Statistics

Random Variables

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Section 1

Random Variables

Random Variables

- In practice we are interested in certain numerical measurements pertaining to a random experiment.
- For example,

$$\Omega = \{H, T\}$$

$$X = \begin{cases} 1, & \text{if } H \\ -1, & \text{if } T \end{cases}$$

- Then X is called a random variable.

Random Variables: A Formal Definition

Definition

A random variable X is a real-value function from the sample space to the real numbers:

$$X : \Omega \longmapsto \mathbb{R}$$

and it assigns to each element $\omega \in \Omega$ one and only one real number $X(\omega) = x$.

- Small letter x denotes the possible value of a random variable X.

Example: Flip a Fair Coin Twice

- The sample space is

$$\Omega = \{HH, HT, TH, TT\}$$

- Let X be the number of heads.
- The mapping is

ω	$X(\omega)$
$\{HH\}$	2
$\{HT\}$	1
$\{TH\}$	1
$\{TT\}$	0

Example: Flip a Fair Coin Twice

- How to assign probability $P(X = x)$?
- For example,

$$\begin{aligned}
P(X=1) &= P(\{\omega : X(\omega) = 1\}) \\
&= P(\{HT, TH\}) \\
&= P(\{HT\} \cup \{TH\}) \\
&= P(\{HT\}) + P(\{TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}$$

Random Variables

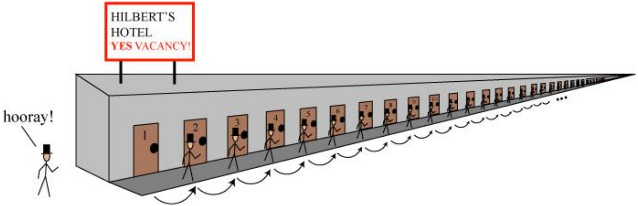
- In the previous example, the set of all possible values that X can assume are finite.
- According to the set of all possible values that a random variable can assume, we define two types of random variables.
  (1) Discrete random variables
  (2) Continuous random variables

Section 2

Discrete Random Variables

Discrete Random Variables

- A random variable X is a discrete random variable if:
  - there are a finite number of possible values of X, or
  - there are a countably infinite number of possible values of X.
- Recall that a countably infinite number of possible values means that there is a one-to-one correspondence between the values and the set of positive integers.



Examples

- The number of defective light bulbs in a box of six.

$$\text{Set of all possible values of } X = \{0, 1, 2, 3, 4, 5, 6\}$$

- The number of tails until the first heads comes up.

$$\text{Set of all possible values of } X = \{0, 1, 2, 3, \ldots\}$$

- We use the probability distribution to describe the likelihood of obtaining the possible values that a random variable can assume.

"I wish I could be as calm as JB when it comes to making decisions."

Probability Distribution

Definition (Probability Distribution)

Let X be a random variable. The probability distribution of X is to specify all probabilities involving X.

- One way to specify the probability distribution of discrete random variables is the probability mass function.

Probability Mass Function

Definition (Probability Mass Function)

Given a discrete random variable X. A probability mass function (pmf), $f(x) : \mathbb{R} \mapsto [0, 1]$ is defined by

$$f(x) = P(X = x)$$

- A probability mass function is also called a discrete probability density function (discrete pdf).
- A preferable notation: $f_X(x)$

Probability Mass Function

Definition (Support)

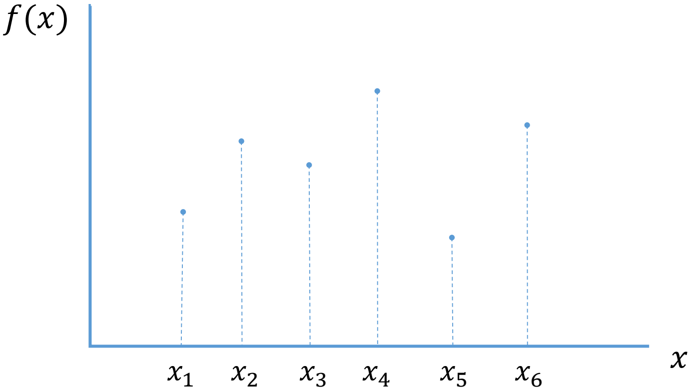
The support of a random variable X is defined as:

$$\text{supp}(X) = \{x : f(x) > 0\}$$

- Properties:

$$\sum_{x \in \text{supp}(X)} f(x) = 1$$

An Example of pmf



Example 1: Flip a Fair Coin Twice

- The sample space is $\Omega = \{HH, HT, TH, TT\}$
- Let X be the number of heads.

Table: Mapping and Probability Distribution

ω	$P(\{\omega\})$	$X(\omega)$
TT	1/4	0
TH	1/4	1
HT	1/4	1
HH	1/4	2

x	$f(x) = P(X = x)$
0	1/4
1	1/2
2	1/4

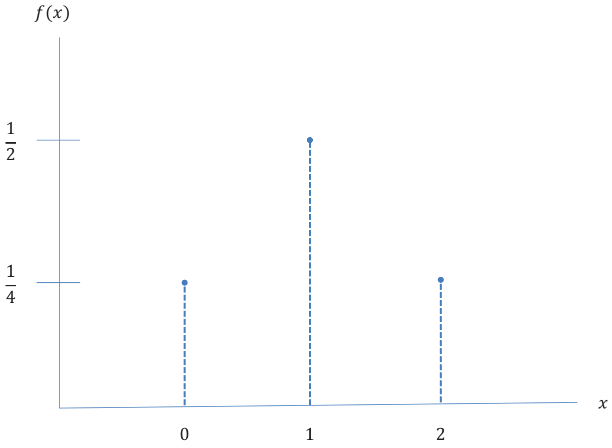
- Clearly, $\text{supp}(X) = \{0, 1, 2\}$, and $\sum_{x \in \text{supp}(X)} f(x) = 1$
- Q: let $A = \{X \leq 1\}$, what is $P(X \in A)$?

Example 1: Flip a Fair Coin Twice

- Probability mass function (alternative notation)

$$f(x) = P(X = x) = \begin{cases} 1/4 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \end{cases}$$

Example 1: Flip a Fair Coin Twice



Example 2: Bernoulli Random Variable

Definition (Bernoulli Random Variable)

A random variable X is said to have a Bernoulli distribution with success probability p if X can only assume the values 0 and 1, with probabilities

$$f(1) = P(X = 1) = p, \quad f(0) = P(X = 0) = 1 - p$$

We write $X \sim \text{Bernoulli}(p)$.

- Find its support, and probability mass function.

Example 3: Binomial Random Variables

Definition (Binomial Random Variable)

A random variable Y has the binomial distribution with parameters n and p if the probability mass function is

$$f(y) = \binom{n}{y} p^y (1-p)^{n-y}, \ \ \text{supp}(Y) = \{y | y = 0, 1, 2, \ldots, n\}$$

It is denoted by $Y \sim \text{Binomial}(n, p)$

- By the binomial theorem, $(a+b)^n = \sum_{y=0}^{n} \binom{n}{y} a^y b^{n-y}$,

$$\sum_{y=0}^{n} f(y) = 1$$

Example 3: Binomial Random Variables

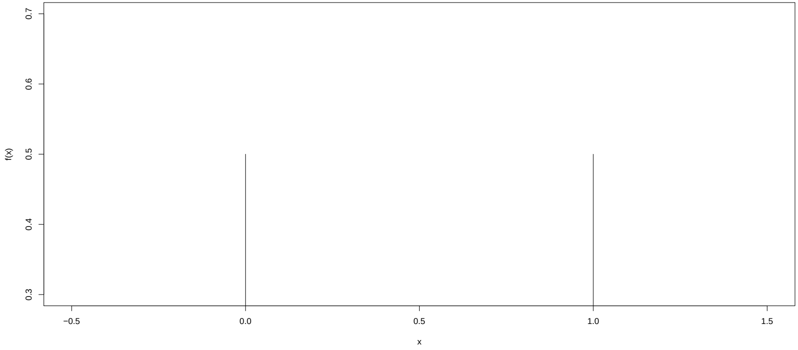
- Bernoulli vs. Binomial
  - Bernoulli(p) is used to model a single coin toss experiment.
  - Binomial(n, p) is used to model the number of heads in a sequence of n independent coin toss experiment.
- Clearly, they are linked by

$$Y = X_1 + X_2 + \cdots + X_n,$$

where $Y \sim$Binomial(n, p), and $X_1, X_2, \ldots, X_n$ are independent Bernoulli(p) variables.

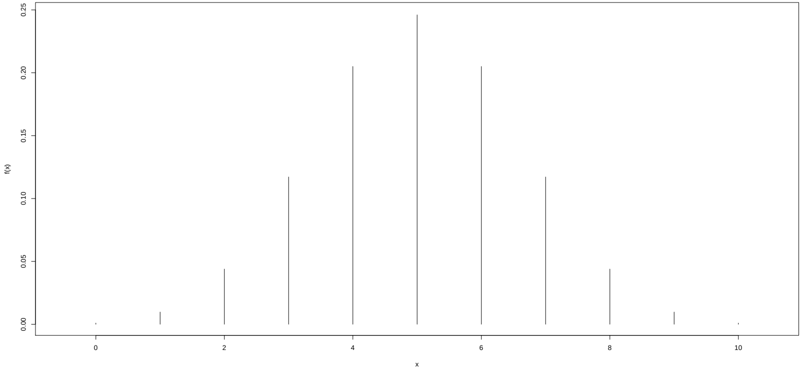
Probability Distribution

Consider $X \sim \text{Bernoulli}(0.5)$



Probability Distribution

Consider $X \sim \text{Binomial}(10, 0.5)$



Section 3

Continuous Random Variables

Continuous Random Variables

- A random variable is called continuous if it can takes on an uncountably infinite number of possible values.
  - The percentage of exam complete after 1 hour
  - The weight of a randomly selected quarter-pound burger

Continuous Random Variables

- Though a continuous variable can take any possible value in an interval, its measured value cannot. This is because no measuring device has infinite precision.
- Nevertheless, continuous random variables offer reasonable approximations to the underlying process of interest even though virtually all phenomena are, at some level, ultimately discrete.

How to Assign Probability?

- Discrete: flip a coin or roll a die.
- How about spinning a spinner?
    - Let X be the result of the spin.
    - Warning! Impossible to assign each outcome positive probability. Why?

How to Assign Probability?

- Suppose NOT, and let the spinner be fair.
- Each outcome has probability $p > 0$.
- Let $A \subset \Omega$ be an event that contains n distinct outcomes.

  $\Rightarrow$ Choose n large enough s.t. $p > \frac{1}{n}$.
- Then $P(X \in A) = np > 1$ (Big Trouble!)

How to Assign Probability?

- Hence p must be zero!
- That is, if X is a continuous random variable,

$$P(X = c) = 0$$

  - How can $P(X = c) = 0$ make sense? Can many nothings make something?
  - Think about the length of a point vs. the length of an interval.
  - A Zero-probability event is NOT an impossible event.

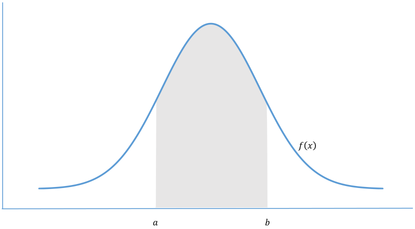
Continuous Random Variables

Definition (Continuous Random Variables)

A random variable X is continuous if there exists a function $f:\mathbb{R}\mapsto\mathbb{R}$ and for any number $a\le b$,

$$P(a\le X\le b)=\int_a^b f(x)dx$$

- The function $f(\cdot)$ is called the probability density function (pdf).



Probability Density Function

- In general, if the support for a continuous random variable is not specified, we assume that

$$\text{supp}(X) = \{x : -\infty < x < \infty\}$$

- The pdf is nonnegative

$$f(x) \geq 0, \quad \forall x$$

- The integral over the support of X is one

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Probability Density Function

- Since $P(X = c) = 0$ for any real value c,

$$
\begin{aligned}
&P(a \leq X \leq b) \\
&\quad = P(a < X < b) \\
&\quad = P(a \leq X < b) \\
&\quad = P(a < X \leq b)
\end{aligned}
$$

Probability Mass Function vs. Probability Density Function

- pmf (discrete pdf):

$$f(x) : \mathbb{R} \mapsto [0,1], \;\; f(x) = P(X = x)$$

- pdf:

$$f(x) : \mathbb{R} \mapsto \mathbb{R}^{+}, \;\; f(x) \neq P(X = x)$$

- That is, density is not probability.

Example 1: Uniform Random Variable

Definition (Uniform Random Variable)

A random variable X is said to have a uniform distribution on the interval $[l,h]$ if its pdf is given by

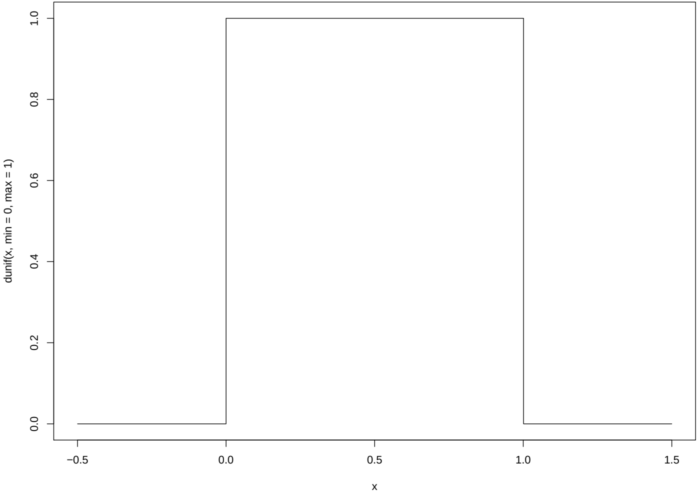
$$f(x)=\frac{1}{h-l},\quad l\leq x\leq h$$

We write $X\sim U[l,h]$.

- The probability that X falls in the sub-interval $[a,b]$ is

$$P(a\leq X\leq b)=\frac{b-a}{h-l}.$$

Uniform Distribution



Section 4

Cumulative Distribution Function

Cumulative Distribution Function

- An alternative way to specify the probability distribution is to give the probabilities of all events of the form

$$\{X \le x\}, \quad x \in \mathbb{R}$$

  - For example, what is the probability that the resulting number by rolling a die is smaller than 3.8?
- This leads to the following definition of cumulative distribution function.

Cumulative Distribution Function

Definition (Cumulated Distribution Function)

Given any real variable x, a function $F(x) : \mathbb{R} \mapsto [0, 1]$:

$$F(x) = P(X \leq x)$$

is called a cumulated distribution function (CDF), or distribution function.

- A preferable notation: $F_X(x)$
- It should be emphasized that the cumulative distribution function is defined as above for every random variable X, regardless of whether the distribution of X is discrete, or continuous.

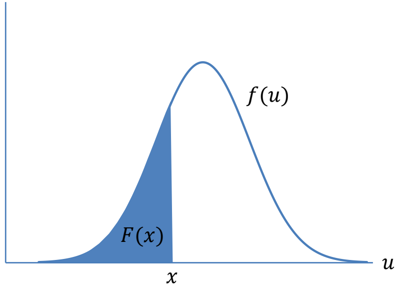
Cumulative Distribution Function

- If X is discrete,

$$F(x) = P(X \leq x) = \sum_{u \leq x} P(X = u) = \sum_{u \leq x} f(u)$$

- If X is continuous,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u) du$$



Cumulative Distribution Function

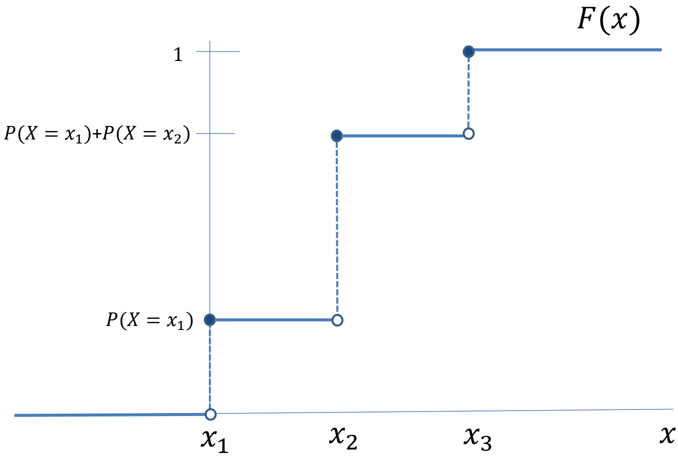
Theorem (Properties of CDF)

Let $F(x)$ be the CDF of a random variable X. Then,

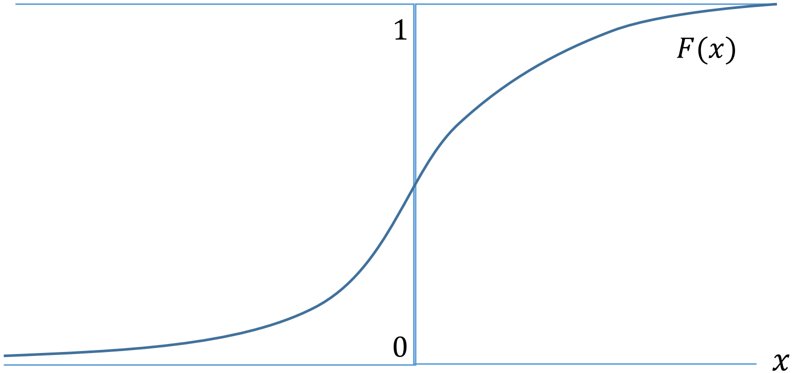
- *If $a < b$, then $F(a) \leq F(b)$ and $P(a < X \leq b) = F(b) - F(a)$.*
- $\lim_{x \to -\infty} F(x) = 0$, *and* $\lim_{x \to \infty} F(x) = 1$
- $F(x) = \lim_{\delta \to 0} F(x + \delta)$

CDF: Discrete Random Variable

$\text{supp}(X) = \{x_1, x_2 x_3\}$



CDF: Continuous Random Variable



Example 1: Bernoulli(p)

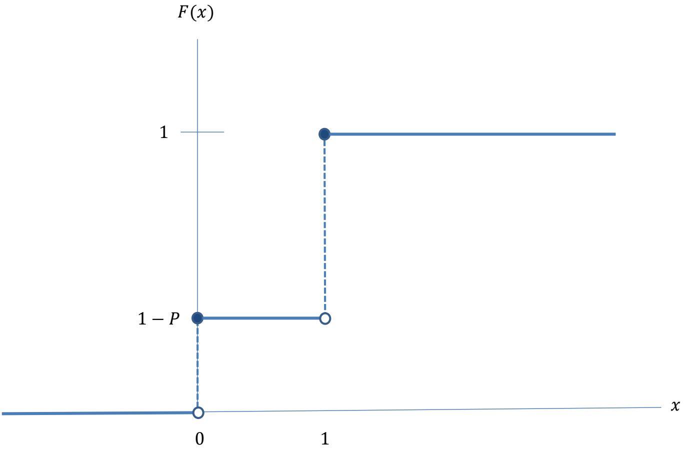
- Given the pmf of X~Bernoulli(p),

$$f(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

- The CDF is

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

Example 1: Bernoulli(p)



Example 2: Uniform $[l, h]$

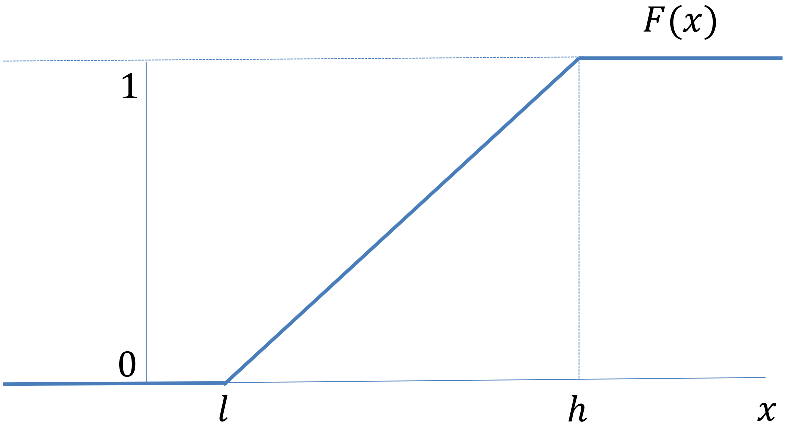
- Given $X \sim \text{Uniform}[l, h]$, the pdf is

$$f(x) = \frac{1}{h - l}$$

- The CDF is

$$F(x) = \int_l^x f(u)du = \int_l^x \frac{1}{h - l}du = \frac{x - l}{h - l}$$

Example 2: Uniform $[l, h]$



Section 5

Quantiles

Quantiles

Definition (Quantiles)

Let F denote the CDF of a random variable X. The function

$$\pi_p = F^{-1}(p) = \inf\{x | F(x) \geq p\}$$

is called the 100 p-th quantile of X. $F^{-1}(\cdot)$ is called the inverse distribution function.

- Given $p = 0.5$, the 50-th quantile, $\pi_{0.5}$, is called the median.
- If F is strictly increasing,

$$\pi_p = F^{-1}(p) = \{x | F(x) = p\}$$