Tutorial on Itô's Formula

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Agenda

- Wiener process.
- Generalized Wiener process (Itô process).
- Itô integral.
- Martingale.
- Quadratic variation of Wiener process.
- Itô's formula.
- Calculation examples.
- Black-Scholes option pricing theory.

Wiener Process

- A process W is called the Wiener process if the following conditions hold:
  (1) $W(0) = 0$.
  (2) The process W has independent increments: if $r < s \leq t < u$, then $W(u) - W(t) \perp W(s) - W(r)$.
  (3) For $s < t$, $W(t) - W(s) \sim N(0, t - s)$.
  (4) W has continuous trajectories.
- Note that W has a nowhere-differentiable trajectory (see the next page).

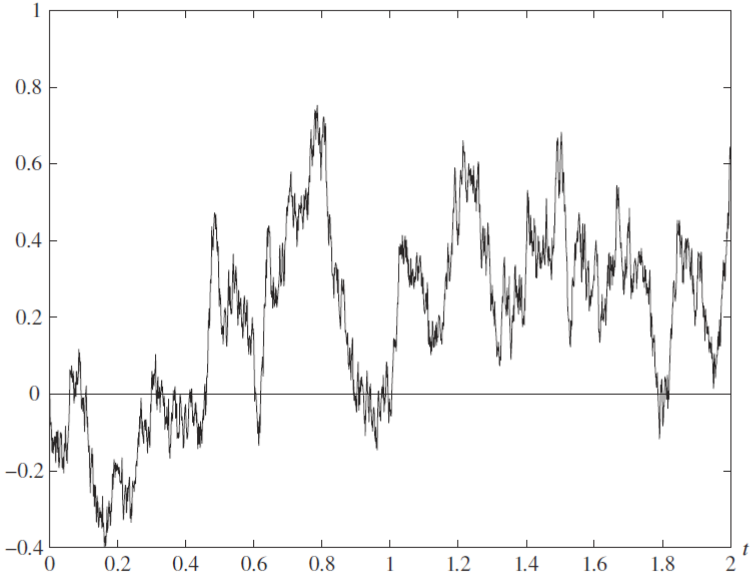

FIG. 4.1. A Wiener trajectory

Itô Process

- A stochastic process $X(t)$ is given by

$$X(t) = a + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \quad (1)$$

  where a is the initial condition of $X(0)$, $\mu(t, X(t))$ and $\sigma(t, X(t))$ are two adapted[1] processes, and $W(t)$ is a Wiener process.
- The third item at the right-hand side of Equation (1) is to be defined.

[1] Let X and Y be stochastic processes. Y is adapted to $\mathcal{F}_t^X$-filtration if Y is $\mathcal{F}_t^X$-measurable.

Itô Integral

- Let g be a process satisfying the following conditions:
  - g is square-integrable, that is,

$$\int_a^b \mathbf{E}\left[g^2(s)\right] ds < \infty.$$

  - g is adapted to the $\mathcal{F}_t^W$-filtration.
- We define the Itô integral as follows:

$$\int_a^b g(s) dW(s) \triangleq \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} g(t_k) \left[W(t_{k+1}) - W(t_k)\right].$$

  - Why using the forward increments?
  - Because we cannot foresee the future.

- Then the following relations hold:

$$\mathbf{E}\left[\int_a^b g(s)dW(s)\right] = 0, \tag{2}$$

$$\mathbf{E}\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = \int_a^b \mathbf{E}\left[g^2(s)\right] ds, \tag{3}$$

and $\int_a^b g(s)dW(s)$ is $\mathcal{F}_b^W$-measurable.[2]

[2]We could say that the integral is deterministic at time b.

Sketch of Proof for Equation (2)

$$\begin{aligned}
\mathbf{E}\left[\int_a^b g(s)dW(s)\right] &\approx \mathbf{E}\left[\sum_{k=0}^{n-1} g(t_k)\Delta W(t_k)\right] \\
&= \sum_{k=0}^{n-1} \mathbf{E}\left[g(t_k)\right]\mathbf{E}\left[\Delta W(t_k)\right] \;\; (\because W(t_k) \perp \Delta W(t_k)) \\
&= 0. \; (\because \mathbf{E}[\Delta W(t_k)] = 0)
\end{aligned}$$

Sketch of Proof for Equation (3)

- For all i, j with $i \neq j$, we first calculate

$$\mathbf{E}\left[\Delta W(t_i)\Delta W(t_j)\right] = \mathbf{E}\left[\Delta W(t_i)\right]\mathbf{E}\left[\Delta W(t_j)\right] = 0.$$

- Note that the first equality results from the property of independent increments.
- Then Equation (3) is proved as follows.

$$\mathbf{E}\left[\left(\int_a^b g(s)dW(s)\right)^2\right] \approx \mathbf{E}\left[\left(\sum_{k=0}^{n-1} g(t_k)\Delta W(t_k)\right)^2\right]$$

$$= \sum_{k=0}^{n-1} \mathbf{E}\left[g^2(t_k)\right] \mathbf{E}\left[\Delta W^2(t_k)\right] +$$

$$\sum_i \sum_j \mathbf{E}\left[g(t_i)g(t_j)\right] \mathbf{E}\left[\Delta W(t_i)\Delta W(t_j)\right]$$

$$= \sum_{k=0}^{n-1} \mathbf{E}\left[g^2(t_k)\right] \mathbf{E}\left[\Delta W^2(t_k)\right]$$

$$= \sum_{k=0}^{n-1} \mathbf{E}\left[g^2(t_k)\right] (t_{k+1} - t_k)$$

$$\to \int_a^b \mathbf{E}\left[g^2(s)\right] ds.$$

Martingale[3]

- A stochastic process X is called an $\mathcal{F}_t$-martingale if the following condition hold.
  - For all t, $\mathbf{E}\left[\,|X(t)|\,\right] < \infty$.
  - X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
  - For all s and t with $s \leq t$, $\mathbf{E}\left[X(t)\,|\,\mathcal{F}_s\right] = X(s)$.
- Now let $X(t) = \int_0^t g(s)dW(s)$ with $0 \leq t' < t$.
- Then we have

$$
\begin{aligned}
\mathbf{E}\left[X(t)\middle|\mathcal{F}_{t'}^W\right] &= X(t') + \mathbf{E}\left[\int_{t'}^{t} g(s)dW(s)\middle|\mathcal{F}_{t'}^W\right] \\
&= X(t').
\end{aligned}
$$

- By Equation (2), every stochastic integral is a martingale.

[3] It is a notion of fair games.

Digression: Is the Market a Martingale?

- For stock markets, the stock prices are not martingales.
- Consider that you deposit $S(0)$ in the bank with $r \geq 0$.
- Then $S(t) = S(0)e^{rt}$, which is riskless.
- Or we rewrite the equation above like

$$\mathbf{E}[S(t) \mid \mathcal{F}_0] = e^{rt} S(0) \geq S(0).$$

- This implies that the riskless asset is a submartingale.
- Because of risk aversion, one should expect a higher return for taking higher risk, that is,

$$\mathbf{E}[S'(t) \mid \mathcal{F}_0] > e^{rt} S'(0),$$

where $S'(t)$ is a process of one risky asset.

Digression: Risk-Neutral Valuation & Martingale

- Under a physical measure $\mathbb{P}$, it is known that

$$\mathbf{E}^{\mathbb{P}}\left[S'(t)\middle|\mathcal{F}_0\right] > e^{rt}S'(0).$$

- Let $Y(t) = e^{-rt}S'(t)$.
- Under the risk-neutral measure $\mathbb{Q}$, the discounted asset price is a martingale because

$$\mathbf{E}^{\mathbb{Q}}\left[Y(t)|\mathcal{F}_0\right] = Y(0).$$

- This result is used to price derivatives as follows:

$$p = \mathbf{E}^{\mathbb{Q}}\left[e^{-rT}\Pi(S'(T))\middle|\mathcal{F}_0\right],$$

where p is the derivative price and Π is a stochastic contingent claim for S' with the time to maturity T.

Quadratic Variation of Wiener Process

- Define $\Delta t = t - s$ and $\Delta W = W(t) - W(s)$ with $s < t$.
- By definition, we have
  - $\mathbf{E}[\Delta W] = 0$, $(\because \Delta W \sim N(0, \Delta t))$
  - $\mathbf{Var}[(\Delta W)] = \Delta t$.
- Now we are interested in the quadratic variation $(\Delta W)^2$, which has:
  - $\mathbf{E}[(\Delta W)^2] = \Delta t$,
  - $\mathbf{Var}[(\Delta W)^2] = 2(\Delta t)^2$. (Why?)
- This is because the trajectory of W is rough!
- In differential form, it reads

$$(dW)^2 = dt.$$

- This identity will be used in the Itô's formula.

Itô Formula

- For convenience, notations are simplified unless necessary.
    - For example, $X(t)$ and $\mu(t, X(t))$ are replaced by X and μ, respectively.
- In a differential form, Equation (1) is equivalent to

$$dX = \mu dt + \sigma dW. \tag{4}$$

- Let f be a C^2-function.[4]
- Define the process Z by $Z = f(t, X)$.
- Then Z has a stochastic differential given by

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW. \tag{5}$$

---

[4] The function f is said to be of (differentiability) class C^k if the derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous.

Sketch of Proof for Itô Formula

- It is known that the second-order Taylor expansion for f is

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 \boxed{+ \frac{\partial^2 f}{\partial t \partial x} dt dX + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2}.$$

- We then calculate $(dX)^2$ with the identity $(dW)^2 = dt$ so that

$$\begin{aligned} (dX)^2 &= \boxed{\mu^2 (dt)^2 + 2\mu\sigma dt dW} + \sigma^2 (dW)^2 \\ &\sim \sigma^2 dt. \end{aligned}$$

- Note that $\boxed{\cdots}$ is negligible compared to the dt-term.

- As a result,

$$
\begin{aligned}
df &= \boxed{\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dW) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}dt \\
&= \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW.
\end{aligned}
$$

- Hence the proof is complete.
- Note that $\boxed{\cdots}$ is used as the second form of Itô's formula.

Example 1: $\mathbf{E}\left[W^4(t)\right] = ?$

- Define Z by $Z(t) = W^4(t)$.
- Then we have
$$\frac{\partial Z}{\partial W} = 4W^3,$$
and
$$\frac{\partial^2 Z}{\partial W^2} = 6W^2.$$
- By the Itô formula,
$$dZ = 6W^2 dt + 4W^3 dW \text{ with } Z(0) = 0.$$

- Written in integral form, this reads

$$Z(t) = 0 + 6\int_0^t W^2(s)ds + 4\int_0^t W^3(s)dW(s).$$

- Taking the expected value on the equation above, the stochastic-integral term will vanish.
- So we have

$$\mathbf{E}\left[W^4(t)\right] = 6\int_0^t \mathbf{E}\left[W^2(s)\right]ds = 6\int_0^t sds = 3t^2.$$

- Note that the exchange between doing an integration and taking an expected value works in most cases of financial math.[5]
- This result could be used to prove $\mathbf{Var}[(\Delta W)^2] = 2(\Delta t)^2$.

[5] See Fubini's theorem.

Example 2: $\mathbf{E}\left[e^{\alpha W(t)}\right] = ?$

- Define Z by $Z(t) = e^{\alpha W(t)}$ with $Z(0) = 1$.
- The Itô formula gives us

$$\begin{aligned} dZ(t) &= \frac{1}{2}\alpha^2 e^{\alpha W(t)} dt + \alpha e^{\alpha W(t)} dW \\ &= \frac{1}{2}\alpha^2 Z(t) dt + \alpha Z(t) dW(t). \end{aligned}$$

- In integral form, this reads

$$Z(t) = 1 + \frac{1}{2}\alpha^2 \int_0^t Z(s) ds + \alpha \int_0^t Z(s) dW(s).$$

- Why bother?[6]

[6] One can rewrite the stochastic process in form of $\boxed{\cdots} dt + \boxed{\cdots} dW$ via the Itô formula. Starting from this form, it is easier to derive the expected values associated with the stochastic process. For most time, you cannot derive these expected values without this form.

- Now define $m(t) = \mathbf{E}[Z(t)]$ and differentiate the resulting equation as follows:

$$dm(t) = \frac{1}{2}\alpha^2 m(t)dt.$$

- Using the ODE technique[7], we have

$$m(t) = \mathbf{E}\left[e^{\alpha W(t)}\right] = e^{\frac{1}{2}\alpha^2 t}.$$

- Note that $\mathbf{E}\left[e^{\alpha W(t)}\right]$ is the moment-generating function (MGF)[8] of $W(t)$ so that you may follow the definition of MGF to produce the same result.

[7] To be more specific, you need the identity $\frac{dx}{x} = d\ln x$.

[8] See https://en.wikipedia.org/wiki/Moment-generating_function.

Example 3: $\int_0^t W(s)dW(s) = ?$

- Define Z by $Z(t) = W^2(t)$.
- By the Itô formula,

$$dZ(t) = dt + 2W(t)dW(t).$$

- In integral form this reads

$$Z(t) = W^2(t) = t + 2\int_0^t W(s)dW(s).$$

- So we have

$$\int_0^t W(s)dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}.$$

- The second term in the RHS differs from the ordinary calculus!

Example 4: Geometric Brownian Motion (GBM)

- Let μ and σ be constant, and W be under the $\mathbb{P}$ measure.
- A GBM is given by

$$dS = \mu S dt + \sigma S dW.$$

- Now take $X = \ln S$ with $X(0) = \ln S_0$.
- It is easy to see that

$$\frac{\partial X}{\partial S} = \frac{\partial (\ln S)}{\partial S} = \frac{1}{S},$$

and

$$\frac{\partial^2 X}{\partial S^2} = \frac{\partial^2 (\ln S)}{\partial S^2} = -\frac{1}{S^2}.$$

- By the Itô's formula,

$$
\begin{aligned}
dX &= \frac{\partial(\ln S)}{\partial t}dt + \frac{\partial(\ln S)}{\partial S}dS + \frac{1}{2}\frac{\partial^2(\ln S)}{\partial S^2}(dS)^2 \\
&= \frac{1}{S}dS + \frac{1}{2}\left(\frac{-1}{S^2}\right)S^2\sigma^2 dt \\
&= \frac{1}{S}(\mu S dt + \sigma S dW) - \frac{1}{2}\sigma^2 dt. \\
&= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW.
\end{aligned}
$$

- In integral form, this reads

$$\begin{aligned} \ln S &= \ln S_0 + \int_0^t (\mu - \frac{1}{2}\sigma^2)dt + \int_0^t \sigma dW \\ &= \ln S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t). \end{aligned}$$

- This gives us

$$\ln S(t) \sim N\left(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right).$$

- Note that the price volatility of one asset is $\sigma\sqrt{t}$.

- In the end, we have

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)},$$

which follows a so-called lognormal distribution with

$$\mathbf{E}\left[S(t)\right] = S_0 e^{\mu t},$$
$$\mathbf{Var}\left[S(t)\right] = S_0^2 \left(e^{(2\mu+\sigma^2)t} - e^{2\mu t}\right).\text{(Why?)}$$

Exercise: Futures Price

- Assume that $S(t)$ follows a GBM.
- It is known that the futures price $F(t)$ is given by

$$F(t) = S(t)e^{r(T-t)}.$$

- By the Itô's formula,

$$dF = (\mu - r)Fdt + \sigma FdW.$$

- If we shift to the $\mathbb{Q}$ measure (i.e., μ is replaced by r), then

$$dF = \sigma FdW$$

with $\mathbf{E}\left[F(t)\right] = F_0$, which is a martingale.

Exercise: Product of GBM Processes

- Let Y and Z be two GBM processes:

$$\frac{dY}{Y} = a\,dt + b\,dW_Y,$$
$$\frac{dZ}{Z} = f\,dt + g\,dW_Z,$$

  where dW_Y and dW_Z has correlation ρ.
- Consider the product of two GBM processes,

$$U = YZ.$$

- By the Itô's formula,

$$
\begin{aligned}
dU &= Z\,dY + Y\,dZ + dY\,dZ \\
&= YZ(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) + \\
&\qquad\qquad\qquad\qquad\qquad YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\
&= U\left[(a + f + bg\rho)dt + b\,dW_Y + g\,dW_Z\right].
\end{aligned}
$$

- Rewrite the above equation as below:

$$
\frac{dU}{U} = (a + f + bg\rho)dt + b\,dW_Y + g\,dW_Z.
$$

- We show that the product of correlated GBM processes thus remains a GBM.
- In particular, we can also show that S^n is also a GBM process for $n \in \mathbb{N}$.

Exercise: Quotients of GBM Processes

- Consider the quotient of two GBM processes,

$$U = \frac{Y}{Z},$$

  where Y and Z are drawn from Example 6.

- By the Itô formula,

$$\begin{aligned} dU &= \frac{1}{Z}dY - \frac{Y}{Z^2}dZ - \frac{1}{Z^2}dYdZ + \frac{Y}{Z^3}(dZ)^2 \\ &\vdots \\ &= U\left[(a - f + g^2 - bg\rho)dt + b\,dW_Y - g\,dW_Z\right]. \end{aligned}$$

- This example reminds us to collect all dt-terms.

Example 5: Vasicek Model[9]

- X is a Vasicek process, defined by

$$dX = \kappa(\theta - X)dt + \sigma dW,$$

  with $\theta, \kappa, \sigma > 0$.
- Let $Y = e^{\kappa t}X$.
- By the Itô's formula, we then have

$$\begin{aligned} dY &= \kappa e^{\kappa t}Xdt + e^{\kappa t}dX \\ &= \kappa e^{\kappa t}Xdt + e^{\kappa t}(\kappa(\theta - X)dt + \sigma dW) \\ &= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW. \end{aligned}$$

[9] Vasicek (1977). It is one of extension of the Ornstein-Uhlenbeck process, proposed by Ornstein and Uhlenbeck in 1930. Now the Vasicek model is out-of-date. The main focus aims at the LIBOR market model (LMM).

- So it reads

$$e^{\kappa t} X = X_0 + \int_0^t \kappa\theta e^{\kappa s} ds + \int_0^t \sigma e^{\kappa s} dW.$$

- Moreover, we could calculate

$$\mathbf{E}[X] = X_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}),$$
$$\mathbf{Var}[X] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

- As $t \to 0$, it is easy to see that $\mathbf{E}[X] = X_0$ and $\mathbf{Var}[X] = 0$.
- As $t \to \infty$, $\mathbf{E}[X] = \theta$ and $\mathbf{Var}[X] = \frac{\sigma^2}{2\kappa}$, which is finite due to the mean-reverting property!
- Note that X is a process following a normal distribution. (Why?)

Black-Scholes Option Pricing Theory

- Assume that the stock price S_t follows a GBM (see p. 22).
- For this stock, we now consider to sell a European call option which expires in time T and has the payoff function

$$\Phi(S_T) = (S_T - K)^+ .$$

- Insert a figure as an illustration of options.

- Define the call price $C_t = f(t, S_t)$.
- By the Itô's formula,

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} (dS_t)^2 \\
&= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} (\mu S_t dt + \sigma S_t dW) + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 dt \\
&= \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2} \right) dt + \frac{\partial f}{\partial S_t} \sigma S_t dW.
\end{aligned}
$$

- What is the fair price of this call option?
- The no-arbitrage principle comes into play.

- Construct a riskless portfolio as follows: buy $\Delta = \frac{\partial f}{\partial S_t}$ shares of the stock and sell one European call.
- The portfolio value V is $V = \Delta \times S_t - f$.
- For a small variation of S_t,

$$dV = \Delta \times dS_t - df. \tag{6}$$

- If the market is free of arbitrage, then the risk-free asset must earn the risk-free rate, denoted by $r > 0$.
- This gives us

$$dV = rVdt = r(\Delta \times S_t - f)dt. \tag{7}$$

- Now equate (6) and (7):

$$r(\Delta \times S_t - f)dt = -\left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial S_t^2}\right)dt.$$

- Hence we derive the famous Black-Scholes PDE as follows:

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2} = rf. \tag{8}$$

- Define $\Delta = \frac{\partial f}{\partial S_t}$, $\Theta = \frac{\partial f}{\partial t}$, and $\Gamma = \frac{\partial^2 f}{\partial S_t^2}$.
- Then we have another representation of BS-PDE as follows:

$$\Theta + rS_t\Delta + \frac{\sigma^2 S_t^2}{2}\Gamma = rf.$$

- If one considers the delta neutral ($\Delta = 0$), then the previous equation becomes

$$\Theta + \frac{\sigma^2 S_t^2}{2}\Gamma = rf.$$

Feynman-Kac[10] Theorem

- This discovery bridges two research domains (PDE and SDE)!
- If $f(t,x)$ with $t \in [0, T]$ is a solution to

$$\frac{\partial f}{\partial t} + \mu(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2} = rf,$$
$$f(T,x) = \Phi(x),$$

then $f(t,x)$ has a representation

$$f(t,X) = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}\left[\,\Phi(X_T)\,|\,X_t = x\,\right],$$

where X follows a Itô process.

[10] Mark Kac (1914–1984), a Polish American mathematician.

- Now replace X by S.
- Then the call price is

$$C = f(0, S_0) = e^{-rT}\mathbf{E}^{\mathbb{Q}}[\,(S_T - X)^+\,].$$

- This is called risk-neutral valuation.
- The price of European call options is

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2), \tag{9}$$

where $N(\cdot)$ is a cdf of a standard normal distribution,

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

and $d_2 = d_1 - \sigma\sqrt{T}$.

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[11] See https://www.csie.ntu.edu.tw/~lyuu/finance1.html.