Python Programming in Finance

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Lecture 6

Pricing Theory

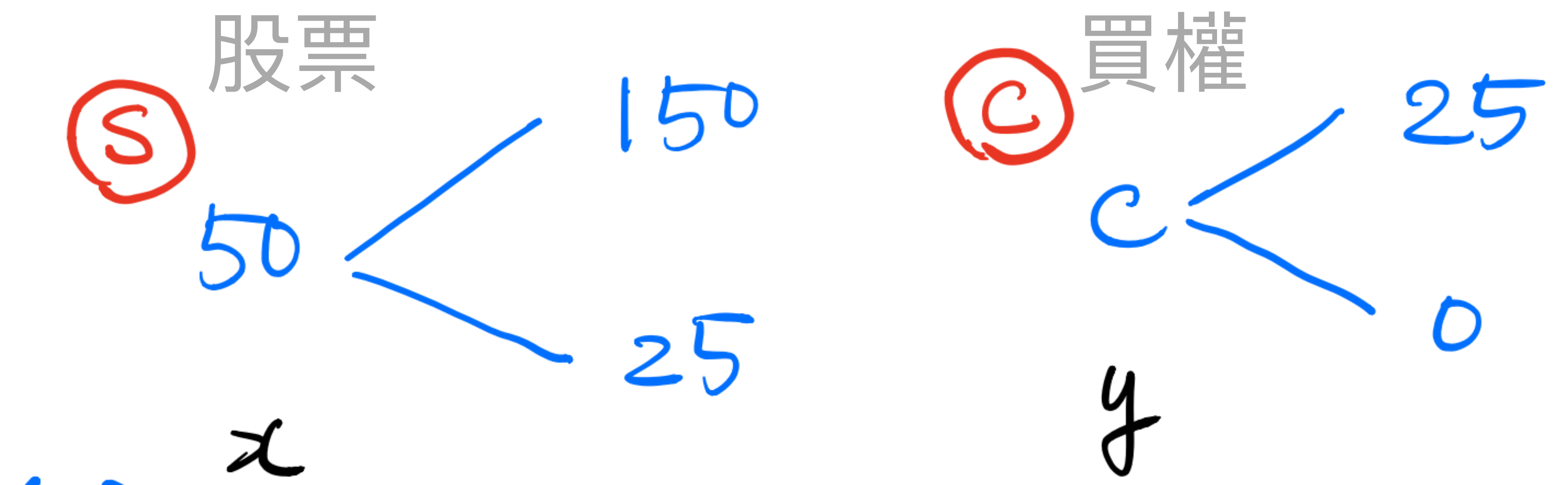
套利：在沒有承擔風險的前提下，可以賺取**超額報酬**。

風險趨避者

定價的精神就是
無套利原則。

無風險資產只能賺取
無風險利率的報酬。

天下沒有白吃的午餐



(a)

$$xS + yC, \quad y = -1$$

$$x\,50 - 1C \quad \begin{cases} x \cdot 150 - 1 \cdot 25 = 5 \\ x \cdot 25 - 1 \cdot 0 = 5 \end{cases}$$

you pay: $x\,50 - 1C$

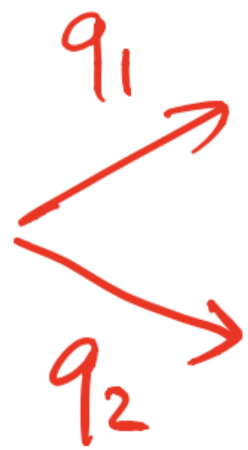
$\therefore x = \frac{1}{5}$, total value $= 5$ (you get)

no arbitrage

$\therefore$ return $= 0$, $x\,50 - 1C = 5$, with $x = \frac{1}{5}$

$\therefore C = 5$

arbitrage theorem



does there exist such a prob measure $Q = (q_1, q_2)$?

$$\begin{cases} 50 = q_1 \cdot 150 + q_2 \cdot 25 \\ 5 = q_1 \cdot 25 + q_2 \cdot 0 \end{cases}$$

$\Rightarrow$ $q_1 = 0.2$, $q_2 = 0.8$

Q measure exists $\Rightarrow$ no arbitrage

可以套利的情況 1

(b) if $C = 4 < 5 \Rightarrow -\frac{1}{5}S + C$ 做空這個組合

$= -(\frac{1}{5}S - C)$

4 ↓ too low

$$\underbrace{\overset{\text{receive}}{\frac{1}{5}\times 50} - \overset{\text{pay}}{4}}_{\text{time } 0} + \underbrace{\overset{\text{get}}{(-5)}}_{\text{time } T} = \underset{\substack{\Uparrow \\ \text{arbitrage}}}{1}$$

arbitrage theorem

$$\begin{cases} 50 = q_1 \cdot 150 + q_2 \cdot 25 \\ 4 = q_1 \cdot 25 + q_2 \cdot 0 \end{cases}$$

$$\Rightarrow \quad q_1 = \frac{4}{25}, \quad q_2 = \frac{26}{25} > 1$$

(c) if $C = 10 > 5 \Rightarrow +\frac{1}{5}S - 1C$

可以套利的情況 2

$C = 10$ ↓ too high

$+\frac{1}{5}S - 1C$ $\parallel$ $+(\frac{1}{5}S - 1C)$ 做多這個組合

pay	receive		get		
$-\frac{1}{5}\cdot 50$	$+ 10$	$+$	5	$=$	5

$\underbrace{-\frac{1}{5}\cdot 50 + 10}_{\text{time } 0} + \underbrace{5}_{\text{time } T} = 5$

arbitrage theorem

$$50 = q_1 \cdot 150 + q_2 \cdot 25$$

$$10 = q_1 \cdot 25 + q_2 \cdot 0$$

$$\Rightarrow q_1 = \frac{10}{25}, \quad q_2 = -\frac{10}{25} < 0$$

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from (b) (c) we see that

Arbitrage $\Longleftrightarrow$ $\not\exists$ Q measure

$= (q_1, q_2)$

(Q measure

= risk-neutral measure

= martingale measure)

第一定理

無套利

↕

存在一組合法的機率

(風險中立的機率測度，

即 Q-measure)

為什麼要談定價？

避險

避險 = 複製

完全市場

任何商品都可以被複製的市場即完全市場。

第二定理

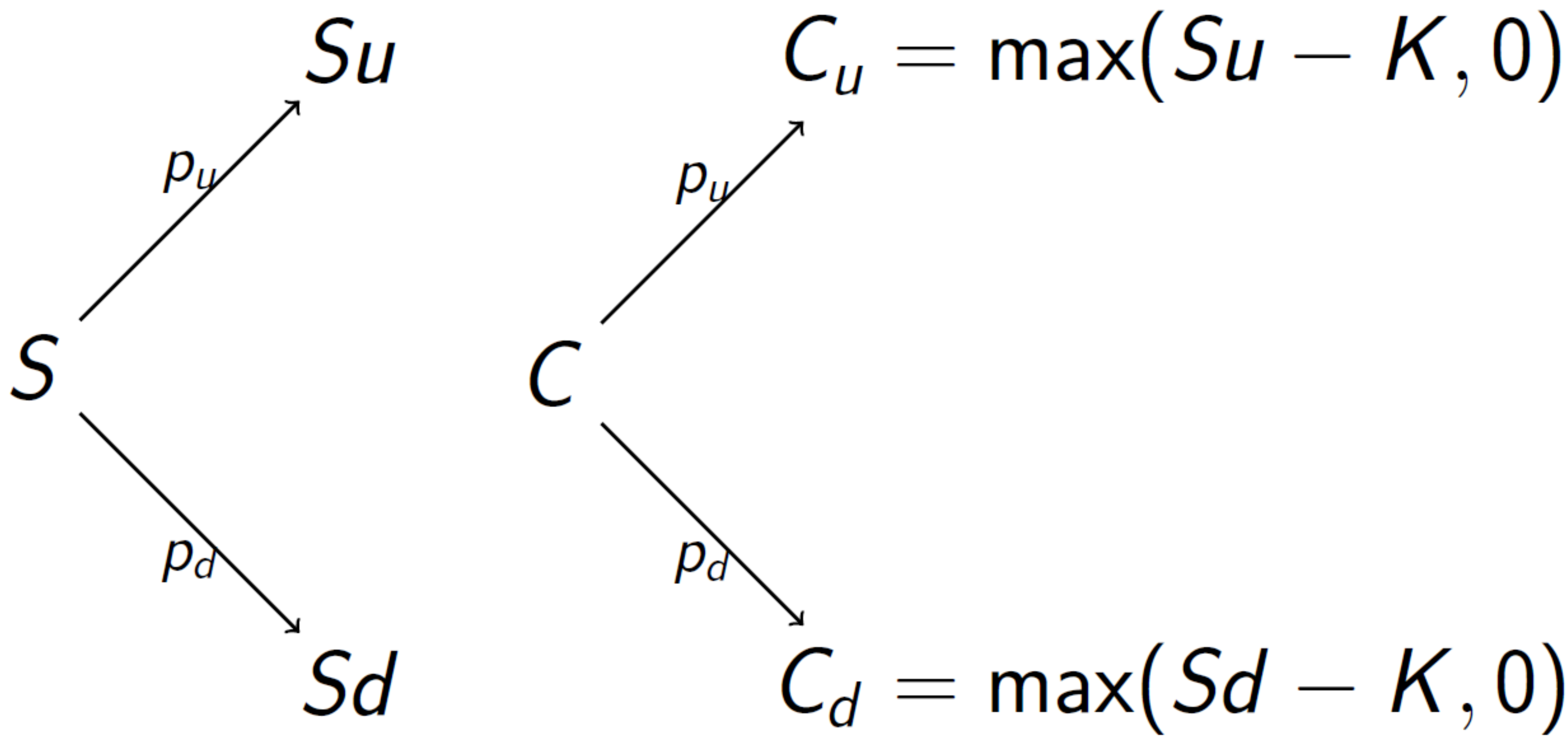
完全市場

↕

存在唯一合法的機率

Binomial Option Pricing Model

一期模型

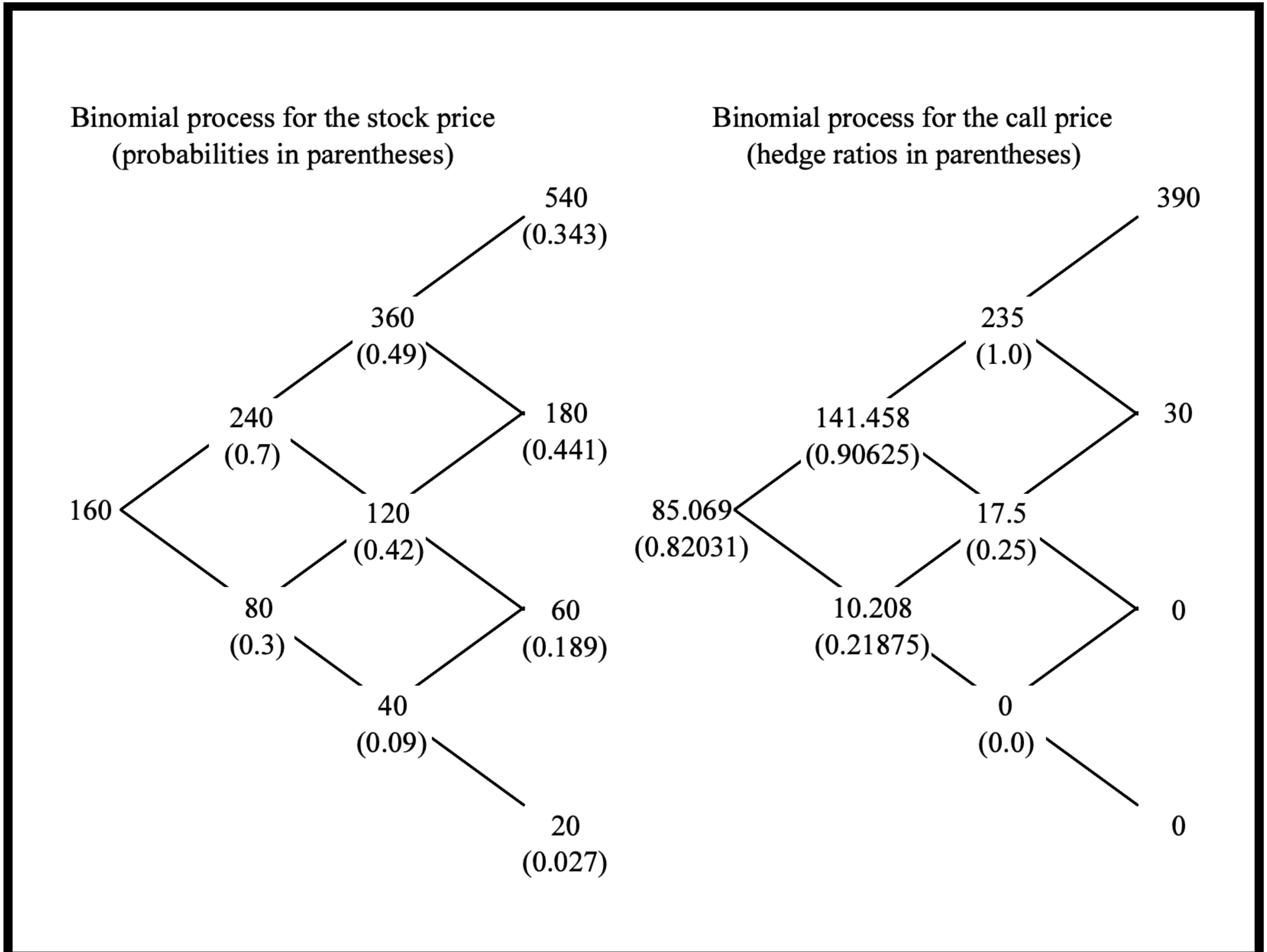


三期模型

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

$$C = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(S_T - X)^+\right]$$

履約價

折現因子

風險中立測度

$$C = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(S_T - X)^+\right]$$

買權價格

報酬函數

期望值

到期日的標的物價格

但，S_t是個隨機變數！

未來的價格是**不確定**的(?)

隨機過程

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布朗運動

Wiki

Wiener Process

Def 4.1

① $W(0) = 0$ — Gaussian

② $\Delta W_{r \to s}$, $\Delta W_{t \to u}$ indep

③ $\Delta W_{s \to t} \sim N(0, \Delta t)$ normal

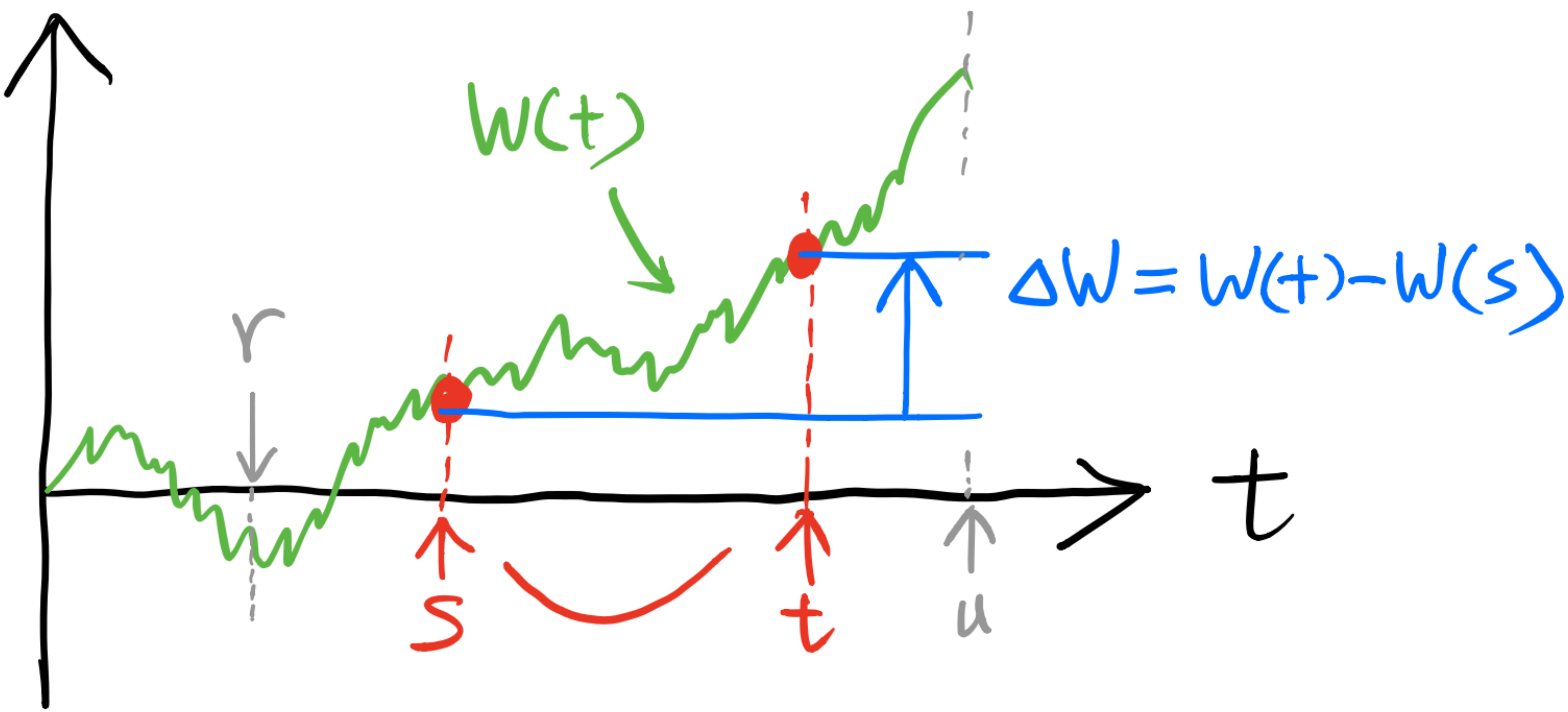
$\Delta t = t - s$ → variance

④ continuous

It's nowhere differentiable ⇒

notation $\frac{dW(t)}{dt}$ carries no meaning!

Wiener process



隨機微分方程式

SDE (stochastic differential eq.)

differential form :

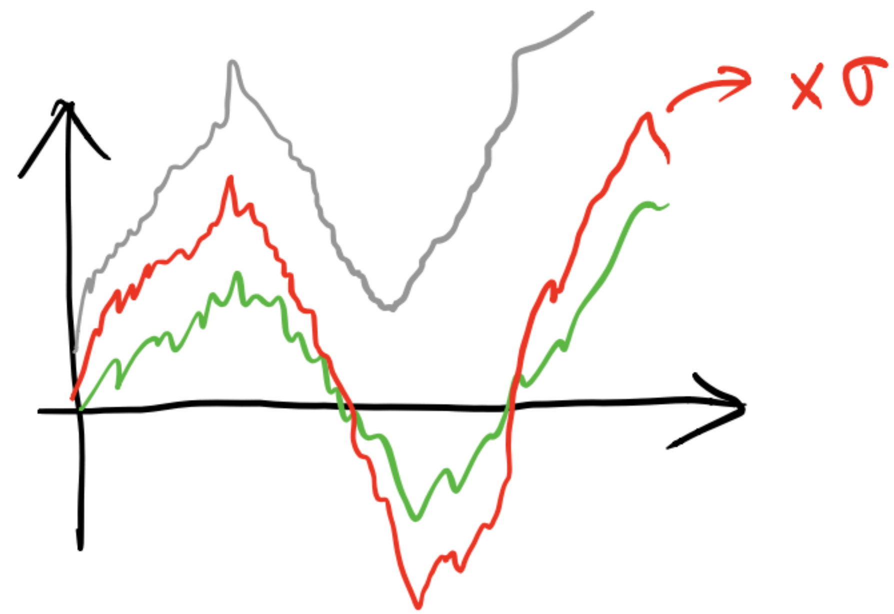
$$dX(t) = \mu\, dt + \sigma\, dW(t)$$

- $\mu \to \mu(t, X(t))$: drift — deterministic
- $\sigma \to \sigma(t, X(t))$: volatility (variance parameter) — stochastic

integral form

$$X(t) = X(0) + \int_0^t \mu \, ds + \underline{\underline{\int_0^t \sigma \, dW(s)}}$$

↑ stochastic integral (Ito integral)



① $X(t) = W(t)$

$X(t) \sim N(0, t)$ ← variance

② $X(t) = \sigma \cdot W(t)$

$X(t) \sim N(0, \sigma^2 t)$

③ $X(t) = \mu t + \sigma W(t)$

$X(t) \sim N(\mu t, \sigma^2 t)$

Black, Scholes, and Merton

假設報酬率是布朗運動

報酬率

市場波動率

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

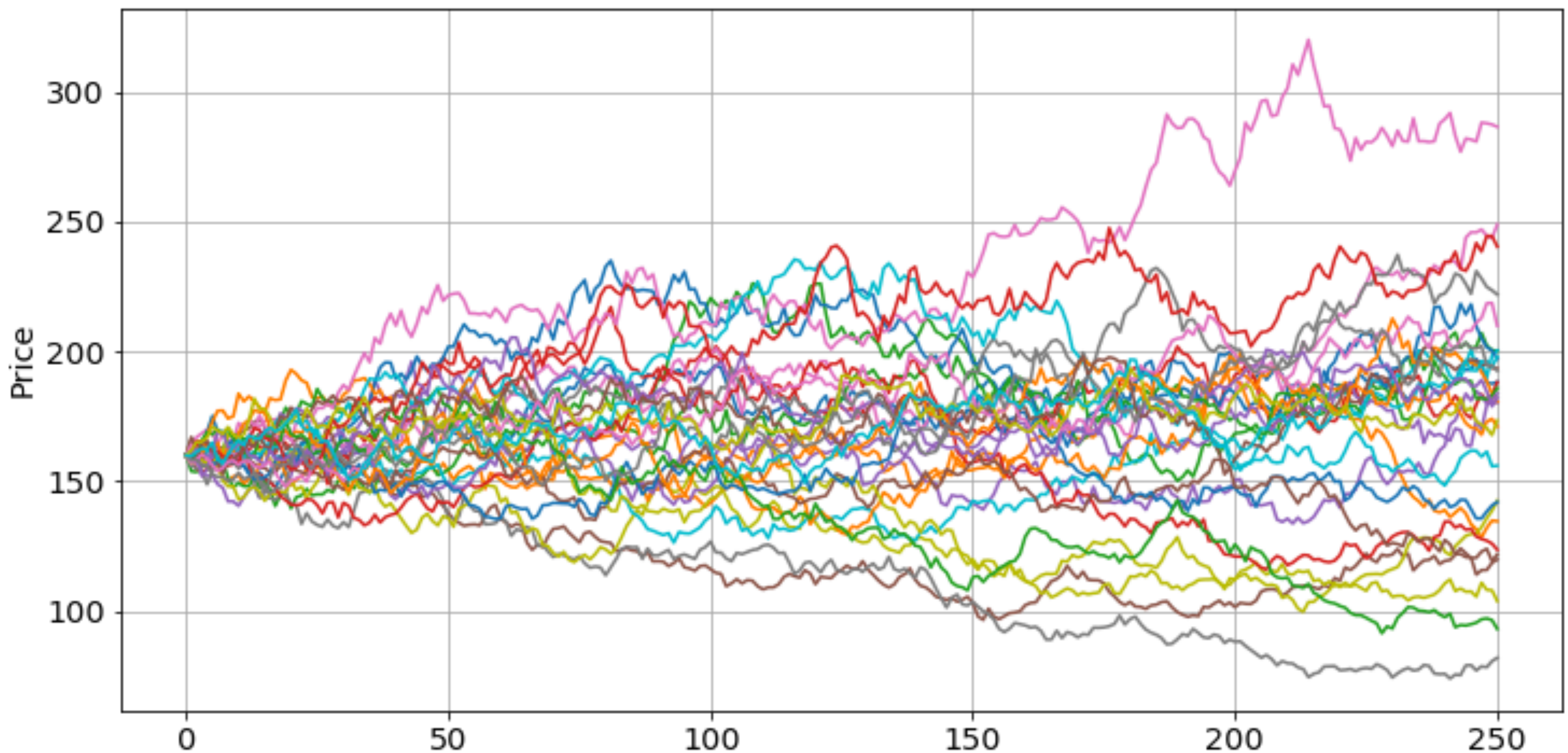
期望報酬率

不確定性的來源

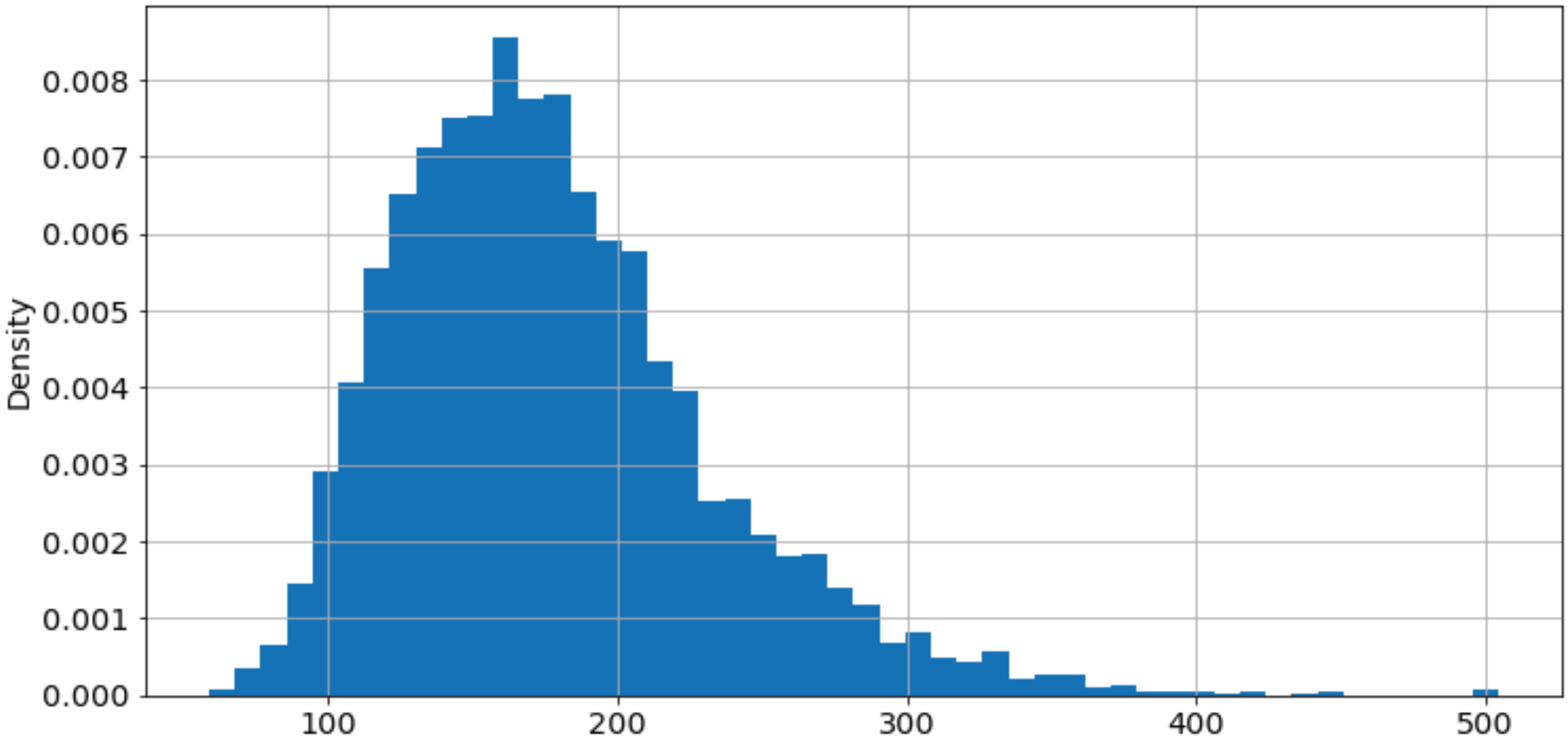
By Ito's lemma,

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$$

假設r = 0.1, v = 0.3, s0 = 160，模擬30個平行宇宙的價格路徑：



在第250日時，股價的機率分佈 (五千個平行宇宙)：



價格的分佈很類似常態，精確的說為一個對數常態分配 (log-normal)。

根據定價公式

$$C = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(S_T - X)^+\right]$$

我們可以透過蒙地卡羅法模擬價格路徑，
進而計算出該選擇權的價格！

歐式買權封閉解

$$C(S_0, T) = S_0 N(d_1) - X e^{-rT} N(d_2)$$

with

$$d_1 = \frac{\ln(\frac{S_T}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

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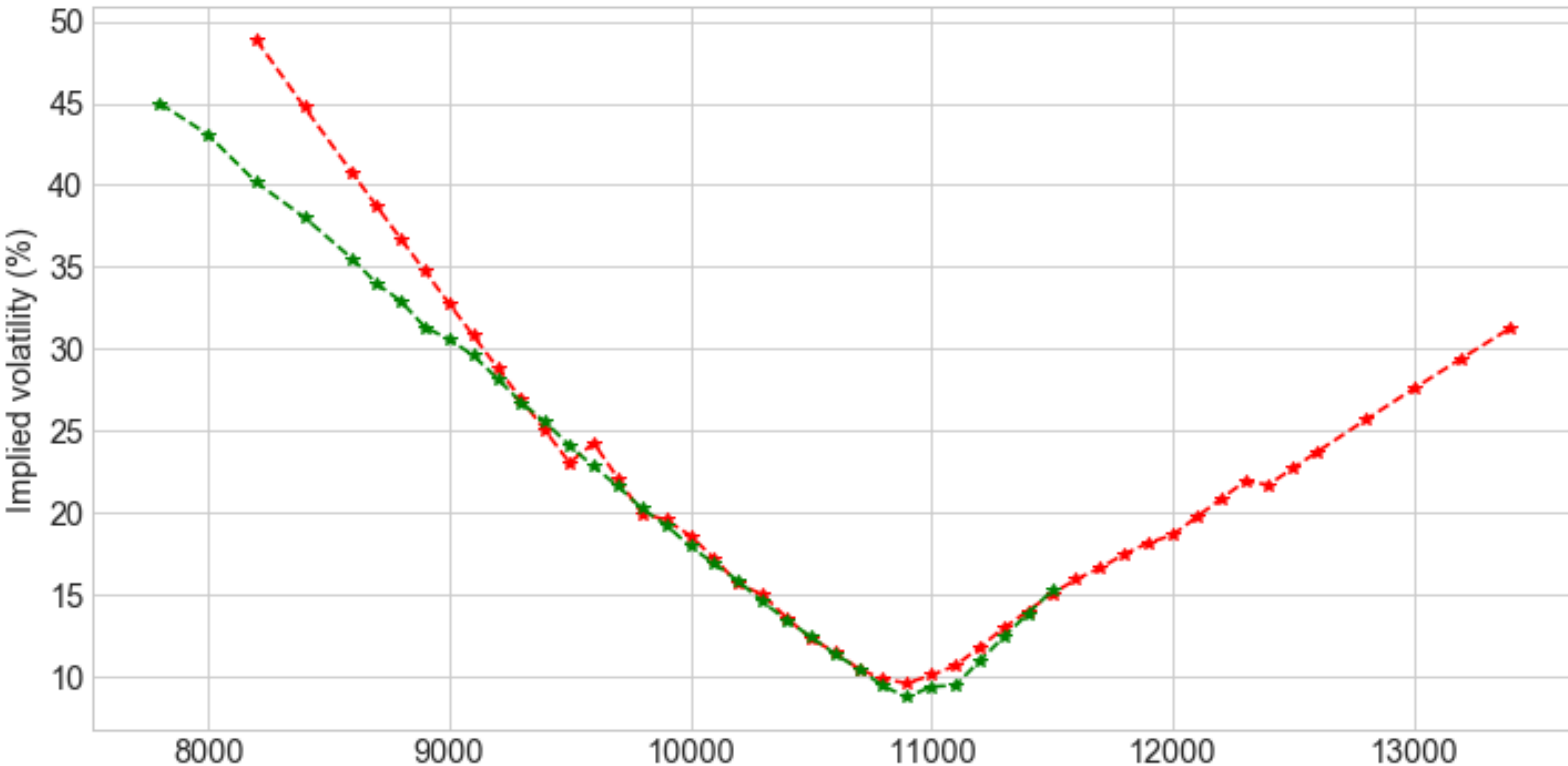
均值回歸

波動率微笑曲線


Implied volatility (%)
50
45
40
35
30
25
20
15
10
8000
9000
10000
11000
12000
13000

隨機波動率模型

黑天鵝事件

https://www.moneydj.com/KMDJ/Wiki/WikiViewer.aspx?KeyID=43c13f1c-68b9-45e8-b4e1-ed6a255cb6fe

Jump Model

模型校準

隱含波動率

$$C = f(S_0, X, T, r, \sigma)$$

不可直接觀測

牛頓法