Fundamental Analysis of Securities Trading
(IV) Pairs Trading A

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Short Biodata

- Research interests:
  - time series models.
  - simulation modeling.
  - portfolio choice.
- Central themes of my application:
  - multivariate pairs trading in real time.
  - assets searching with a long-run equilibrium.
  - riskless portfolio building.
- Current work:
  - cointegration test.
  - structural change analysis.
  - the probability estimation of mean reversion.
Consider asset price has some principal factors.

\[
\text{price}(t) = f(\text{factor}_1(t), \ldots, \text{factor}_p(t)) + \epsilon(t) \\
= a + \sum_{i=1}^{p} \sum_{j=1}^{\infty} b_{ij}\text{factor}_i^j(t) + \epsilon(t).
\]

E.g., \( p = 1 \)

\[
\text{price}(t) = a + b_1\text{factor}(t) + b_2\text{factor}^2(t) + \cdots + \epsilon(t).
\]

Consider two asset prices has only one same principal factor. That is,

\[
\begin{align*}
\text{price}_A(t) &= a_A + b_A\text{factor}(t) + \epsilon_A(t); \\
\text{price}_B(t) &= a_B + b_B\text{factor}(t) + \epsilon_B(t).
\end{align*}
\]

Assume \( \epsilon_A \) and \( \epsilon_B \) are stable.

**Warning**

Arbitrage pricing theory shows this assumption is not good. But why we assume the above? (Hint: see the first slide.)

Assumption (3/4)

Wiki: Arbitrage Pricing Theory

Risky asset returns are said to follow a factor intensity structure if they can be expressed as:

\[ r_j = a_j + b_{j1}F_1 + b_{j2}F_2 + \cdots + b_{jn}F_n + \epsilon_j \]

where

- \( a_j \) is a constant for asset \( j \),
- \( F_n \) is a systematic factor,
- \( b_{jn} \) is the sensitivity of the \( j \)-th asset to factor \( n \), also called factor loading,
- and \( \epsilon_j \) is the risky asset’s idiosyncratic random shock with mean zero.

Assumption (4/4)

Example

Consider the price series \( \text{Price}_A(t) \) and \( \text{Price}_B(t) \)

\[
\begin{align*}
\text{Price}_A(t) &= a_A + b_A \cdot \text{Factor}(t) + \epsilon_A(t) \\
1 &= 3 + 3 \cdot 1 - 5 \\
33 &= 3 + 3 \cdot 10 + 0 \\
308 &= 3 + 3 \cdot 100 + 5
\end{align*}
\]

and

\[
\begin{align*}
\text{Price}_B(t) &= a_B + b_B \cdot \text{Factor}(t) + \epsilon_B(t) \\
2 &= 4 + 4 \cdot 1 - 6 \\
44 &= 4 + 4 \cdot 10 + 0 \\
410 &= 4 + 4 \cdot 100 + 6
\end{align*}
\]
If we buy $b_A$ A and short $b_B$ B at time $t_0$, then our cash flow is

$$\text{Cash Flow}(t_0) = -b_B \text{Price}_A(t_0) + b_A \text{Price}_B(t_0)$$

$$= -b_B (a_A + b_A \text{factor}(t_0) + \epsilon_A(t_0))$$

$$+ b_A (a_B + b_B \text{factor}(t_0) + \epsilon_B(t_0))$$

$$= (-a_A b_B + a_B b_A)$$

$$+ (-b_A b_B \text{factor}(t_0) + b_A b_B \text{factor}(t_0))$$

$$+ (-b_B \epsilon_A(t_0) + b_A \epsilon_B(t_0))$$

$$= (-a_A b_B + a_B b_A) + (-b_B \epsilon_A(t_0) + b_A \epsilon_B(t_0)).$$

We remark that $b_A = 3$, $b_B = 4$ and

$$\text{Price}_A(t) : 1, \ 33, \ 308;$$

$$\text{Price}_B(t) : 2, \ 44, \ 410.$$

Then

$$\text{Cash Flow}(1) = - \frac{b_B}{2} \cdot \text{Price}_A + \frac{b_A}{2} \cdot \text{Price}_B$$

$$= - \frac{4}{2} \cdot 1 + \frac{3}{2} \cdot 2.$$
Cash Flow (3/10)

That is, the portfolio value is
\[ \text{Price}_{\text{Portfolio}}(t) = -\text{Cash}_\text{Flow}(t) = (a_B b_B - a_A b_A) + (b_B \epsilon_A(t) - b_A \epsilon_B(t)). \]

However, in fact, our cash flow is
\[ \text{Cash}_\text{Flow}(t_0) = -|b_B| \text{Price}_A(t_0) - |b_A| \text{Price}_B(t_0) \leq -\text{Price}_{\text{Portfolio}}(t_0) \]

and
\[ \text{Cash}_\text{Flow}(t_1) = b_B (\text{Price}_A(t_1) - \text{Price}_A(t_0)) + b_A (-\text{Price}_B(t_1) + \text{Price}_B(t_0)) + |b_B| \text{Price}_A(t_0) + |b_A| \text{Price}_B(t_0). \]

Example—Continued

We remark that \( b_A = 3 \), \( b_B = 4 \), and
\[
\begin{align*}
\text{Price}_A(t) & : 1, \ 33, \ 308; \\
\text{Price}_B(t) & : 2, \ 44, \ 410; \\
\text{Factor}(t) & : 1, \ 10, \ 100.
\end{align*}
\]

Then
\[
\begin{align*}
\text{Price}_{\text{Portfolio}}(t) &= b_B \cdot \text{Price}_A - b_A \cdot \text{Price}_B \\
-2 &= 4 \cdot 1 - 3 \cdot 2 \\
0 &= 4 \cdot 33 - 3 \cdot 44 \\
2 &= 4 \cdot 308 - 3 \cdot 410
\end{align*}
\]
Example—Continued

We remark that $b_A = 3$, $b_B = 4$, and

$$
\begin{align*}
\text{Price}_A(t) & : 1, 33, 308; \\
\text{Price}_B(t) & : 2, 44, 410.
\end{align*}
$$

Then

$$
\begin{align*}
\text{Cash}_\text{Flow}(1) &= - |b_B| \cdot \text{Price}_A - |b_A| \cdot \text{Price}_B \\
-10 &= - |4| \cdot 1 - |3| \cdot 2
\end{align*}
$$

and

$$
\text{Cash}_\text{Flow}(1) = -10 < 2 = \text{Price}_{\text{Portfolio}}(t).
$$

So, if we know the values $b_A$ and $b_B$, then we can build a portfolio independent of principal factor.

Similarly, we can consider any number of factor.

Why do we need to know this?
If we buy one this portfolio at time $t_0$ and sell it at time $t_1$, then our cash flow is

$$\text{Cash}_\text{Flow}(t_0) + \text{Cash}_\text{Flow}(t_1)$$

$$= -\text{Price}_\text{Portfolio}(t_0) + \text{Price}_\text{Portfolio}(t_1)$$

$$= -((a_A b_B - a_B b_A) + (b_B \epsilon_A(t_0) - b_A \epsilon_B(t_0)))$$

$$+ ((a_A b_B - a_B b_A) + (b_B \epsilon_A(t_1) - b_A \epsilon_B(t_1)))$$

$$= -(a_A b_B - a_B b_A) + (a_A b_B - a_B b_A)$$

$$+ (-b_B \epsilon_A(t_0) - b_A \epsilon_B(t_0)) + (b_B \epsilon_A(t_1) - b_A \epsilon_B(t_1))$$

$$= -(b_B \epsilon_A(t_0) - b_A \epsilon_B(t_0)) + (b_B \epsilon_A(t_1) - b_A \epsilon_B(t_1)).$$
Remark: Return of This Portfolio

Return of this portfolio over \([t_0, t_1]\) is

\[
\text{Return}_{[t_0, t_1]} = -(b_B \epsilon_A(t_0) - b_A \epsilon_B(t_0)) + (b_B \epsilon_A(t_1) - b_A \epsilon_B(t_1)).
\]

That shows we only trade a stable noise.
We can wait for the low value to buy it or wait for the high value to sell it.

Example–Continued

We remark that

\[
\text{Price}_{\text{Portfolio}}(t) = -2, 0, 2.
\]

Then

\[
\text{Return}_{[t_0, t_1]} = \text{Price}_{\text{Portfolio}}(t_1) - \text{Price}_{\text{Portfolio}}(t_0)
\]

\[
\begin{align*}
2 &= 0 - (-2) \\
2 &= 2 - 0
\end{align*}
\]
Definition: Matrix

A rectangular arrangement

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

of elements \(a_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\), in \(m\) rows and \(n\) columns is called an \(m \times n\) matrix \(A\).

Example: Matrix [1]

Some examples of matrices are

\[
\begin{pmatrix}
1 & 2 \\
3 & 0 \\
-1 & 4
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 1 & -3
\end{pmatrix},
\begin{pmatrix}
e & \pi & -\sqrt{2} \\
0 & \frac{1}{2} & 1 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 \\
3
\end{pmatrix},
\begin{pmatrix}
4
\end{pmatrix}.
\]
Definition: Matrix Product [1]

If $A$ is an $m \times r$ matrix and $B$ is an $r \times n$ matrix, then the product $AB$ is the $m \times n$ matrix whose entries are determined as follows:

To find the entry in row $i$ and column of $AB$, single out row $i$ from the matrix $A$ and column $j$ from the matrix $B$. Multiple the corresponding entries from the row and column together, and then add up the resulting products.

Example: Multiplying Matrices [1]

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{pmatrix}$$

Since $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 4$ matrix, then product $AB$ is a $2 \times 4$ matrix. To determine, for example the entry in row 2 and column 3 of $AB$, we single out row 2 from $A$ and column 3 from $B$. 

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Example: Multiplying Matrices—Continued [1]

Then, as illustrated below, we multiply corresponding entries together and add up these products.

\[
\begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & 26 & ? \end{pmatrix}
\]

The computations for the remaining entries are

\[
\begin{align*}
(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\
(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\
(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\
(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\
(2 \cdot 1) + (6 \cdot 1) + (0 \cdot 7) &= -4 \\
(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12
\end{align*}
\]

and

\[
AB = \begin{pmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{pmatrix}
\]
Consider one equation

\[ y(t) = a + bx(t) \]

\[ \implies y(t) = \left(1 \  x(t)\right) \begin{pmatrix} a \\ b \end{pmatrix} \]

\[ \implies y(t) = X(t)\beta \]

Then, we have the following equations

\[
\begin{cases} 
\text{Price}_A(t) \approx \left(1 \ \text{factor}(t)\right) \begin{pmatrix} a_A \\ b_A \end{pmatrix} \\
\text{Price}_B(t) \approx \left(1 \ \text{factor}(t)\right) \begin{pmatrix} a_B \\ b_B \end{pmatrix}
\end{cases}
\]

We remark that \( a_A = 3, \ b_A = 3, \) and

\[
\begin{align*}
\text{Price}_A(t) & : 1, \ 33, \ 308; \\
\text{Factor}(t) & : 1, \ 10, \ 100.
\end{align*}
\]

Then

\[
\begin{align*}
\text{Price}_A(t) & \approx \left(1 \ \text{factor}(t)\right) \begin{pmatrix} a_A \\ b_A \end{pmatrix} \\
1 & \approx \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
33 & \approx \begin{pmatrix} 1 & 10 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
308 & \approx \begin{pmatrix} 1 & 100 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}
\end{align*}
\]
The System of Equations (3/13)

Example—Continued

We remark that $a_B = 4$, $b_B = 4$, and

\[
\begin{align*}
\text{Price}_B(t) & : 2, 44, 410; \\
\text{Factor}(t) & : 1, 10, 100.
\end{align*}
\]

Then

\[
\text{Price}_B(t) \approx (1 \text{ factor}(t)) \begin{pmatrix} a_A \\ b_A \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}
\]

\[
\begin{align*}
2 & \approx (1 \ 1) \begin{pmatrix} a_A \\ b_A \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{pmatrix} \\
44 & \approx (1 \ 10) \begin{pmatrix} a_A \\ b_A \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{pmatrix} \\
410 & \approx (1 \ 100) \begin{pmatrix} a_A \\ b_A \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}
\]

The System of Equations (4/13)

Definition: Linear Equation [1]

We define a linear equation in the $n$ variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

\[a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,\]

where $a_1, a_2, \ldots, a_n$ and $b$ are constants, and the $a$'s are not all zeros.

Definition: System of Linear Equations [1]

A finite set of linear equations is called a system of linear equations or, more briefly, a linear system.
Example: a General Linear System [1]

A general linear system of \( m \) equations in the \( n \) unknowns \( x_1, x_2, \ldots, x_n \) can be written as

\[
\begin{align*}
    a_{11}x_1 & + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
    a_{21}x_1 & + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
    \vdots & \vdots \vdots \vdots \vdots \\
    a_{m1}x_1 & + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.
\end{align*}
\]

Moreover, it can be written as

\[
Ax = b,
\]

where \( A = [a_{ij}]_{m \times n} \), and \( b = (b_1, b_2, \cdots, b_m)' \).

Consider

\[
\begin{align*}
    y(1) &= a + bx(1) \\
    y(2) &= a + bx(2) \\
    \vdots \\
    y(T) &= a + bx(T)
\end{align*}
\]

\[
\begin{bmatrix}
    y(1) \\
    \vdots \\
    y(T)
\end{bmatrix} =
\begin{bmatrix}
    1 & x(1) \\
    \vdots & \vdots \\
    1 & x(T)
\end{bmatrix}
\begin{bmatrix}
    a \\
    b
\end{bmatrix}
\]

\[
\begin{bmatrix}
    y(1) \\
    \vdots \\
    y(T)
\end{bmatrix} =
\begin{bmatrix}
    X(1) \\
    \vdots \\
    X(T)
\end{bmatrix}
\begin{bmatrix}
    a \\
    b
\end{bmatrix}
\]

\[y = X\beta.\]
The System of Equations (7/13)

This is, we have the following equation

\[
\begin{align*}
\begin{cases}
\text{Price}_A(1) = (1 \times (1)) (a_A) \\
\vdots \\
\text{Price}_A(T) = (1 \times (T)) (a_A) \\
\text{Price}_B(1) = (1 \times (1)) (a_B) \\
\vdots \\
\text{Price}_B(T) = (1 \times (T)) (a_B)
\end{cases}
\end{align*}
\]

Example–Continued

We remark that \(a_A = 3\), \(b_A = 3\), and

\[
\begin{align*}
\text{Price}_A(t) & : 1, 33, 308; \\
\text{Factor}(t) & : 1, 10, 100.
\end{align*}
\]

Then

\[
\begin{pmatrix}
1 \\
33 \\
308
\end{pmatrix}
\approx
\begin{pmatrix}
1 & 1 \\
1 & 10 \\
1 & 100
\end{pmatrix}
\begin{pmatrix}
3 \\
3
\end{pmatrix}
\]
Example–Continued

We remark that \( a_B = 4, \ b_B = 4, \) and

\[
\begin{align*}
  \text{Price}_B(t) & : 2, \ 44, \ 410; \\
  \text{Factor}(t) & : 1, \ 10, \ 100.
\end{align*}
\]

Then

\[
\begin{pmatrix}
  2 \\
  44 \\
  410 
\end{pmatrix}
\approx
\begin{pmatrix}
  1 & 1 \\
  1 & 10 \\
  1 & 100 
\end{pmatrix}
\begin{pmatrix}
  4 \\
  4 
\end{pmatrix}
\]

That is, we have the following equation

\[
(\text{Price}_A(t) \ \text{Price}_B(t)) \approx (1 \ \text{factor}(t))
\begin{pmatrix}
  a_A & a_B \\
  b_A & b_B 
\end{pmatrix}
\]

\( \hspace{1cm} \)

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Example–Continued

We remark that \( a_B = 3, \ b_B = 3, \ a_B = 4, \ b_B = 4, \) and

\[
\text{Price}_B(t) : 1, \ 33, \ 308; \\
\text{Price}_B(t) : 2, \ 44, \ 410; \\
\text{Factor}(t) : 1, \ 10, \ 100.
\]

Then

\[
\begin{pmatrix}
1 & 2 \\
33 & 44 \\
308 & 410
\end{pmatrix}
\approx
\begin{pmatrix}
1 & 1 \\
1 & 10 \\
1 & 100
\end{pmatrix}
\begin{pmatrix}
3 & 4 \\
3 & 4
\end{pmatrix}
\]

OK, this form is \( y = X\beta. \)
The System of Equations (13/13)

Example–Continued

We remark that $a_B = 3$, $b_B = 3$, $a_B = 4$, $b_B = 4$, and

$$
\begin{align*}
\text{Price}_B(t) & : 1, 33, 308; \\
\text{Price}_B(t) & : 2, 44, 410; \\
\text{Factor}(t) & : 1, 10, 100.
\end{align*}
$$

Then

$$
\begin{bmatrix}
1 \\
33 \\
308 \\
2 \\
44 \\
410
\end{bmatrix}
\approx
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 10 & 0 & 0 \\
1 & 100 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 10 \\
0 & 0 & 1 & 100
\end{bmatrix}
\begin{bmatrix}
3 \\
3 \\
4 \\
4
\end{bmatrix}
$$

The Kronecker Product (1/4)

Definition: Kronecker Product [2]

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $(m \times m)$ and $(p \times q)$ matrices, respectively. The $(mp \times nq)$ matrix

$$
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{pmatrix}
$$

is the Kronecker product (or direct product) of $A$ and $B$. 
The Kronecker Product (2/4)

Example: An Example for Kronecker Product [2]

The Kronecker product of \( A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 5 & -1 \\ 3 \end{pmatrix} \) is

\[
A \otimes B = \begin{pmatrix} 15 & -3 \\ 9 & 9 \\ 10 & -2 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} 20 & -4 \\ 12 & 12 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ -3 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 15 & -3 & 20 & -4 & -5 & 1 \\ 9 & 9 & 12 & 12 & -3 & -3 \\ 10 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

The Kronecker Product (3/4)

Example: An Example for Kronecker Product–Continued [2]

The Kronecker product of \( B = \begin{pmatrix} 5 & -1 \\ 3 \end{pmatrix} \) and \( A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 0 \end{pmatrix} \) is

\[
B \otimes A = \begin{pmatrix} 15 & 20 & -5 \\ 10 & 0 & 0 \\ 9 & 12 & -3 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & -4 & 1 \\ -2 & 0 & 0 \\ 9 & 12 & -3 \\ 6 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 15 & 20 & -5 & -3 & -4 & 1 \\ 10 & 0 & 0 & -2 & 0 & 0 \\ 9 & 12 & -3 & 9 & 12 & -3 \\ 6 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}
\]
### The Kronecker Product (4/4)

**Example–Continued**

We remark that

\[ \text{Factor}(t) : 1, 10, 100. \]

Then

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 10 & 0 & 0 \\
1 & 100 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 10 \\
0 & 0 & 1 & 100 \\
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 10 \\ 1 & 100 \end{pmatrix}.
\]

### The vec Operator (1/3)

**Definition: vec Operator [2]**

Let \( A = (a_1, \ldots, a_n) \) be an \((m \times n)\) matrix with \((m \times 1)\) columns \(a_i\). The **vec operator** transforms \(A\) into an \((mn \times 1)\) vector by stacking the columns, that is,

\[
\text{vec}(A) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.
\]
The vec Operator

Example: An Example for vec Operator [2]

For instance, if $A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -1 \\ 3 & 3 \end{pmatrix}$, then

$$\text{vec}(A) = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and $\text{vec}(B) = \begin{pmatrix} 5 \\ 3 \\ -1 \\ 3 \end{pmatrix}$.

We remark that $a_A = 3$, $b_A = 3$, $a_B = 4$, and $b_B = 4$. Then

$$\text{vec} \begin{pmatrix} a_A \\ a_B \\ b_A \\ b_B \end{pmatrix} = \text{vec} \begin{pmatrix} 3 & 4 \\ 3 & 4 \\ 3 & 4 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a_A \\ b_A \\ a_B \\ b_B \end{pmatrix}.$$
The simplify notation (1/3)

**Definition: Identity (or Unit) Matrix**

A matrix $I = I_n$ is called identity (or unity) matrix if

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
$$

**Rule: the $\otimes$ and vec Operator [2]**

Let $A, B, C$ be matrices with appropriate dimensions. We have

- $\text{vec}(AB) = (I \otimes A) \text{vec}(B)$.

The simplify notation (2/3)

That is, we have the following equation

$$
\text{vec}((\text{Price}_A(t) \ \text{Price}_B(t))) \approx (b \otimes (1 \ \text{factor}(t))) \text{vec}\left(\begin{pmatrix} a_A & a_B \\ b_A & b_B \end{pmatrix}\right)
$$

and

$$
\text{vec}\left(\begin{pmatrix} \text{Price}_A(1) & \text{Price}_B(1) \\ \vdots & \vdots \\ \text{Price}_A(T) & \text{Price}_B(T) \end{pmatrix}\right) \approx (b \otimes (1 \ \text{factor}(1))) \text{vec}\left(\begin{pmatrix} a_A & a_B \\ b_A & b_B \end{pmatrix}\right).
$$
Example—Continued

We remark that $a_B = 3$, $b_B = 3$, $a_B = 4$, $b_B = 4$, and

$$\begin{align*}
\text{Price}_B(t) & : 1, \ 33, \ 308; \\
\text{Price}_B(t) & : 2, \ 44, \ 410; \\
\text{Factor}(t) & : 1, \ 10, \ 100.
\end{align*}$$

Then

$$\text{vec} \begin{pmatrix} 1 & 2 \\ 33 & 44 \\ 308 & 410 \end{pmatrix} \approx \left( I_2 \otimes \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix} \right) \text{vec} \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}.$$