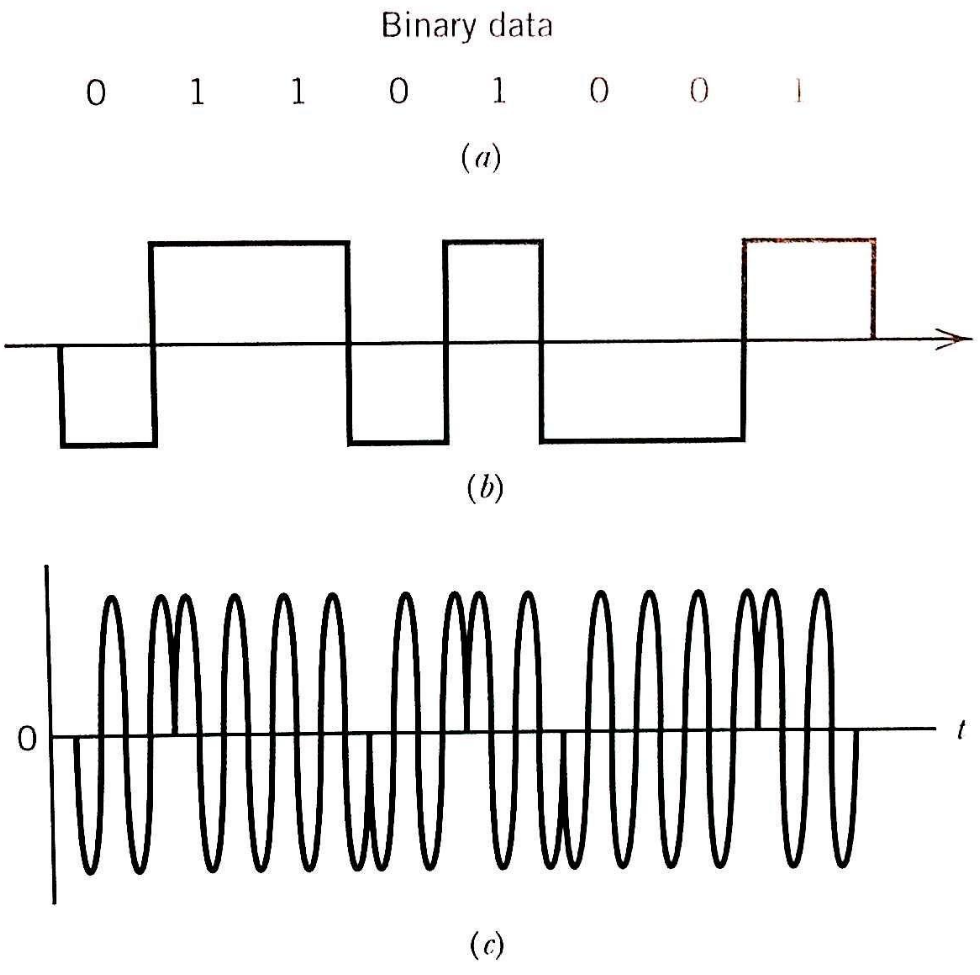
(Binary) Phase Shift Keying (BPSK)



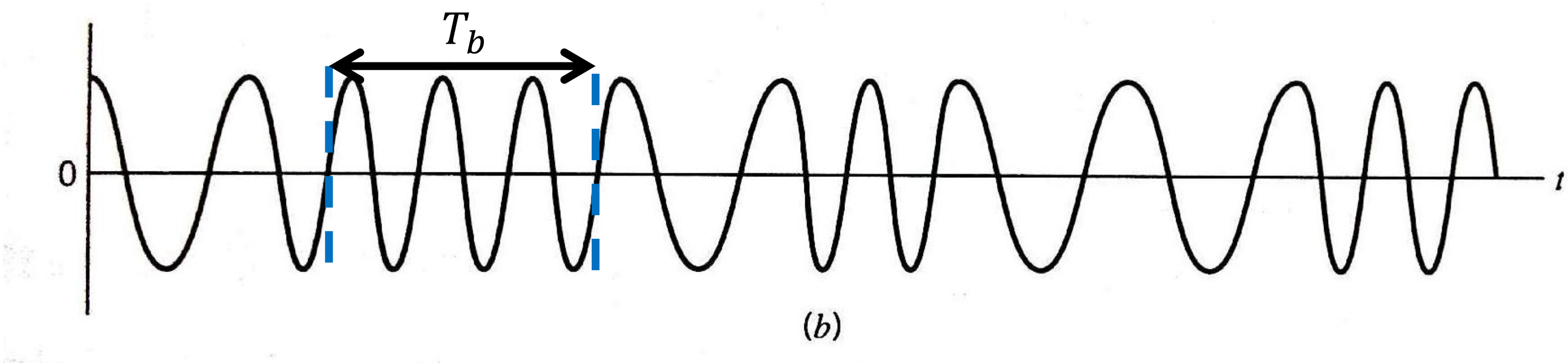
(c)

- **Same amplitude, same frequency**
- **Send the original carrier to represent "1"**
- **Send an inverted carrier (phase difference 180 degrees) to represent "0"**
- **Or, more formally,**
  - $s_1(t) = A_c \cos(2\pi f_c t)$
  - $s_0(t) = A_c \cos(2\pi f_c t + \pi) = -A_c \cos(2\pi f_0 t)$

(Binary) Phase Shift Keying (BPSK)



(Binary) Frequency Shift Keying (BFSK)



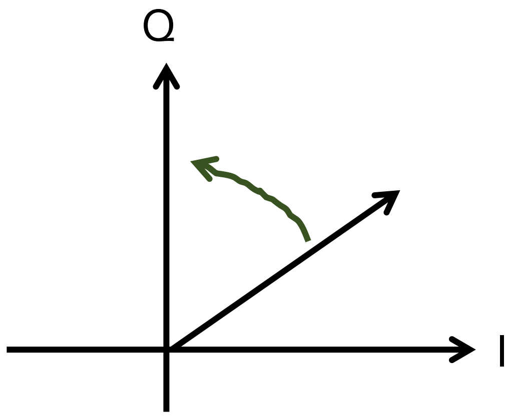
(b)

- **Same amplitude**
- **Send a carrier at f_1 to represent "1"**
- **Send a carrier at f_0 to represent "0"**
- **Or, more formally,**
  - $s_1(t) = A_c \cos(2\pi f_1 t)$
  - $s_0(t) = A_c \cos(2\pi f_0 t)$

for a duration of T_b

(Binary) Frequency Shift Keying (BFSK)

- **Usually we have $\boldsymbol{f_1 = f_c + \Delta f, f_0 = f_c - \Delta f}$**
  - $s_1(t) = A_c \cos[2\pi(f_c + \Delta f)t]$
  - $s_0(t) = A_c \cos[2\pi(f_c - \Delta f)t]$
- **Then,**
  - $s_1(t) = Re\{A_c \exp(j2\pi(f_c + \Delta f)t)\} = g(t)\exp(j2\pi f_c t)$
  - $s_0(t) = Re\{A_c \exp(j2\pi(f_c - \Delta f)t)\} = g(t)\exp(j2\pi f_c t)$
- **So,**
  - For "1", $\tilde{g}(t) = g_I(t) + jg_Q(t) = A_c \exp[-j2\pi\Delta f t]$
  - For "0", $\tilde{g}(t) = g_I(t) + jg_Q(t) = A_c \exp[+j2\pi\Delta f t]$



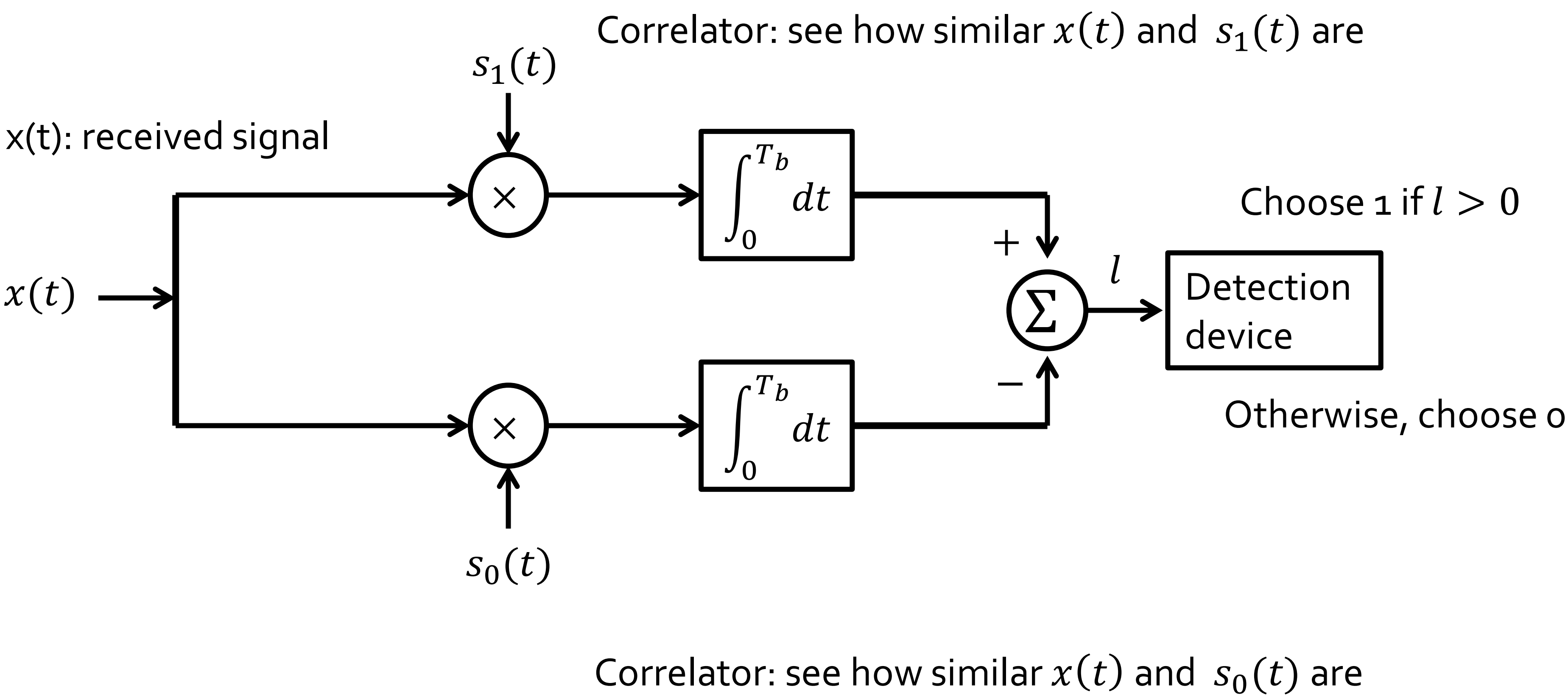
Coherent Detection of FSK and PSK signals

- **Since f_c is large compared to $\frac{1}{T_b}$ (symbol rate, or bit rate), we can say that the same signal energy E_b is transmitted in a bit interval T_b:**

$$E_b = \int_0^{T_b} s_0^2(t)dt = \int_0^{T_b} s_1^2(t)dt$$

$$= \frac{A_c^2 T_b}{2}$$

Two-path correlation receiver (general case)



Coherent Detection

- $\boldsymbol{w(t)}$**: AWGN,** $\boldsymbol{N\left(0, \frac{N_0}{2}\right)}$
- $\boldsymbol{H_0: x(t) = s_0(t) + w(t)}$
- $\boldsymbol{H_1: x(t) = s_1(t) + w(t)}$
- **Receiver output:**

$$l = \int_0^{T_b} x(t)[s_1(t) - s_0(t)]dt$$

- **Decision level: 0**
  - If l is larger than 1, than $x(t)$ is "more similar" to $s_1(t)$
  - If l is smaller than 1, than $x(t)$ is "more similar" to $s_0(t)$

$$\boxed{l = \int_0^{T_b} x(t)[s_1(t) - s_0(t)]dt}$$

Coherent Detection

- $\boldsymbol{H_1}$:

$$l = \int_0^{T_b} s_1(t)[s_1(t) - s_0(t)]dt - \int_0^{T_b} w(t)[s_1(t) - s_0(t)]dt$$

- **Since the noise w(t) is zero-mean,**

L: the random variable whose value is l

$$E[L|H_1] = \int_0^{T_b} s_1(t)[s_1(t) - s_0(t)]dt \quad = E_b(1-\rho)$$

- $\boldsymbol{\rho}$**: the correlation coefficient of the signals** $\boldsymbol{s_0(t)}$ **and** $\boldsymbol{s_1(t)}$

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{\left[\int_0^{T_b} s_0^2(t)dt \int_0^{T_b} s_1^2(t)dt\right]^{\frac{1}{2}}} = \frac{1}{E_b}\int_0^{T_b} s_0(t)s_1(t)dt$$

$$\boxed{0 \le \rho \le 1}$$

Coherent Detection

- **Similarly,**

$$E[L|H_0] = -E_b(1-\rho)$$

- **L's variance is the same for $\boldsymbol{H_1}$ and $\boldsymbol{H_0}$. Since $\boldsymbol{s_1(t)}$ and $\boldsymbol{s_0(t)}$ is deterministic given the transmitted bit, we have**

$$Var[L] = \mathrm{E}[\{\mathrm{L} - \mathrm{E}[\mathrm{L}]\}^2]$$

$$= \mathrm{E}\left[\int_0^{T_b}\int_0^{T_b} w(t)w(u)[s_1(t)-s_0(t)][s_1(u)-s_0(u)]dt\,du\right]$$

$$= \int_0^{T_b}\int_0^{T_b} \boldsymbol{E}[\boldsymbol{w}(\boldsymbol{t})\boldsymbol{w}(\boldsymbol{u})][s_1(t)-s_0(t)][s_1(u)-s_0(u)]dt\,du$$

$$= \int_0^{T_b}\int_0^{T_b} \delta(t-u)[s_1(t)-s_0(t)][s_1(u)-s_0(u)]dt\,du$$

$$= \int_0^{T_b} \int_0^{T_b} \delta(t-u)[s_1(t) - s_0(t)][s_1(u) - s_0(u)]dt\,du$$

$$= \int_0^{T_b} \frac{N_0}{2}[s_1(t) - s_0(t)]^2 dt = N_0 E_b(1-\rho)$$

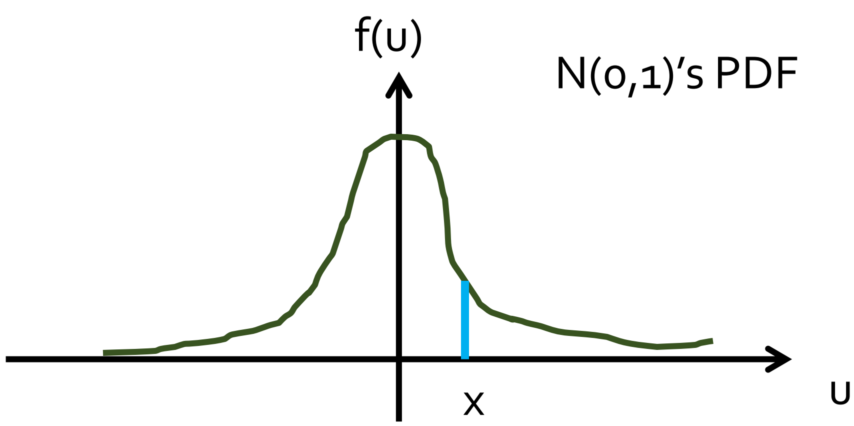
- **Therefore, we know that L conditioned on $\boldsymbol{H_0}$ is a Gaussian distributed random variable: $\boldsymbol{N(E_b(1-\rho), N_0 E_b(1-\rho))}$**

Q Function

- **Q function is defined over the CDF of Gaussian distribution $\boldsymbol{N(0,1)}$**

CDF of $\boldsymbol{N(0,1)}$

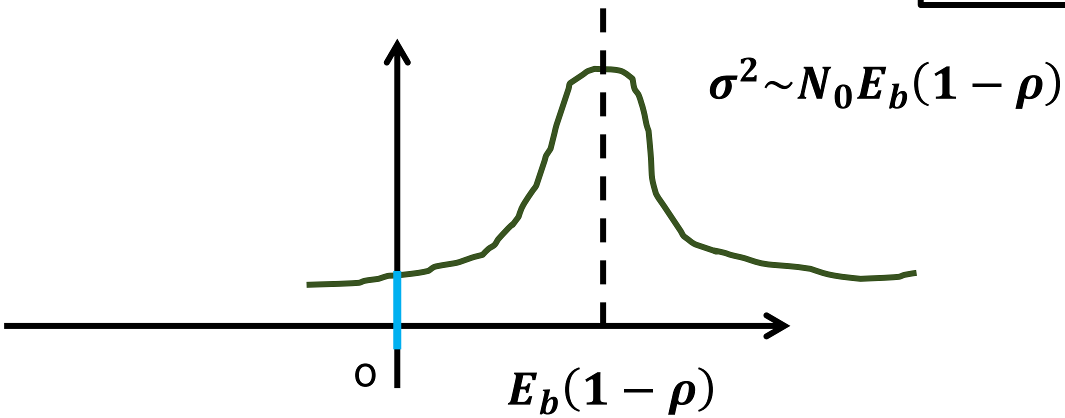
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du = 1 - \Phi(x)$$



Integration (Area under the curve) →

Bit Error Rate

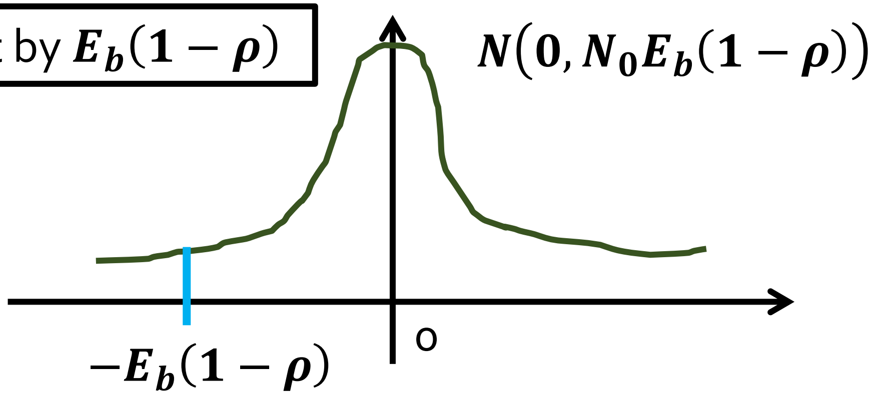
$\boxed{L|H_1 \sim N(E_b(1-\rho), N_0E_b(1-\rho))}$



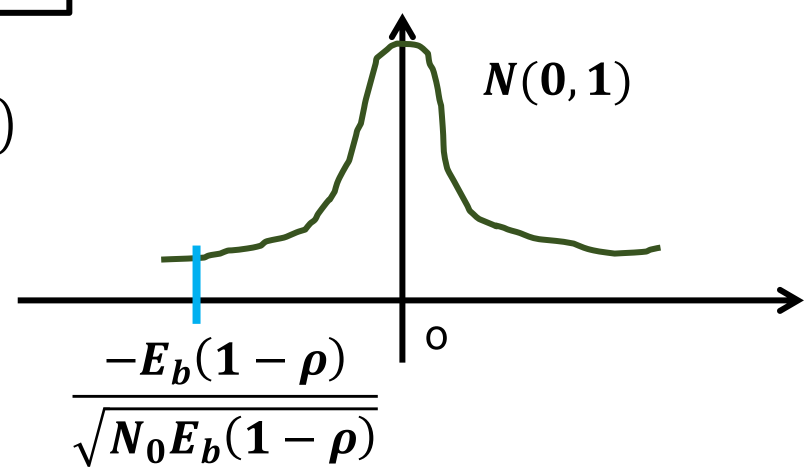
←Integration (Area under the curve)

$\boxed{\text{How to express this area with Q function?}}$

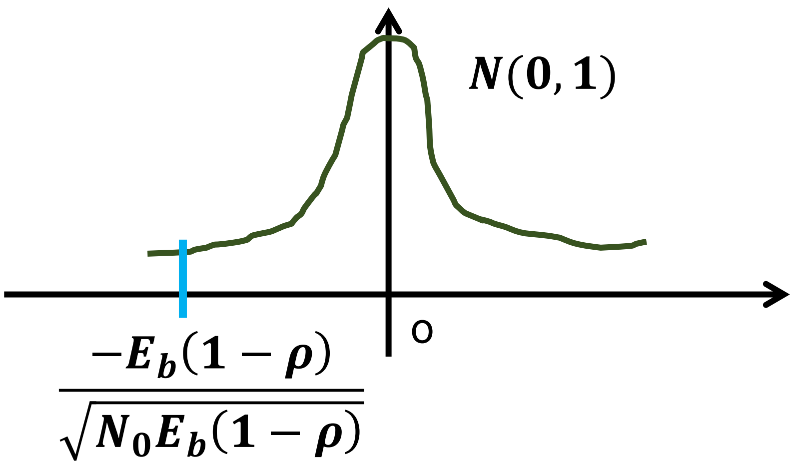
$\boxed{\text{Shift left by } E_b(1-\rho)}$

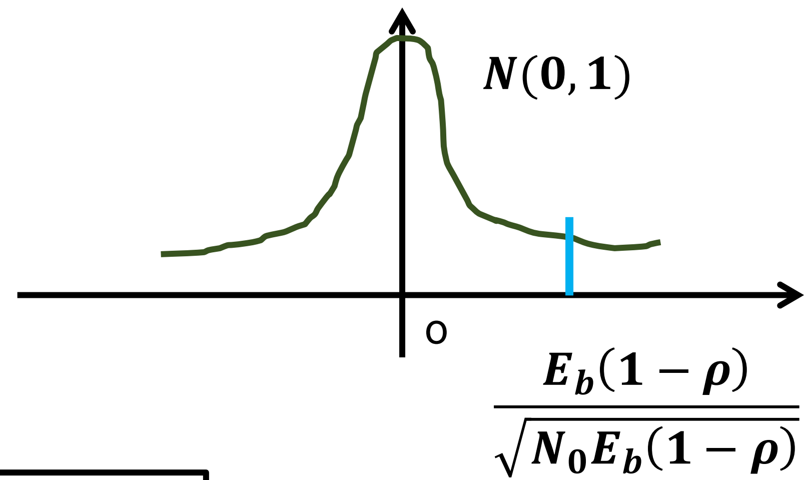


$\boxed{\text{Divide u by } \sqrt{N_0E_b(1-\rho)}}$



Bit Error Rate





$$\boxed{P_e = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)}$$

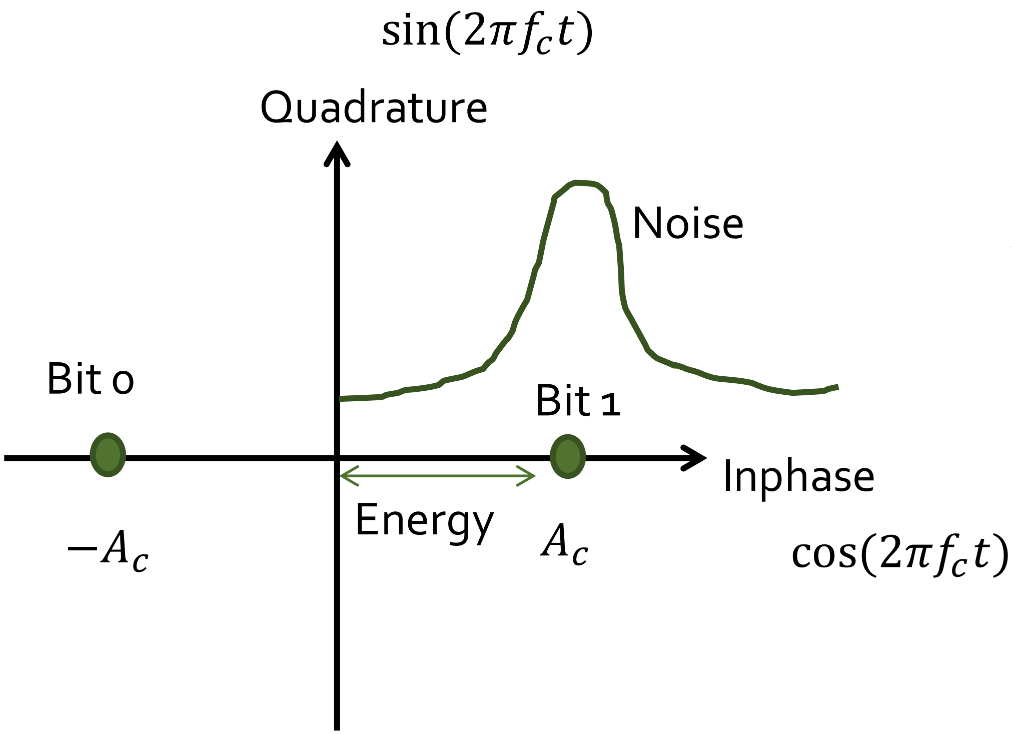
For BPSK, $\rho = -1$

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

For BFSK, $\rho = 0$

$$\boxed{P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

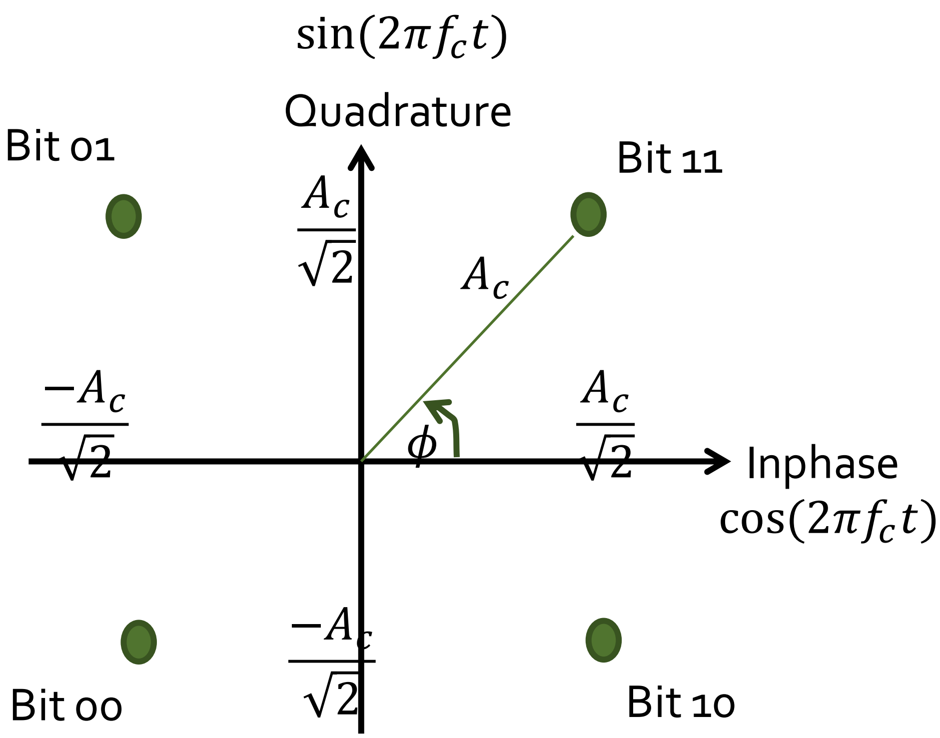
Signal Space - BPSK



$$s_1(t) = A_c \cos(2\pi f_c t)$$

$$s_0(t) = -A_c \cos(2\pi f_c t)$$

Signal Space - QPSK



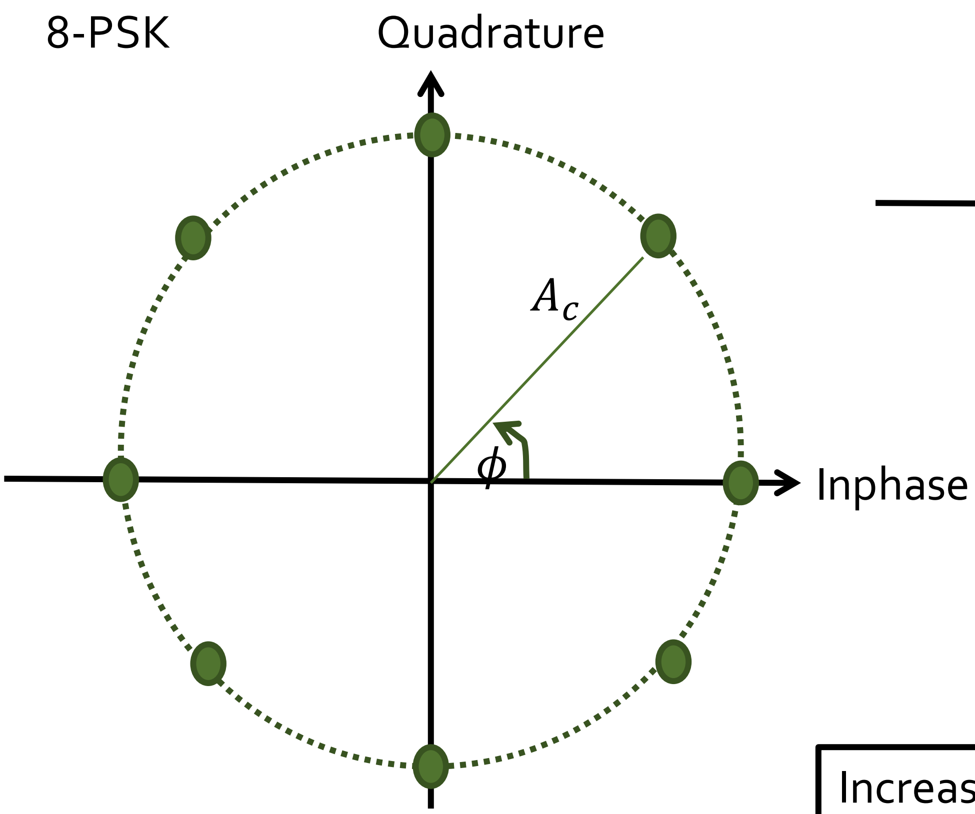
$$s_{11}(t) = A_c \cos\left(2\pi f_c t + \frac{\pi}{4}\right)$$

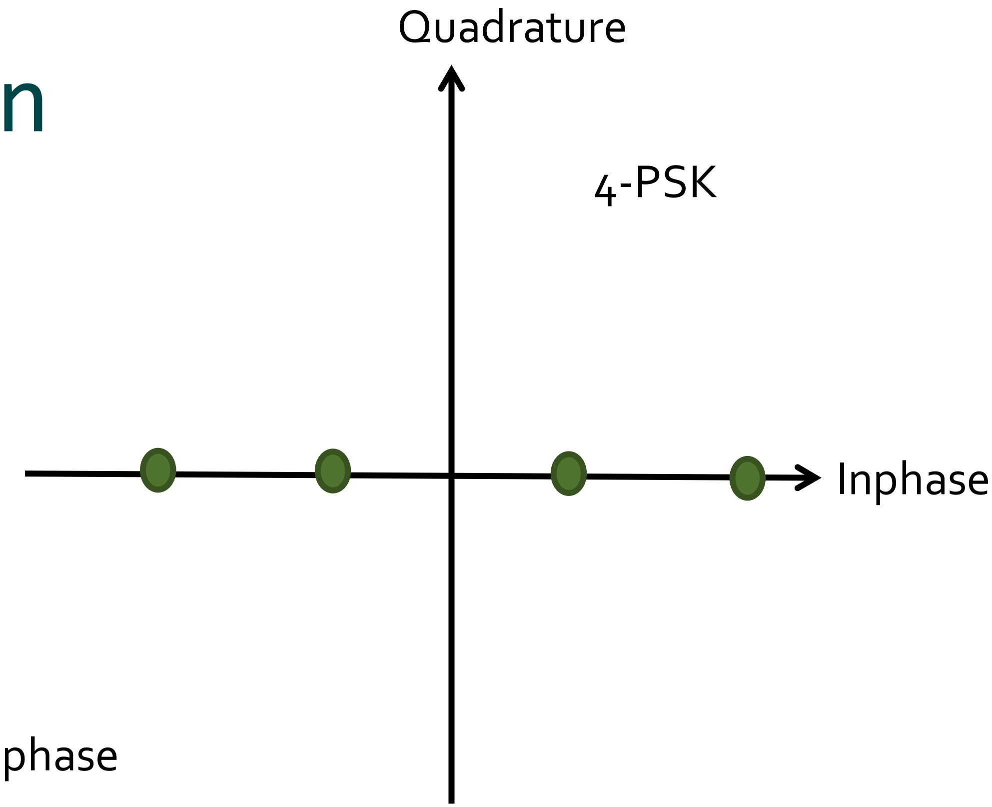
$$s_{01}(t) = A_c \cos\left(2\pi f_c t + \frac{3\pi}{4}\right)$$

$$s_{00}(t) = A_c \cos\left(2\pi f_c t + \frac{5\pi}{4}\right)$$

$$s_{10}(t) = A_c \cos\left(2\pi f_c t + \frac{7\pi}{4}\right)$$

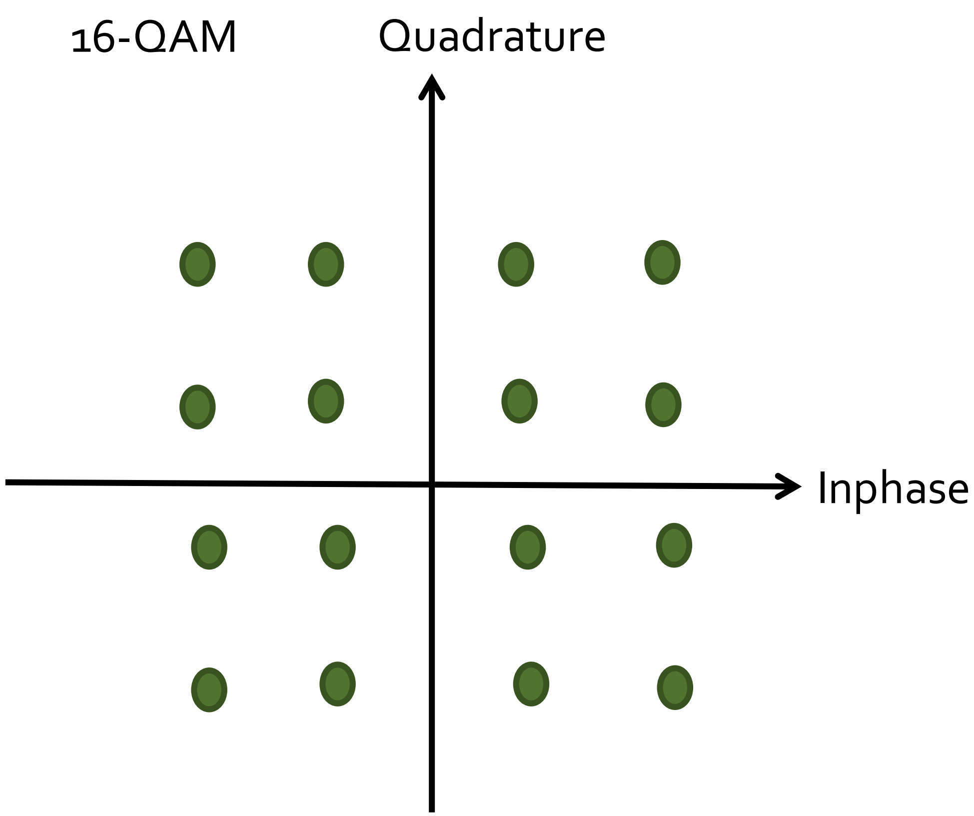
M-ary Modulation



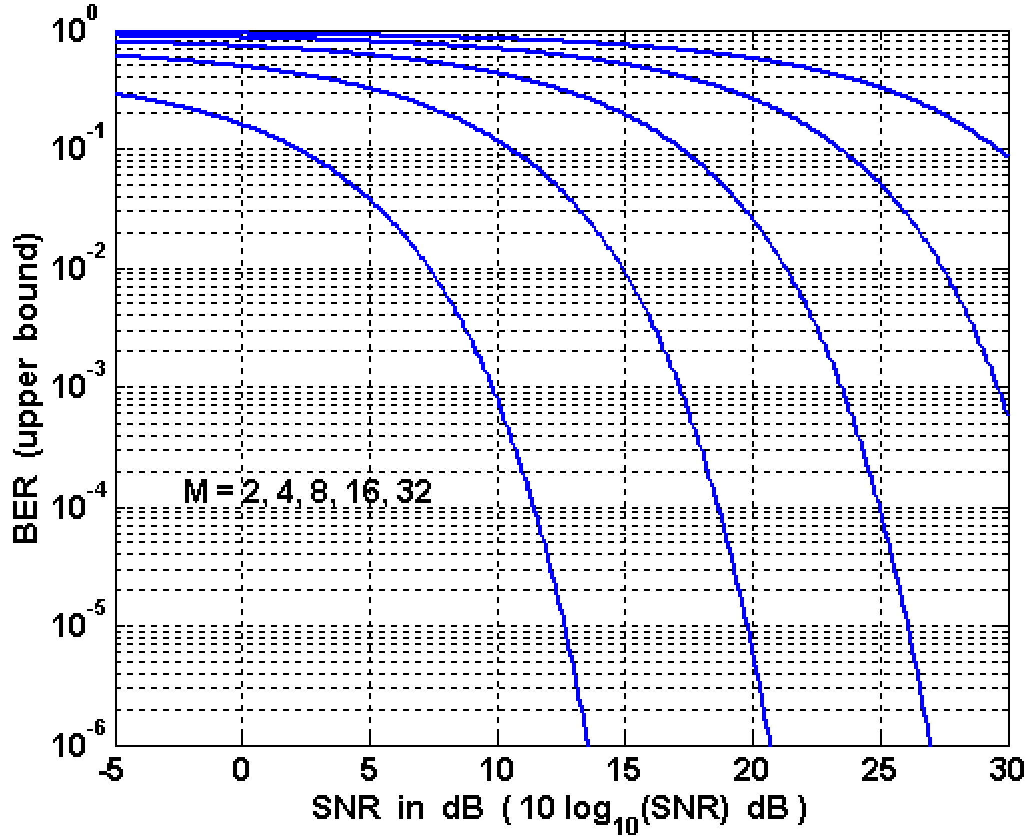


Increasing M would increase the data rate (given the same signal bandwidth)

M-ary Modulation



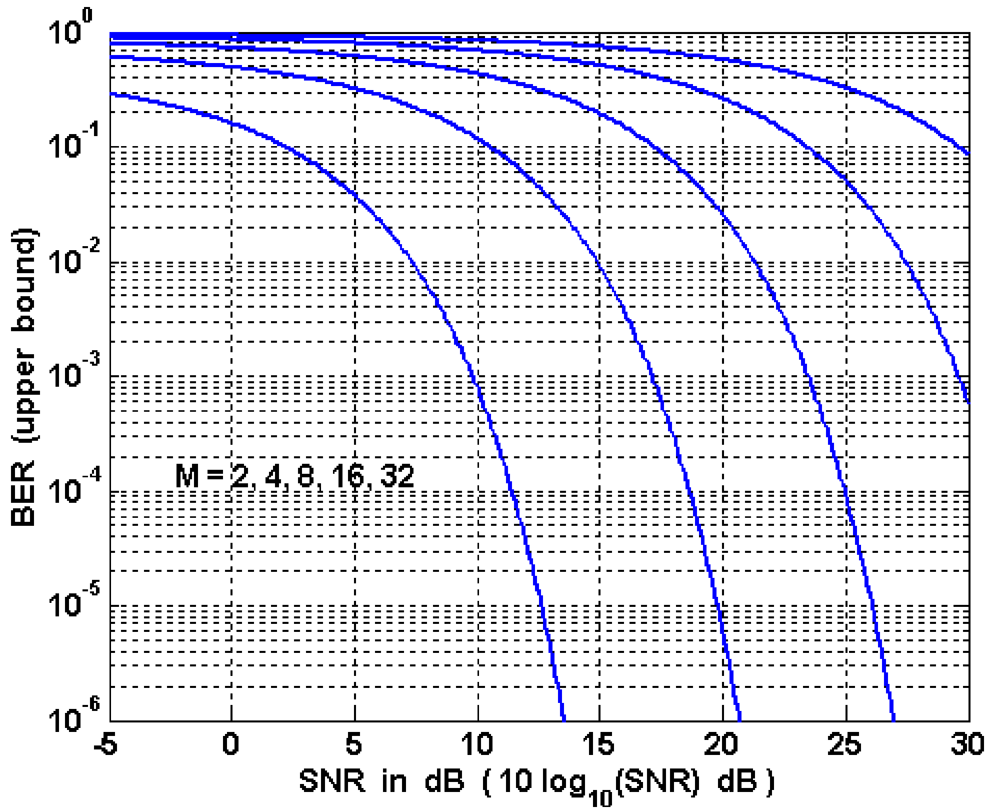
M-PAM BER versus SNR



$$SNR = \frac{\overline{E_b}}{\sigma_n^2}$$

$$BER \leq P_e \leq 2 \times \left(1 - \frac{1}{M}\right) \times Q\left(\sqrt{\frac{3 \times SNR}{M^2 - 1}}\right)$$

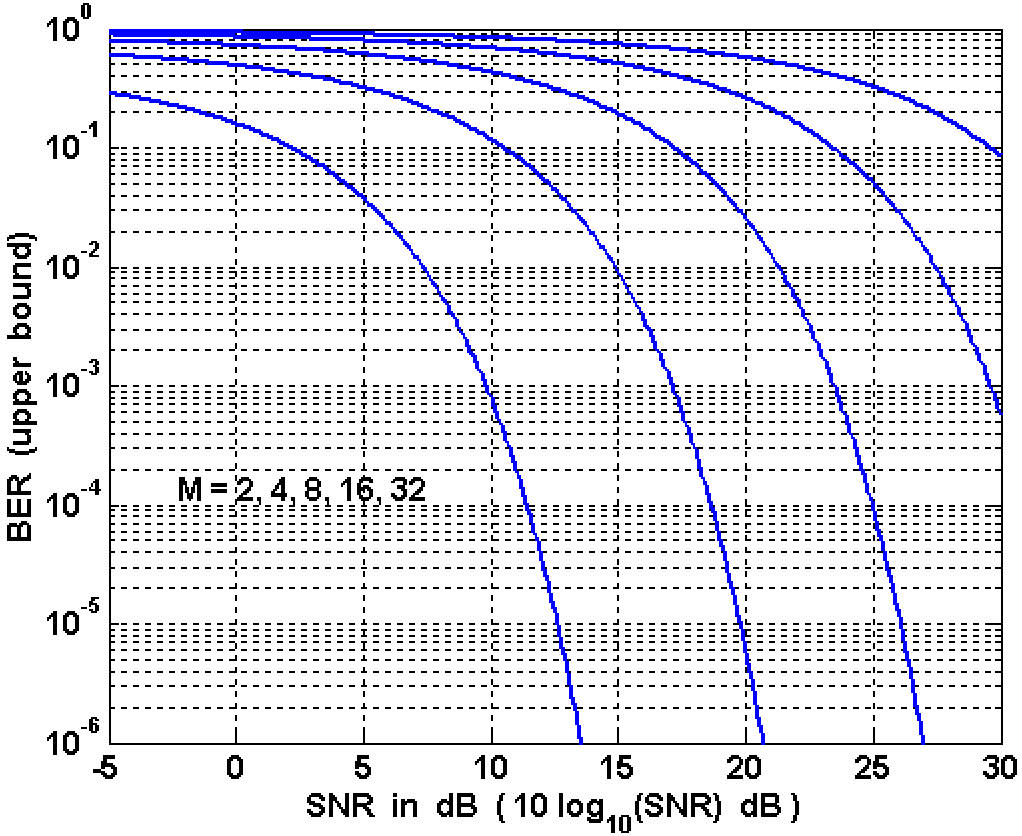
M-QAM BER versus SNR



$$BER \leq P_e \leq 4 \times \left(1 - \frac{1}{\sqrt{M}}\right) \times Q\left(\sqrt{\frac{3 \times SNR}{M - 1}}\right)$$

$$BER \leq P_e \leq 4 \times \left(1 - \frac{1}{\sqrt{2M}}\right) \times Q\left(\sqrt{\frac{3 \times SNR}{31 \times \frac{M}{32} - 1}}\right)$$

M-PSK BER versus SNR



$$BER \leq P_e \leq 2 \times Q\left(\sqrt{2 \times SNR} \times \sin\left(\frac{\pi}{M}\right)\right)$$