Python Programming in Finance

Volatility & The VIX Index

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Geometric Brownian Motion (GBM)

- Let S be the stock price and dW be a Wiener process with its volatility σ.
- Consider a GBM process

$$\frac{dS}{S} = (\mu - q)dt + \sigma dW, \tag{1}$$

where μ is the expected return rate and $q \geq 0$ is the dividend yield rate.

- By the Itô's formula, the log price follows

$$d \ln S = (\mu - q - \frac{\sigma^2}{2})dt + \sigma dW. \tag{2}$$

- By (1) − (2),

$$\frac{\sigma^2}{2} dt = \frac{dS}{S} - d \ln S.$$

- Let $\overline{V}$ be the average variance rate within the period $[0, T]$.
- Then we have

$$\overline{V} = \frac{1}{T} \int_0^T \sigma^2 dt = \frac{2}{T} \int_0^T \frac{dS}{S} - \frac{2}{T} \ln \left(\frac{S_T}{S_0} \right).$$

- Hence the expectation under the $\mathbb{Q}$ measure is

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}} \left[\overline{V} \right] &= \frac{2}{T} (\mu - q) T - \frac{2}{T} \mathbf{E}^{\mathbb{Q}} \left[\ln \left(\frac{S_T}{S_0} \right) \right], \\ &= \frac{2}{T} \ln \left(\frac{F_0}{S_0} \right) - \frac{2}{T} \mathbf{E}^{\mathbb{Q}} \left[\ln \left(\frac{S_T}{S_0} \right) \right], \end{aligned}$$

where $(\mu - q)T = rT$ replaced by $\ln \left(\frac{F_0}{S_0} \right)$ is an immediate result of martingale pricing.

- Consider the integration below:

$$\int_0^\infty \frac{1}{K^2}(K - S_T)^+ \, dK = \int_0^{K^*} \frac{1}{K^2}(K - S_T)^+ \, dK + \int_{K^*}^\infty \frac{1}{K^2}(K - S_T)^+ \, dK.$$

- For $K^* > S_T$, the RHS becomes

$$\ln\left(\frac{K^*}{S_T}\right) + \frac{S_T}{K^*} - 1 + 0.$$

- For $K^* < S_T$, the RHS becomes

$$0 + \ln\left(\frac{K^*}{S_T}\right) + \frac{S_T}{K^*} - 1.$$

- It is easy to see that

$$\ln\left(\frac{S_T}{K^*}\right) = \frac{S_T}{K^*} - 1 - \int_0^{K^*} \frac{1}{K^2}(K - S_T)^+ dK - \int_{K^*}^{\infty} \frac{1}{K^2}(K - S_T)^+ dK.$$

- Therefore the expectation value of the above equation is

$$\mathbf{E}^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{K^*}\right)\right] = \frac{F_0}{K^*} - 1 - \int_0^{K^*} \frac{1}{K^2} e^{rT} p(K) dK - \int_{K^*}^{\infty} \frac{1}{K^2} e^{rT} c(K) dK,$$

where

$$p(K) = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(K - S_T)^+\right]$$

and

$$c(K) = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(S_T - K)^+\right].$$

- Note that we have

$$\mathbf{E}^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right] = \ln\left(\frac{K^*}{S_0}\right) + \mathbf{E}^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{K^*}\right)\right].$$

- In the end, we have

$$\begin{aligned}
\mathbf{E}^{\mathbb{Q}}\left[\overline{V}\right] &= \frac{2}{T}\ln\left(\frac{F_0}{S_0}\right) - \frac{2}{T}\mathbf{E}^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right] \\
&= \frac{2}{T}\ln\left(\frac{F_0}{K^*}\right) - \frac{2}{T}\left(\frac{F_0}{K^*} - 1\right) \\
&\quad + \frac{2}{T}\int_0^{K^*}\frac{1}{K^2}e^{rT}p(K)dK + \frac{2}{T}\int_{K^*}^{\infty}\frac{1}{K^2}e^{rT}c(K)dK.
\end{aligned}$$

- In practice, there are a finite number of strike prices.
- So the aforesaid equation is discretized as follows:

$$\mathbf{E}^{\mathbb{Q}}\left[\overline{V}\right] \approx \frac{2}{T}\ln\left(\frac{F_0}{K_0}\right) - \frac{2}{T}\left(\frac{F_0}{K_0} - 1\right) + \frac{2}{T}\sum_{K=0}^{K_0}\frac{\Delta K}{K^2}e^{rT}p(K) + \frac{2}{T}\sum_{K=K_0}^{\infty}\frac{\Delta K}{K^2}e^{rT}c(K).$$

- By the Taylor's 2^nd^-order expansion to the log term,

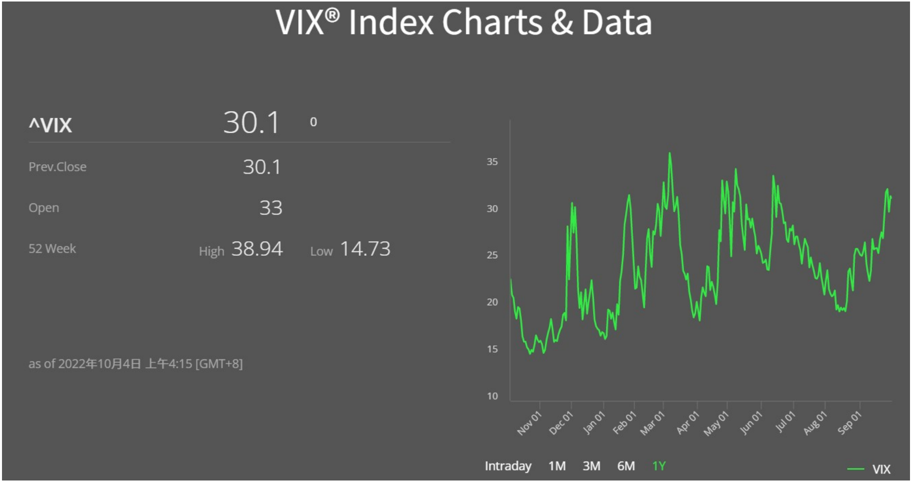
$$\mathbf{E}^{\mathbb{Q}}\left[\overline{V}\right] \approx -\frac{1}{T}\left(\frac{F_0}{K_0} - 1\right)^2 + \frac{2}{T}\sum_{K=0}^{K_0}\frac{\Delta K}{K^2}e^{rT}p(K) + \frac{2}{T}\sum_{K=K_0}^{\infty}\frac{\Delta K}{K^2}e^{rT}c(K). \quad (3)$$

- We follow Equation (3) to calculate the VIX Index.

CBOE VIX Index[1]

- The VIX Index is designed to estimate 30-days expected volatility by aggregating the weighted prices of S&P 500 Index puts and calls over a wide range of strike prices.
- It is often referred to as the "fear gauge."

[1] See http://www.cboe.com/vix.


VIX® Index Charts & Data
^VIX 30.1 0
Prev.Close 30.1
Open 33
52 Week High 38.94 Low 14.73
as of 2022年10月4日 上午4:15 [GMT+8]
35
30
25
20
15
10
Nov 01
Dec 01
Jan 01
Feb 01
Mar 01
Apr 01
May 01
Jun 01
Jul 01
Aug 01
Sep 01
Intraday 1M 3M 6M 1Y
VIX

Short History

- 1989: Brenner and Galai proposed the Sigma Index.[2]
- 1993: first launched; use S&P 100 Index options.
- 2003: changed to S&P 500 Index (SPX) options.
- 2004: launched VIX Futures.
- 2006: launched VIX Options.
- 2008: intraday high of 89.53 on October 24.
- 2020: hit and closed at 75.47 due to a travel ban to the US from Europe was announced.
- 2020: closed at 82.69, the highest level since its inception in 1990.

[2] Brenner and Galai (1989): *New Financial Instruments for Hedging Changes in Volatility*.

Illustration: CBOE VIX Index

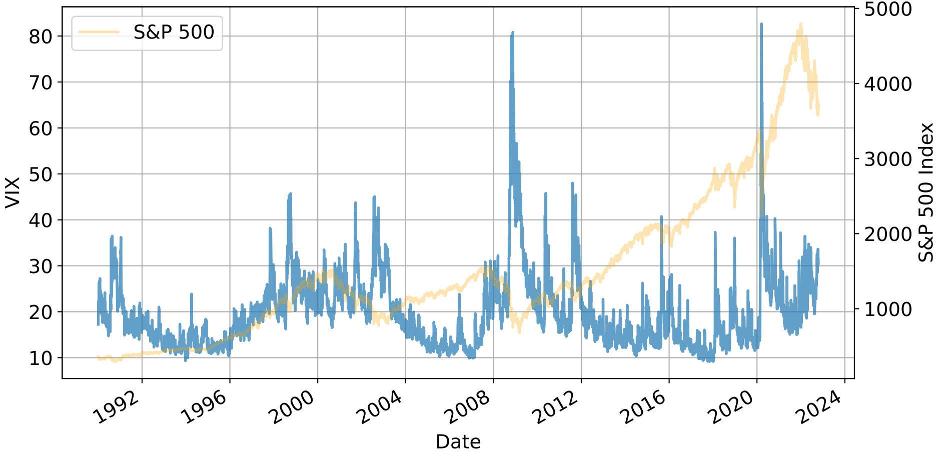
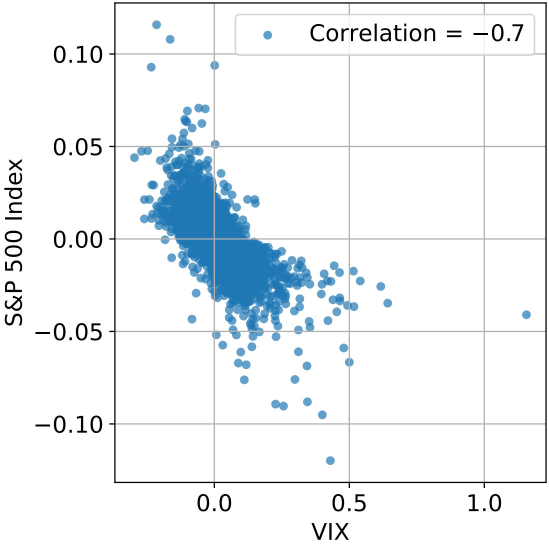


Illustration: Fear Gauge



Caculation Stages[3]

- Stage 1: determine the near-term and the next-term S&P 500 Index options.
- Stage 2: calculate the time to maturity for options.
- Stage 3: determine the forward price of options.
- Stage 4: calculate the variance by Equation (3).
- Stage 5: calculate the resulting VIX Index.

[3] See Volatility Index Methodology: CBOE Volatility Index.

Stage 1

- First determine the near-term and the next-term options which reflect the near 30-day expected volatility of the stock market.
- Assume that D is 2019-3-20.
  - For $D + 24$ (2019-4-13), no option expires.
  - For $D + 30$ (2019-4-19), S&P 500 monthly options are used as the near-term option because they expire on 2019-4-17.
  - For $D + 37$ (2019-4-26), S&P 500 weekly options expire on 2019-4-24 (4-th week) so they are used as the next-term options.

Stage 2

- Calculate the time to maturity, which is

$$T = \frac{M_1 + M_2 + M_3}{M_4},$$

where

  - M_1: minutes remaining until midnight of the current day,
  - M_2: minutes from the midnight to 0830 in the next morning for monthly option (, or 1500 in the next afternoon for weekly options),
  - M_3: total minutes in the days between current day and expiration day,
  - M_4: minutes in one year.

- For example,
  - $T_{\text{near}} = (717 + 510 + 43200) \,/\, 525600 = 0.084526$, and
  - $T_{\text{next}} = (717 + 900 + 53280) \,/\, 525600 = 0.104446$.

Stage 3

- The forward price F can be calculated by

$$F = K^* + e^{rT}\left(C(K^*) - P(K^*)\right),$$

with the ATM strike price

$$K^* = \arg\min_K |\, C(K) - P(K) \,|.$$

- For example,

Near Term Options				Next Term Options			
Strike Price	Call	Put	Difference	Strike Price	Call	Put	Difference
1955	27.60	19.75	7.85	1950	34.05	21.60	12.45
1960	24.25	21.30	2.95	1955	30.60	23.20	7.40
1965	21.05	23.15	2.10	1960	27.30	24.90	2.40
1970	18.10	25.05	6.95	1965	24.15	26.90	2.75
1975	15.25	27.30	12.05	1970	21.10	28.95	7.85

$$F_{near} = 1965 + e^{(0.0305\times 0.084526)} \times (21.05 - 23.15) = 1962.8947$$
$$F_{next} = 1960 + e^{(0.0286\times 0.104446)} \times (27.30 - 24.90) = 1962.4070$$

Elimination Rule

Put Strike	Bid	Ask	Include?	Call Strike	Bid	Ask	Include?
1345	0	0.15	N	2095	0.05	0.35	Y
1350	0.05	0.15	N	2100	0.05	0.15	Y
1355	0.05	0.35	N	2120	0	0.15	N
1360	0	0.35	N	2125	0.05	0.15	Y
1365	0	0.35	N	2150	0	0.1	N
1370	0.05	0.35	Y	2175	0	0.05	N
1375	0.1	0.15	Y	2200	0	0.05	N
1380	0.1	0.2	Y	2225	0.05	0.1	N
				2250	0	0.05	N

- Start from the strike closest to the forward price.
- Include all OTM options until the consecutive two zero bid prices occur.
- Ignore the options which has a bid price of zero.

Near term Strike	Option Type	Mid-quote Price
1370	Put	0.2
1375	Put	0.125
1380	Put	0.15
.	.	.
1950	Put	18.25
1955	Put	19.75
1960	Put/Call Average	22.775
1965	Call	21.05
1970	Call	18.1
.	.	.
2095	Call	0.2
2100	Call	0.1
2125	Call	0.1

Next term Strike	Option Type	Mid-quote Price
1275	Put	0.075
1325	Put	0.15
1350	Put	0.15
.	.	.
1950	Put	21.60
1955	Put	23.20
1960	Put/Call Average	26.10
1965	Call	24.15
1970	Call	21.10
.	.	.
2125	Call	0.1
2150	Call	0.1
2200	Call	0.08

- Calculate the mid-quote price by simply averaging the best bid price and the best ask price.

Stage 4

- For $i \in \{\text{near}, \text{next}\}$,

$$\sigma_i^2 = \frac{2}{T_i} \sum_{j=1}^{n_i} \frac{\Delta K_j}{K_j^2} e^{r_i T_i} Q(K_j) - \frac{1}{T_i} \left(\frac{F_i}{K_{0,i}} - 1 \right)^2, \qquad (4)$$

  where n_i is the number of selected options, $Q(K_j)$ is the price quote with strike K_j, and $K_{0,i}$ is the ATM strike price.
- Note that Equation (4) is identical to Equation (3) except that we need both the near/next terms.

Near term Strike	Option Type	Mid-quote Price	Contribution by Strike
1370	Put	0.2	0.0000005328
1375	Put	0.125	0.0000003306
1380	Put	0.15	0.0000003938
.	.	.	.
1950	Put	18.25	0.0000239979
1955	Put	19.75	0.0000258376
1960	Put/Call Average	22.775	0.0000296432
1965	Call	21.05	0.0000272588
1970	Call	18.1	0.0000233198
.	.	.	.
2095	Call	0.2	0.0000002278
2100	Call	0.1	0.0000003401
2125	Call	0.1	0.0000005536

Near term Strike	Option Type	Mid-quote Price	Contribution by Strike
1275	Put	0.075	0.0000023069
1325	Put	0.15	0.0000032041
1350	Put	0.15	0.0000020577
.	.	.	.
1950	Put	21.6	0.0000284031
1955	Put	23.2	0.0000303512
1960	Put/Call Average	26.1	0.0000339711
1965	Call	24.15	0.0000312732
1970	Call	21.1	0.0000271851
.	.	.	.
2125	Call	0.1	0.0000005536
2150	Call	0.1	0.0000008113
2200	Call	0.075	0.0000007748

$$\sigma_{near}{}^2 = \frac{2}{0.084526} \times 0.00063364 - \frac{1}{0.084526}\left[\frac{1962.6947}{1960} - 1\right]^2 = 0.0149671$$

$$\sigma_{next}{}^2 = \frac{2}{0.104446} \times 0.00083382 - \frac{1}{0.104446}\left[\frac{1962.4070}{1960} - 1\right]^2 = 0.0159520$$

Stage 5

- Let $N_{30} = 43200$ be the minutes in 30 days, $N_{365} = 525600$ be the minutes of one year, and $N_i = M_1 + M_2 + M_3$ for $i \in \{\text{near}, \text{next}\}$.
- The VIX formula is

$$\text{VIX} = 100\sqrt{\frac{N_{365}}{N_{30}}\left(T_{\text{near}}\sigma^2_{\text{near}}\omega_{\text{near}} + T_{\text{next}}\sigma^2_{\text{next}}\omega_{\text{next}}\right)}, \qquad (5)$$

where $\omega_{\text{near}} = \frac{N_{T_{\text{next}}} - N_{30}}{N_{T_{\text{next}}} - N_{T_{\text{near}}}} \approx 1.12172$ and $\omega_{\text{next}} = 1 - \omega_{\text{near}}$.

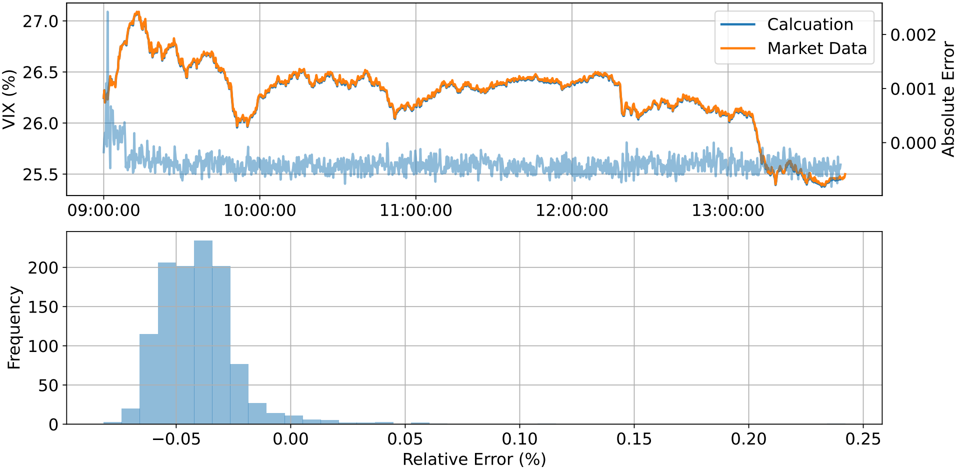
- Hence we have

$$\begin{aligned}\text{VIX} &= 100 \times \sqrt{0.0845 \times 0.01487 \times \omega_{\text{near}} + 0.1044 \times 0.01595 \times \omega_{\text{next}}} \\ &= 12.164787.\end{aligned}$$

Remarks for Calculation of TAIWAN VIX

- Use monthly options (expiration on 3rd Wed.) for the near/next term options.
- Use TAIBOR for the risk-free rate (r).
- No elimination rule.
  - Following this will produce numbers smaller than the market data. (Why?)
- CBOE VIX exploits implied forward price (by put-call parity) in the calculation.
- We find that TAIWAN VIX becomes more stable when using TX futures prices.

Example: TAIWAN VIX Index



Volatility of Volatility

- In 2012, the CBOE introduced the VVIX index (also referred to as "vol of vol"), a measure of the VIX's expected volatility.
- VVIX is calculated using the same methodology as VIX, except the inputs are market prices for VIX options instead of stock market options.
- The VIX can be thought of as the velocity of investor fear while the VVIX can be thought of as the acceleration of investor fear.

- a photo?

Papers

- TBA

Books

- CBOE Volatility Index (VIX Index): http://www.cboe.com/publish/methodology-volatility/vix_methodology.pdf 2019-07-26.

Lectures

- TBA