

THE DUPIRE FORMULA

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1. Introduction. The Dupire formula enables us to deduce the volatility function in a local volatility model from quoted put and call options in the market¹. In a local volatility model the asset price model under a risk-neutral measure takes the form

$$(1.1) \quad dS_t = \mu(t)S_t dt + \tilde{\sigma}(t, S_t)S_t dW_t.$$

Here $\mu(t) = r(t) - q(t)$ in the usual notation, $r(\cdot), q(\cdot)$ are possibly time varying, but *deterministic* and W is Brownian motion. The forward price for delivery at time T is then $F_t = F(t, T) = S_t \exp(\int_t^T \mu(s) ds)$ and it is easily seen that F_t is a martingale, satisfying

$$dF_t = \sigma(t, F_t)F_t dW_t,$$

where

$$\sigma(t, x) = \tilde{\sigma}(t, x e^{-\int_t^T \mu(s) ds}),$$

and of course $F_T = S_T$. If we have a call option with strike K and exercise time T then its *forward price* in this model is

$$(1.2) \quad C(T, K) = \int_K^\infty (x - K) \phi(T, x) dx$$

where $\phi(T, \cdot)$ is the density function of the r.v. S_T (assumed to exist). The actual market price at time 0 would be $p = C(T, K)e^{-\int_0^T r(s) ds}$. If we differentiate (1.2) twice we obtain

$$(1.3) \quad \frac{\partial C}{\partial K} = - \int_K^\infty \phi(x) dx = \Phi(T, K) - 1$$

$$(1.4) \quad \frac{\partial^2 C}{\partial K^2} = \phi(T, x)$$

where $\Phi(T, \cdot)$ is the distribution function of S_T . These relations are known as the Breeden-Litzenberger formulas.

2. The forward equation. For any $t < T$ and (say bounded measurable) function h let $v(t, x) = \mathbb{E}[h(F_T)|F_t = x]$. We have by iterated conditional expectation

$$\begin{aligned} \mathbb{E}[h(F_T)] &= \mathbb{E}[\mathbb{E}[h(F_T)|F_s]] \\ &= \int_0^\infty v(t, x) \phi(t, x) dx. \end{aligned}$$

Since the LHS does not depend on t then neither does the RHS, so differentiating w.r.t. t ,

$$(2.1) \quad 0 = \int_0^\infty \frac{\partial v}{\partial t} \phi dx + \int_0^\infty v \frac{\partial \phi}{\partial t} dx.$$

We know from Itô calculus that v satisfies the *backward equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2(t, x) x^2 \frac{\partial^2 v}{\partial x^2} = 0, \quad v(T, x) = h(x)$$

so with $v' = \partial v / \partial x$ etc. (2.1) becomes

$$0 = - \int_0^\infty \frac{1}{2} \sigma^2 x^2 v'' \phi dx + \int_0^\infty v \frac{\partial \phi}{\partial t} dx.$$

Integrating by parts twice in the first integral gives

$$0 = \int_0^\infty \left(\frac{1}{2} (\sigma^2 x^2 \phi)'' - \frac{\partial \phi}{\partial t} \right) v dx.$$

Since h and hence, essentially, v is arbitrary, we conclude that ϕ satisfies the *forward equation*

$$(2.2) \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(t, x) x^2 \phi).$$

¹WARNING: The calculations presented here are formal. The formulas are correct (I hope!) but no attempt has been made to state conditions under which they are valid.

3. Dupire's equation. From (1.2) we have

$$\begin{aligned}
\frac{\partial C}{\partial T}(T, K) &= \int_K^\infty (x - K) \frac{\partial \phi}{\partial T}(T, x) dx \\
&= \frac{1}{2} \int_K^\infty (\sigma^2 x^2 \phi)''(x - K) dx \quad [\text{using (2.2)}] \\
&= -\frac{1}{2} \int_K^\infty (\sigma^2 x^2 \phi)' 1 dx \quad [\text{integrating by parts}] \\
&= \frac{1}{2} \sigma^2(T, K) K^2 \phi(T, K) \\
&= \frac{1}{2} \sigma^2(T, K) K^2 \frac{\partial^2 C}{\partial x^2}(T, K). \quad [\text{using (1.4)}].
\end{aligned}$$

This gives us Dupire's formula for the local volatility, expressed entirely in terms of the volatility surface $C(\cdot, \cdot)$:

$$(3.1) \quad \boxed{\sigma(T, K) = \frac{1}{K} \sqrt{\frac{2 \partial C / \partial T(T, K)}{\partial^2 C / \partial x^2(T, K)}}.}$$

4. Constructing a local volatility model. The procedure is as follows.

1. Assemble the data, consisting of a matrix of quoted option prices $\{C(T_i, K_j^i), i = 1, \dots, N, j = 1, \dots, M(i)\}$ together with the yield curve (to determine $r(t)$) and dividend information (to determine $q(t)$).
2. Interpolate and extrapolate these prices (or, more likely, the corresponding Black-Scholes implied volatilities) to produce a smooth volatility surface C .
3. Calculate $\sigma(T, F)$ from (3.1) and compute the corresponding $\tilde{\sigma}(T, S)$
4. The price model is S_t given by (1.1).
5. Now we can calculate the prices of other options by finite-difference methods or Monte Carlo.