Monte Carlo Simulation for European Call Option

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Introduction

- We invent new financial instruments to hedge the risk of holding assets, which are volatile from time to time.
- The central question of financial engineering[1] is, "What is the fair price of one instrument?"
- The key idea is very simple: no free lunch.[2]
- We first inscribe the price behavior by stochastic processes.
- In particular, we assume that the return rate of one stock is a Brownian motion[3].

[1] See https://en.wikipedia.org/wiki/Financial_engineering and https://en.wikipedia.org/wiki/Rocket_science_in_finance.

[2] See https://en.wikipedia.org/wiki/Arbitrage.

[3] See https://en.wikipedia.org/wiki/Brownian_motion.

Brownian Motion

- The stock price S_t follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1}$$

where μ is the annual return rate, σ is the annualized volatility, and W_t is a Wiener process.[4]

- By the Itô calculus, the solution of Equation (1) is

$$S_{t+1} = S_t e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z_t},$$

where Δt is the time interval and $Z_t \sim N(0, 1)$.

[4] See https://en.wikipedia.org/wiki/Wiener_process.

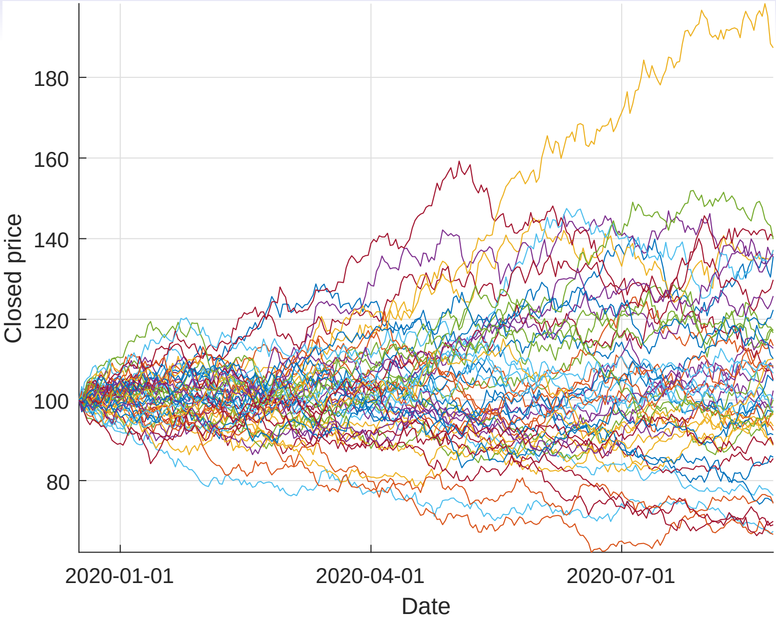
```
clear; clc; close all;

s0 = 100; mu = 0.05; v = 0.2;
T = 1; n = 252; dt = T / n;

A = (mu - 0.5 * v ^ 2) * dt;
B = v * sqrt(dt);

figure; grid on; hold on;
for i = 1 : 50 % 50 price paths
    ds = exp(A + B * randn(1, n));
    S = s0 * cumprod([1 , ds]);
 plot([0 : n] + today, S);
end
xlabel("Date"); ylabel("Closed price");
datetick("x", 29); axis tight;
```

- Use **cumprod** to calculate the cumulative product.
- Use **datetick** and assign dates to the x axis.


Closed price
180
160
140
120
100
80
2020-01-01
2020-04-01
2020-07-01
Date

European Call Option[6]

- A European option is an option that can only be exercised at the strike price X at maturity time T.
- If the asset follows Equation (1) with the spot price S_0, then the call price c is

$$c = e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[(S_T - X)^+\right],$$

where r is the risk-free interest rate and $\mathbf{E}(\cdot)$ is the expected value of the payoff function $(S_T - X)^+$ under the risk-neutral probability measure $\mathbb{Q}$.[5]

[5] Also see https://en.wikipedia.org/wiki/Black_Scholes_model#Black_Scholes_formula.

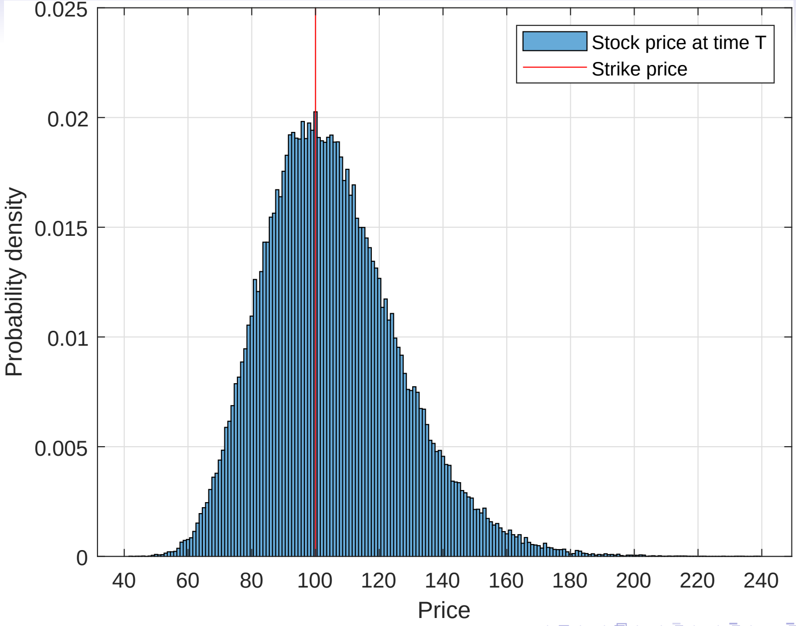
[6] Black and Scholes (1972) and Merton (1973). Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work.

```
clear; clc; close all;

s0 = 100; mu = 0.05;
v = 0.2; T = 1; n = 1e5;
A = (mu - 0.5 * v ^ 2) * T;
B = v * sqrt(T);
ST = s0 * exp(A + B * randn(1, n));

histogram(ST, 20, "binmethod", "integer", ...
                  "Normalization", "probability");
xlabel("Price");
ylabel("Probability density");
hold on; grid on;

X = 100; % strike price
line([X, X], [0, 0.025], "color", "red");
legend({"Stock price at time T", "Strike price"});
```


0.025
0.02
0.015
0.01
0.005
0
Probability density
40
60
80
100
120
140
160
180
200
220
240
Price
Stock price at time T
Strike price

```
...

% MC simulation
c = 0;
for i = 1 : n
    ST = s0 * exp(A + B * randn(1));
    if ST > X
        c = c + ST - X;
    end
end
c = exp(-mu * T) * c / n

% built-in function: see Financial Toolbox
blsprice(s0, X, mu, T, v)
```

- The simulation produces the call price about 10.4579 while the Black-Scholes formula gives us the call price 10.4506.

Vectorized Version

```
clear; clc;

s0 = 100; X = 100; mu = 0.05;
v = 0.2; T = 1; n = 1e6;
A = (mu - 0.5 * v ^ 2) * T;
B = v * sqrt(T);
ST = s0 * exp(A + B * randn(1, n));

% MC simulation by vectorization
c = exp(-mu * T) * sum(ST(ST > X) - X) / n
```