Camera calibration

Digital Visual Effects

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with slides by Richard Szeliski, Steve Seitz,, Fred Pighin and Marc Pollefyes

Outline

- Camera projection models
- Camera calibration
- Nonlinear least square methods
- A camera calibration tool
- Applications

Camera projection models

Pinhole camera

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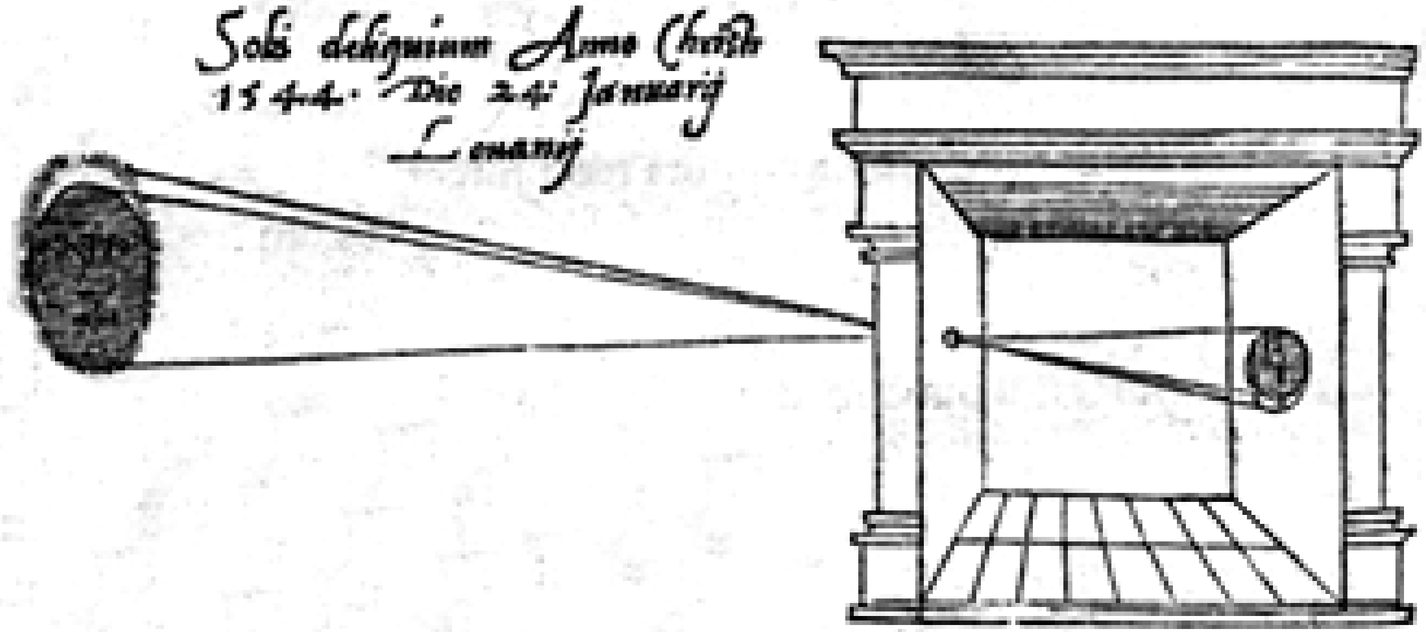

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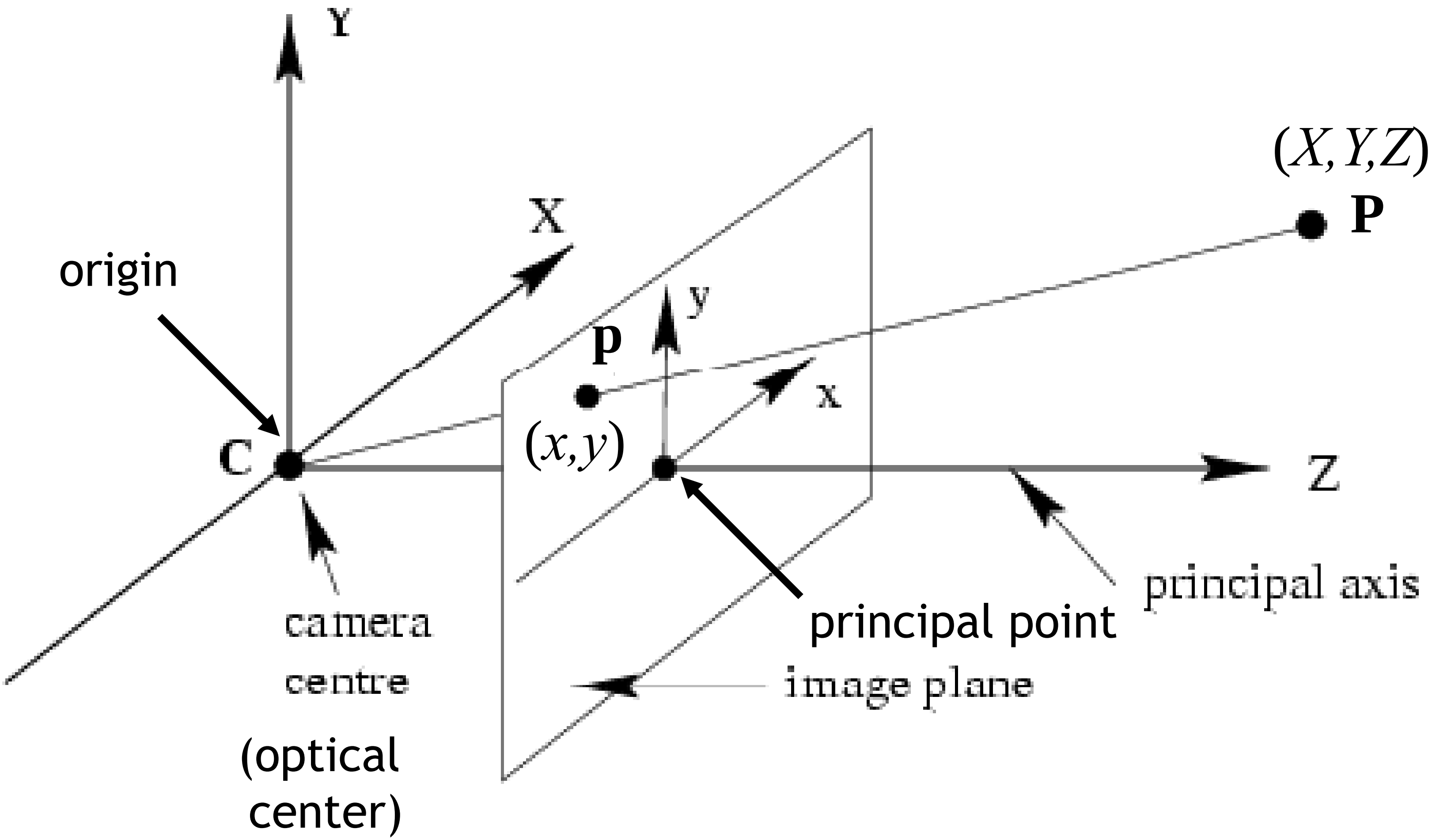
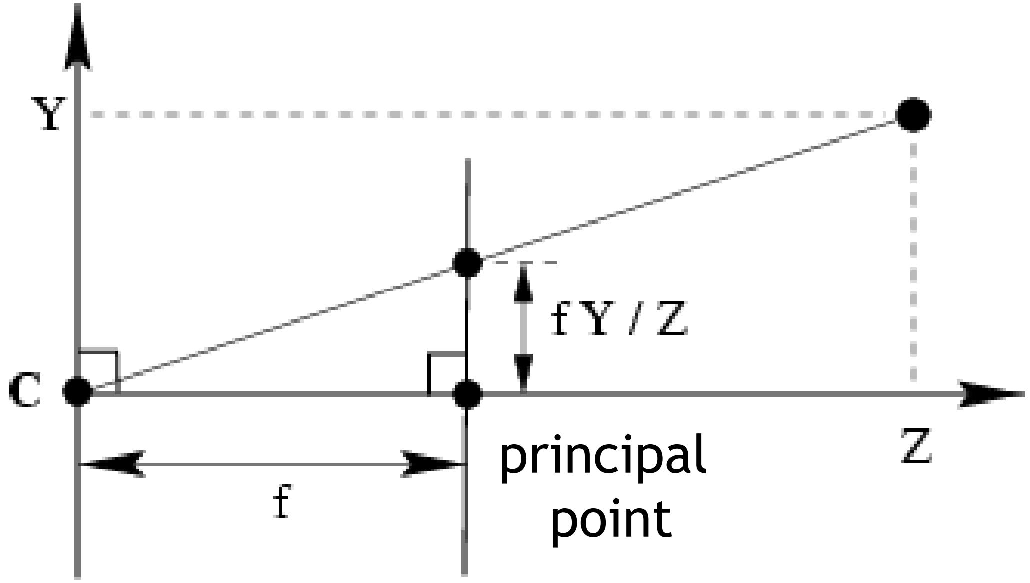
Pinhole camera model



Pinhole camera model

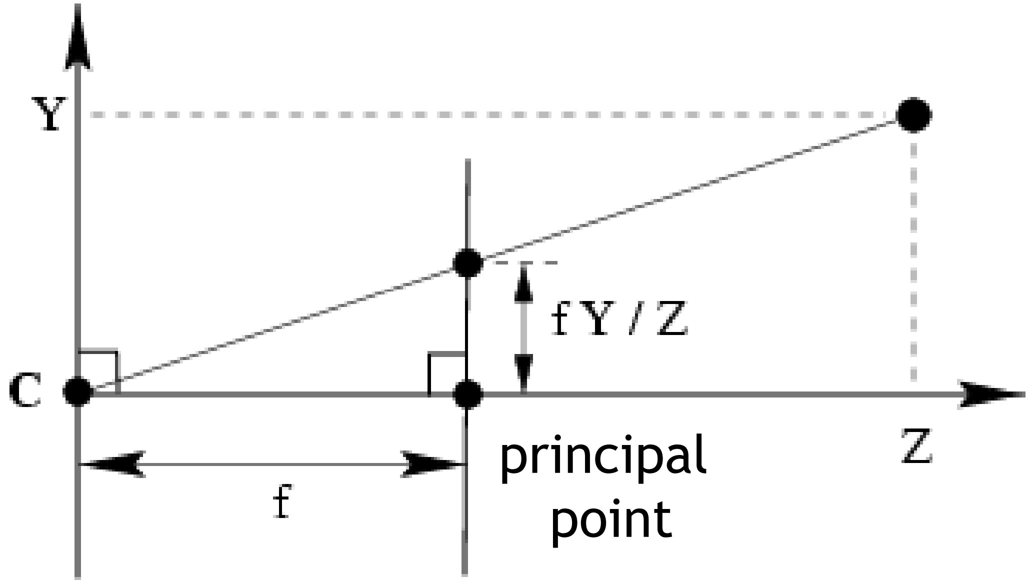


$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

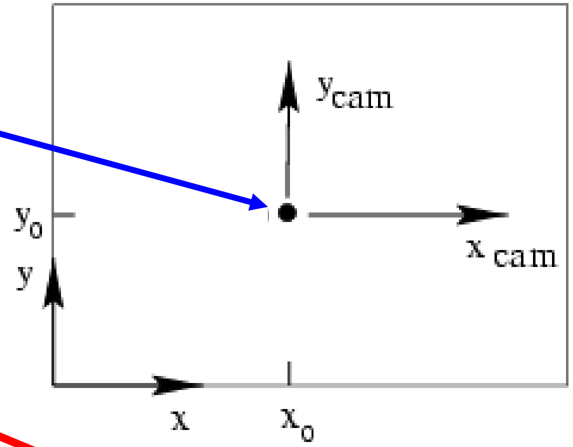
Pinhole camera model



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset

principal point



intrinsic matrix

only related to camera projection

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Intrinsic matrix

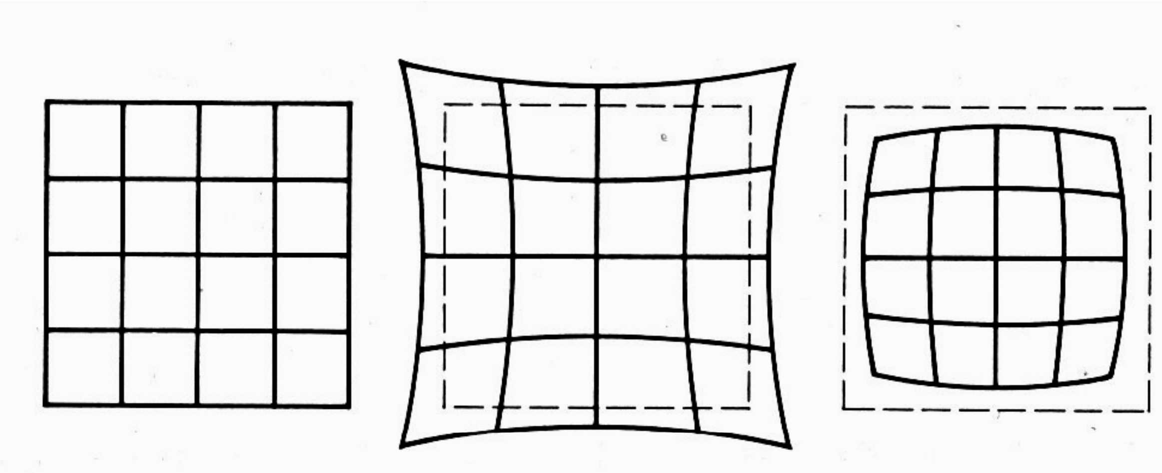
Is this form of $\mathbf{K}$ good enough?

$$\mathbf{K} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

- non-square pixels (digital video)
- skew
- radial distortion

$$\mathbf{K} = \begin{bmatrix} fa & s & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

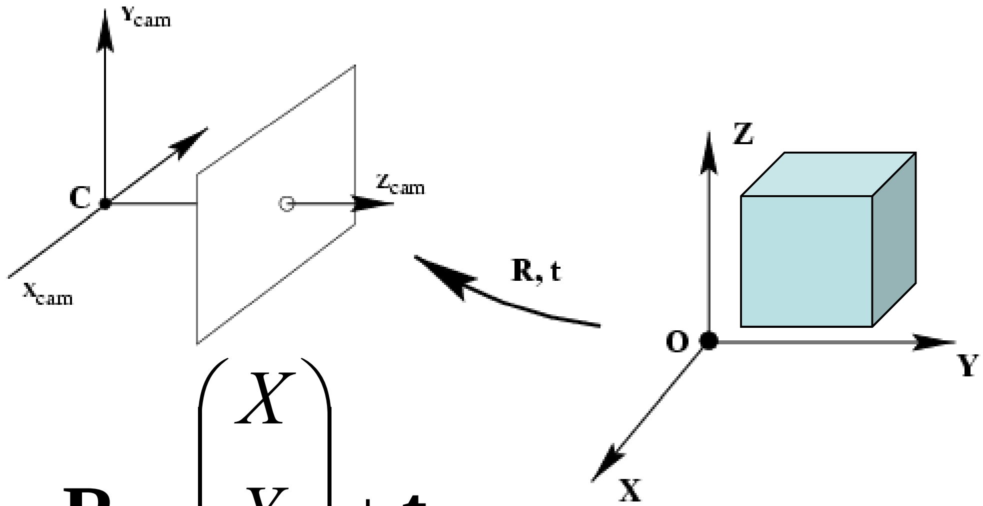
Distortion



No distortion   Pin cushion   Barrel

- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

Camera rotation and translation



$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}_{3\times 3} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mathbf{t}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{R} | \mathbf{t}] \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} \sim \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}$$

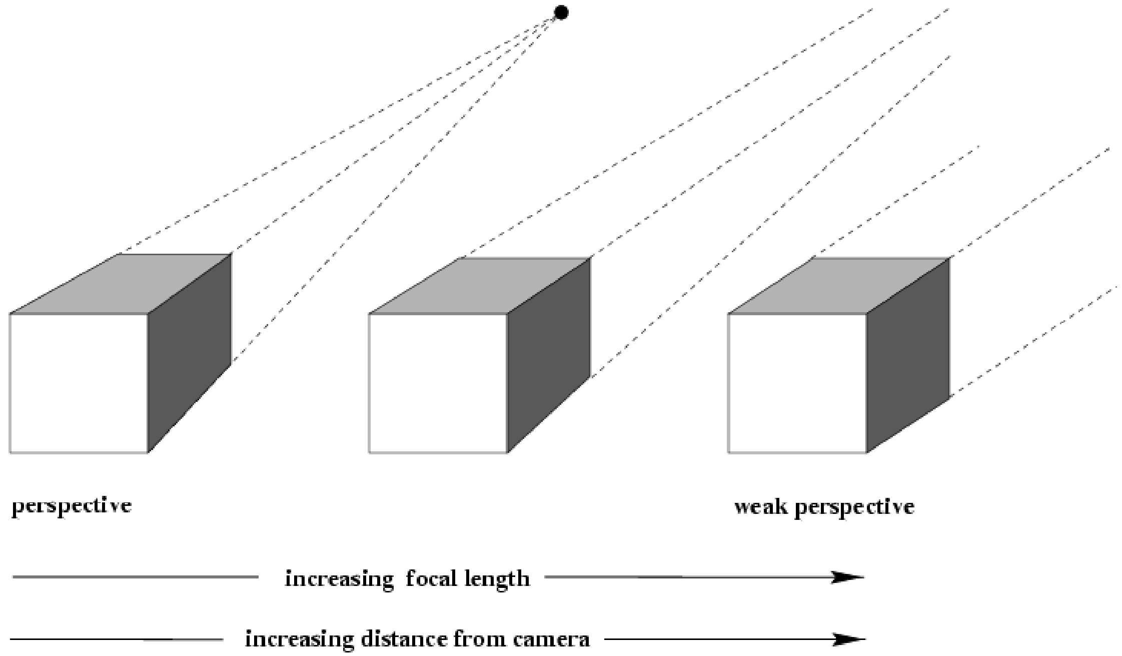
extrinsic matrix

Two kinds of parameters

DigiVFX

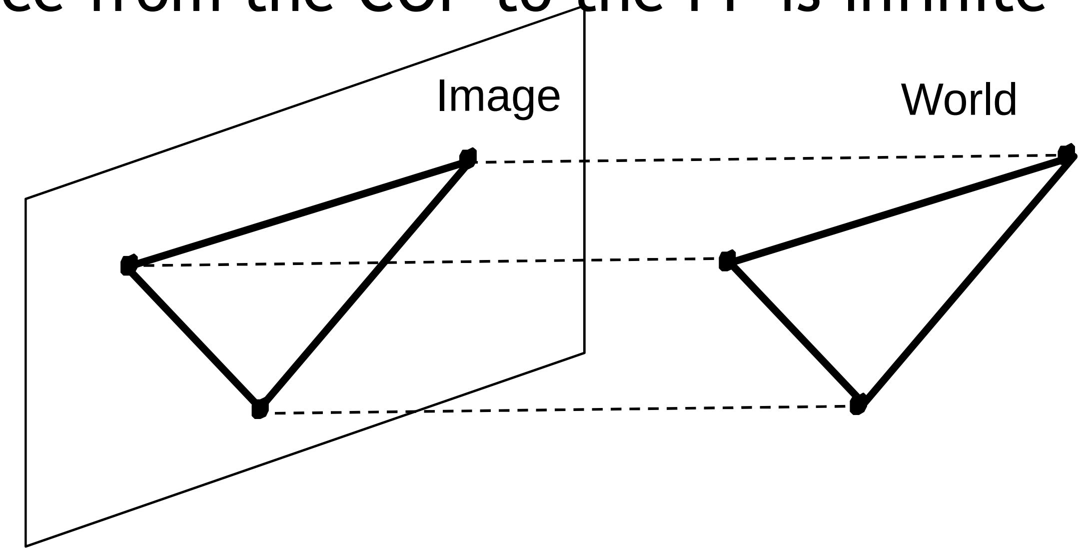
- *internal* or *intrinsic* parameters such as focal length, optical center, aspect ratio: *what kind of camera*?
- *external* or *extrinsic* (pose) parameters including rotation and translation: *where is the camera*?

Other projection models



Orthographic projection

- Special case of perspective projection
  - Distance from the COP to the PP is infinite



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

  - Also called "parallel projection": (x, y, z) $\rightarrow$ (x, y)

Other types of projections

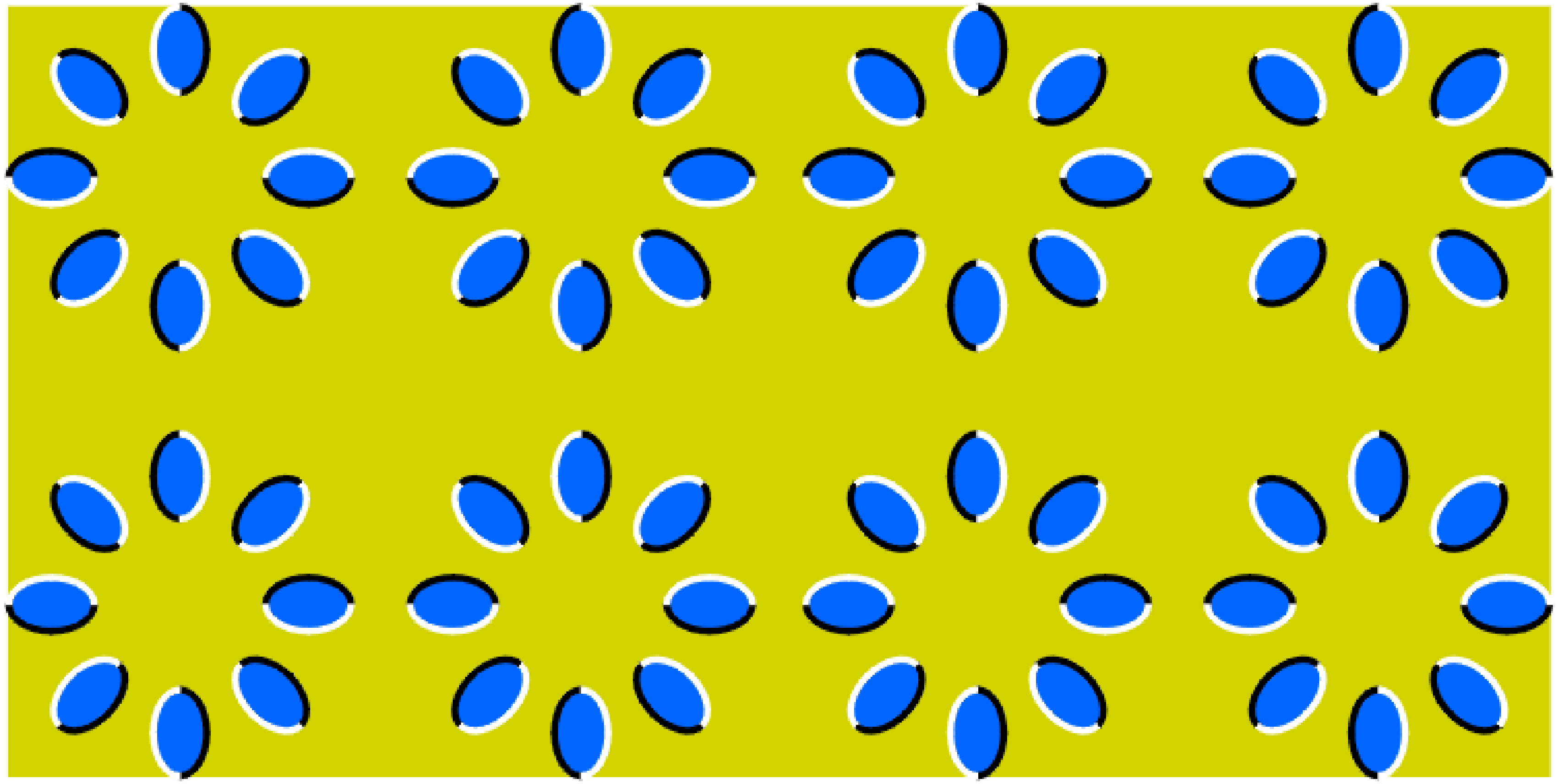
- Scaled orthographic
  - Also called “weak perspective”

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/d \end{bmatrix} \Rightarrow (dx, dy)$$

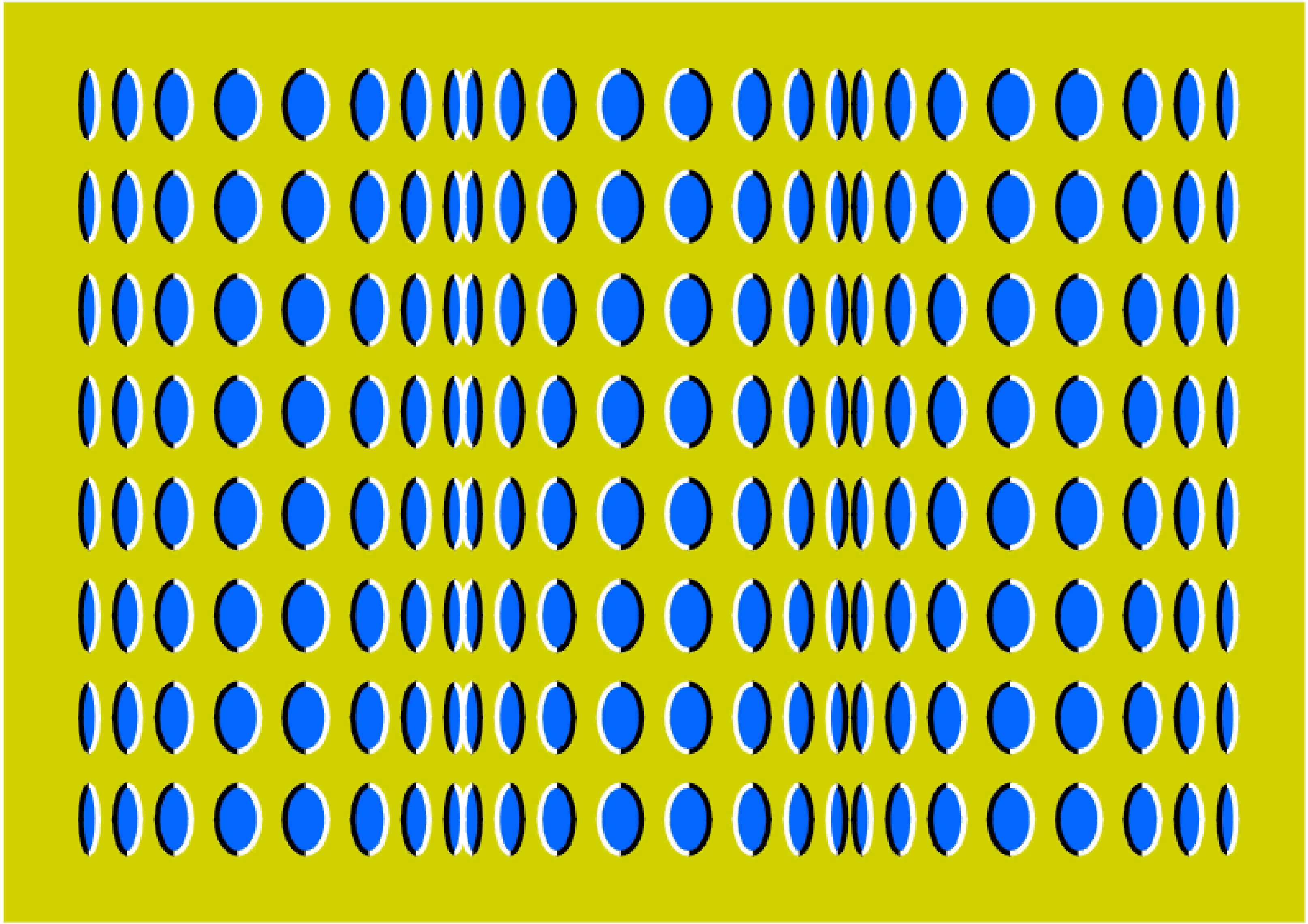
- Affine projection
  - Also called “paraperspective”

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

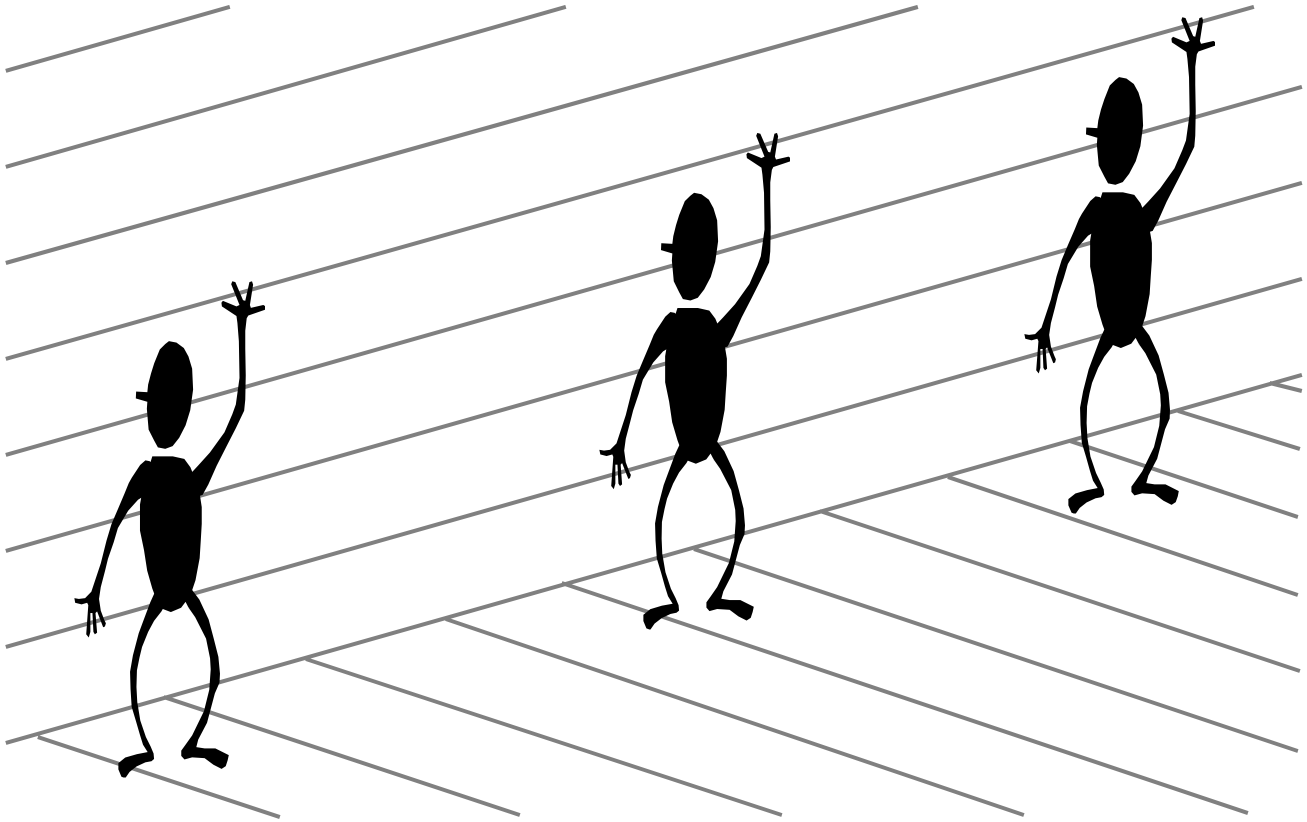
Illusion



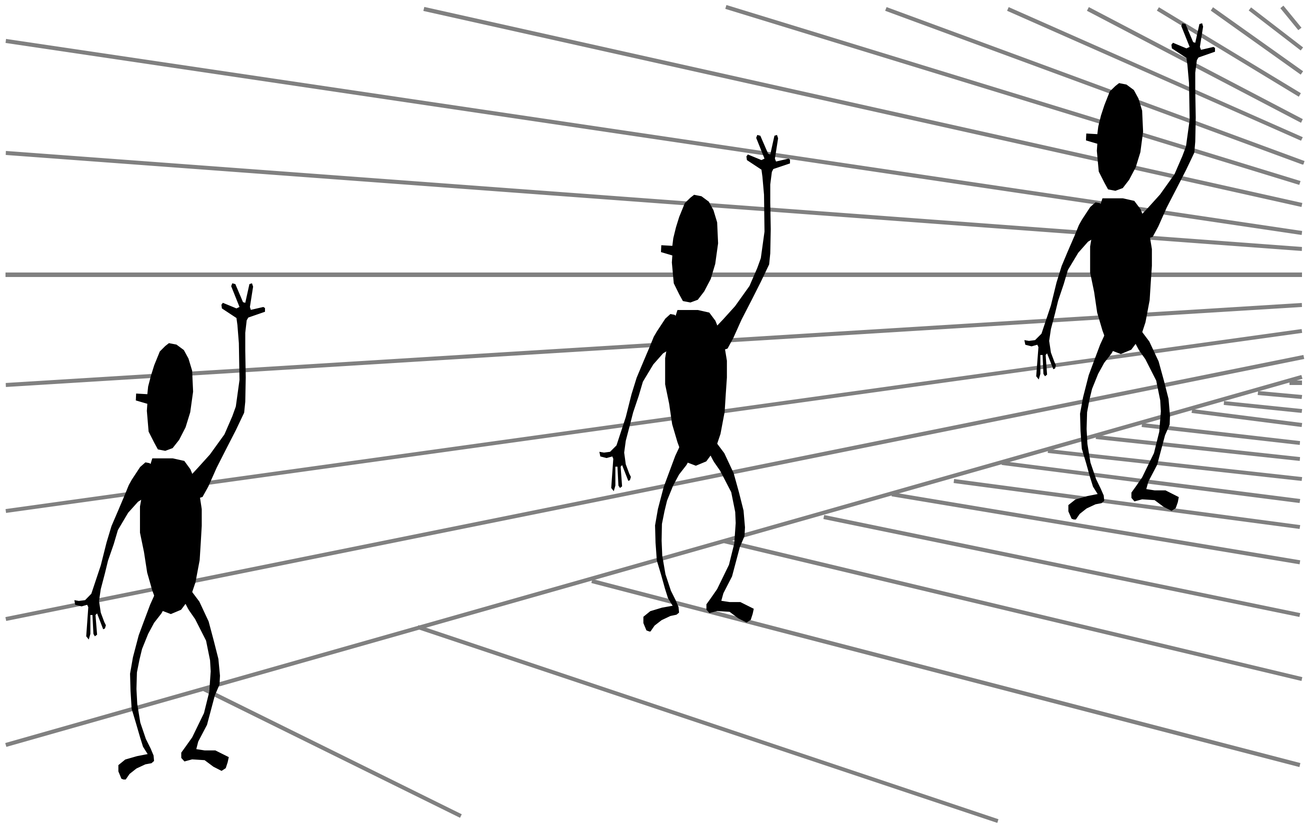
Illusion



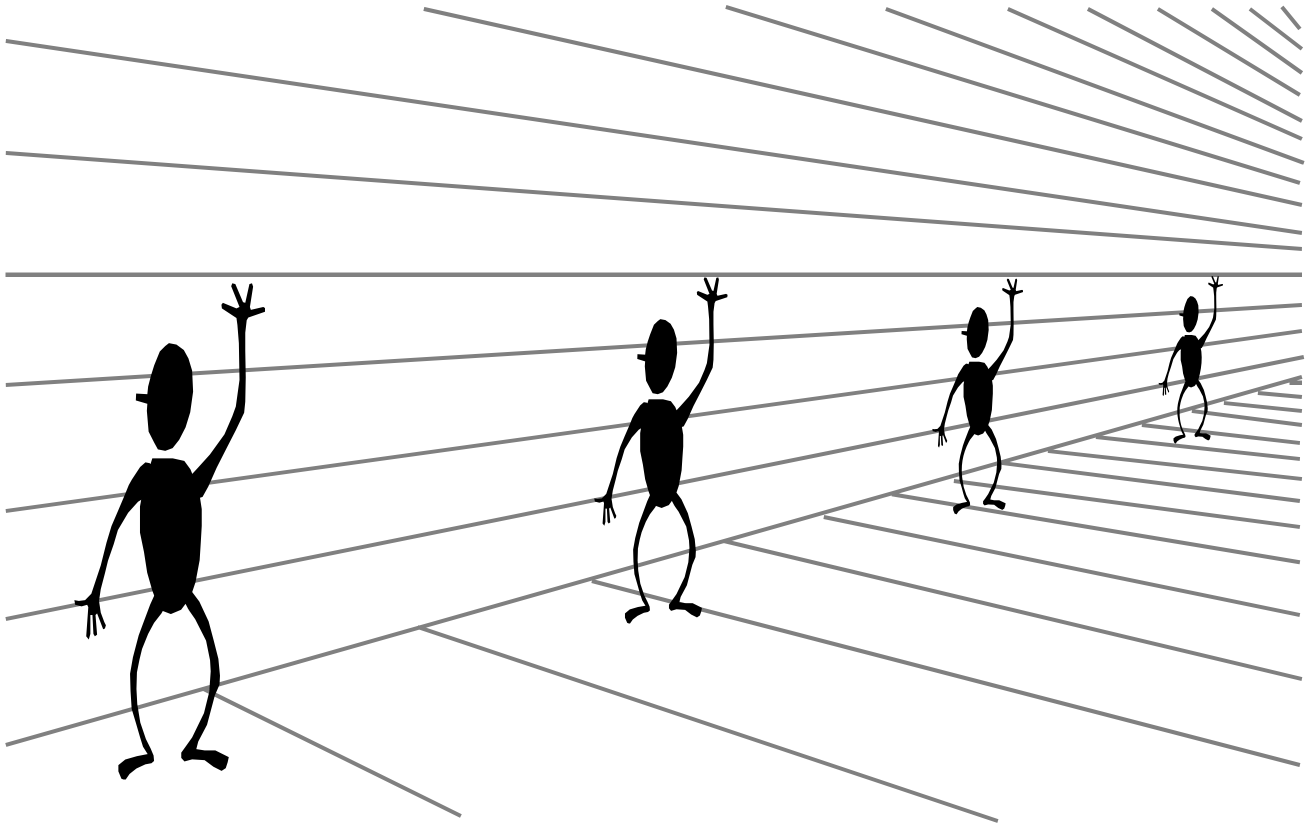
Fun with perspective



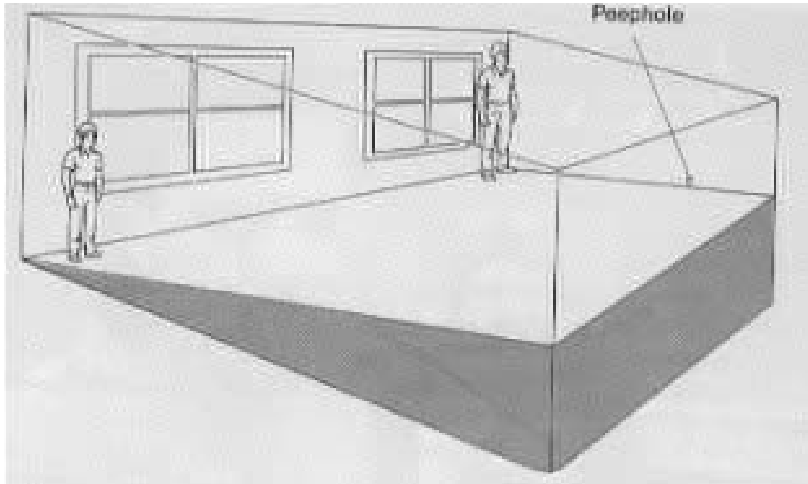
Perspective cues

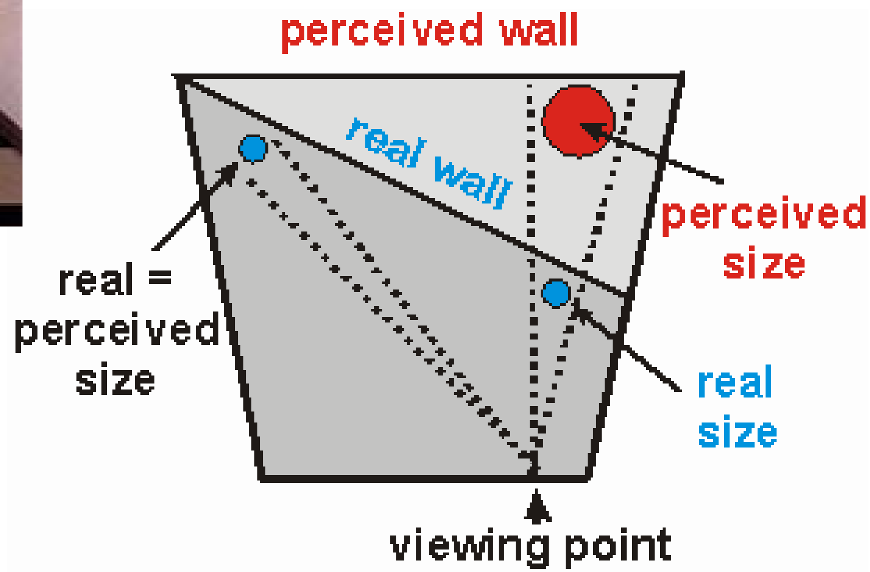


Perspective cues



Fun with perspective





[Ames room]

[Ames video] [BBC story]

Forced perspective in LOTR

Camera calibration

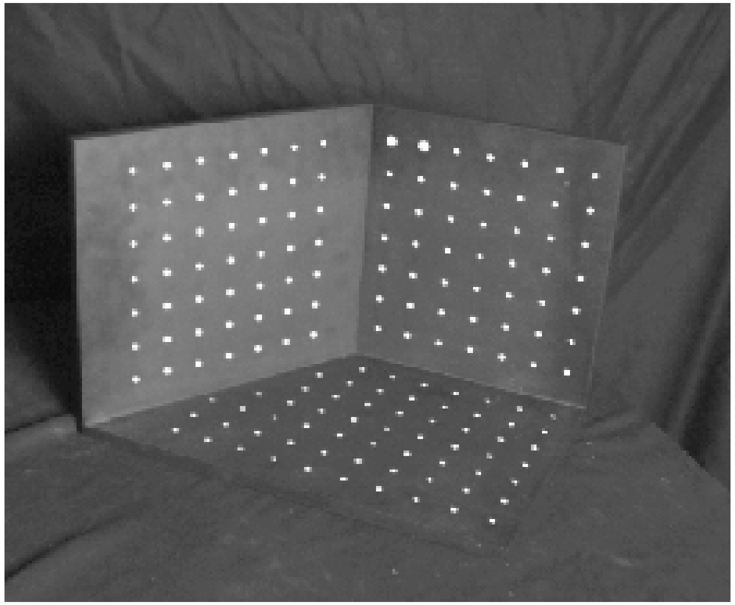
Camera calibration

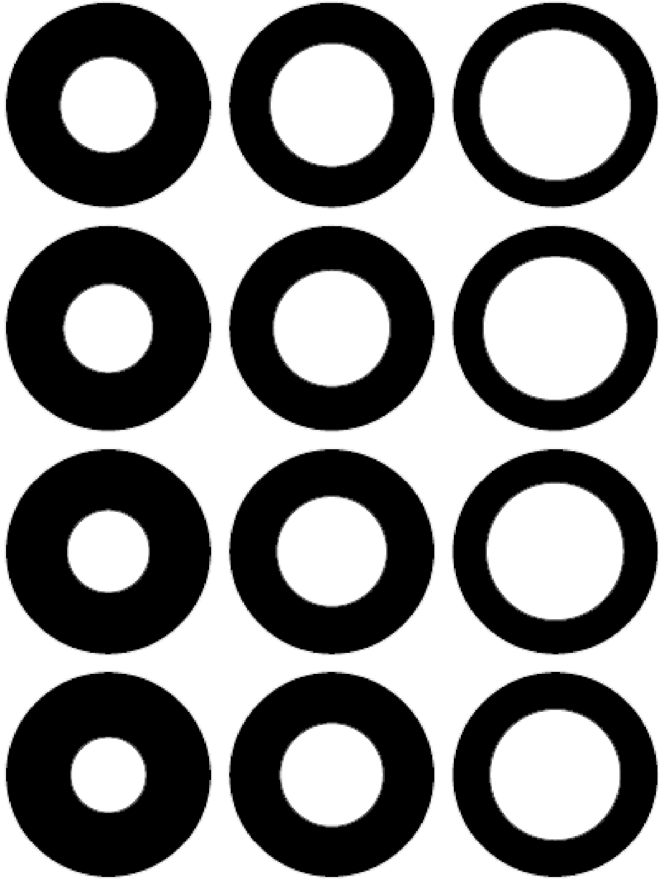
- Estimate both intrinsic and extrinsic parameters. Two main categories:

1. Photometric calibration: uses reference objects with known geometry
2. Self calibration: only assumes static scene, e.g. structure from motion

Camera calibration approaches

1. linear regression (least squares)
2. nonlinear optimization





Chromaglyphs (HP research)

Camera calibration

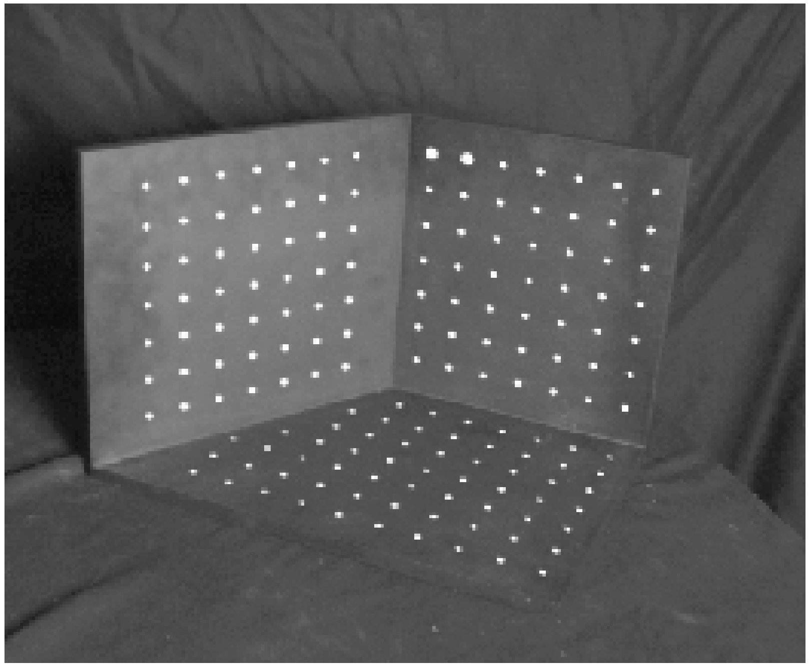
Linear regression

$$\mathbf{x} \sim \mathbf{K}\left[\mathbf{R}|\mathbf{t}\right]\mathbf{X} = \mathbf{M}\mathbf{X}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear regression

- Directly estimate 11 unknowns in the **M** matrix using known 3D points (X_i,Y_i,Z_i) and measured feature positions (u_i,v_i)



Linear regression

$$u_i = \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$v_i = \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$u_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}$$

$$v_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}$$

Linear regression

$$\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i & -u_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i & -v_i \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear regression

$$
\begin{bmatrix}
X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\
0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\
 & & & & & & \vdots & & & & & \\
X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\
0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n
\end{bmatrix}
\begin{bmatrix}
m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ 0
\end{bmatrix}
$$

Solve for Projection Matrix M using least-square techniques

Normal equation

Given an overdetermined system

$$\mathbf{Ax} = \mathbf{b}$$

the normal equation is that which minimizes the sum of the square differences between left and right sides

$$\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Linear regression

- Advantages:
  - All specifics of the camera summarized in one matrix
  - Can predict where any world point will map to in the image
- Disadvantages:
  - Doesn't tell us about particular parameters
  - Mixes up internal and external parameters
    - pose specific: move the camera and everything breaks
  - More unknowns than true degrees of freedom

Nonlinear optimization

- A probabilistic view of least square
- Feature measurement equations

$$u_i = f(\mathbf{M}, \mathbf{x}_i) + n_i = \hat{u}_i + n_i, \quad n_i \sim N(0, \sigma)$$
$$v_i = g(\mathbf{M}, \mathbf{x}_i) + m_i = \hat{v}_i + m_i, \quad m_i \sim N(0, \sigma)$$

- Probability of $\mathbf{M}$ given $\{(u_i, v_i)\}$

$$P = \prod_i p(u_i|\hat{u}_i) p(v_i|\hat{v}_i)$$
$$= \prod_i e^{-(u_i - \hat{u}_i)^2/\sigma^2} e^{-(v_i - \hat{v}_i)^2/\sigma^2}$$

Optimal estimation

- Likelihood of $\mathbf{M}$ given $\{(u_i,v_i)\}$

$$L = -\log P = \sum_i (u_i - \hat{u}_i)^2/\sigma_i^2 + (v_i - \hat{v}_i)^2/\sigma_i^2$$

- It is a least square problem (but not necessarily linear least square)
- How do we minimize L?

Optimal estimation

- Non-linear regression (least squares), because the relations between $\hat{u}_i$ and u_i are non-linear functions of $\mathbf{M}$

unknown parameters

We could have terms like $f\cos\theta$ in this

$$\mathbf{u}-\hat{\mathbf{u}} \sim \mathbf{u}-\mathbf{K}\left[\mathbf{R}\middle|\mathbf{t}\right]\mathbf{X}$$

known constant

- We can use Levenberg-Marquardt method to minimize it

Nonlinear least square methods

Least square fitting

Least Squares Problem

Find $\mathbf{x}^*$, a local minimizer for

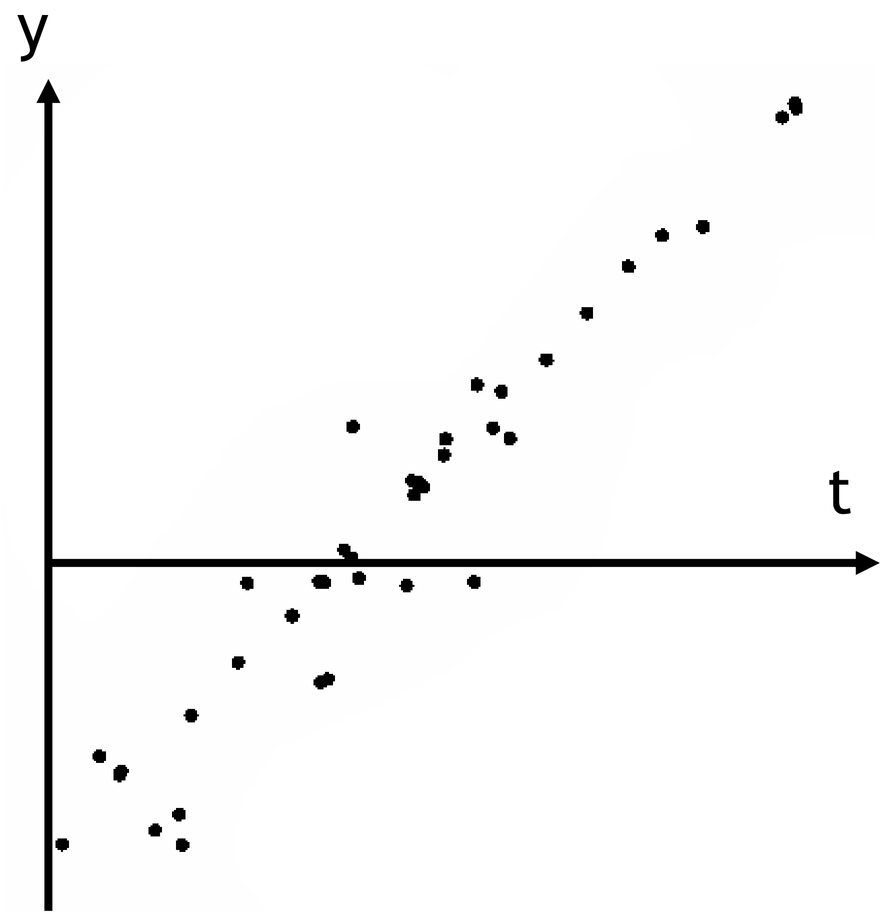
$$F(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}(f_i(\mathbf{x}))^2 ,$$

where $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \ldots, m$ are given functions, and $m \geq n$.

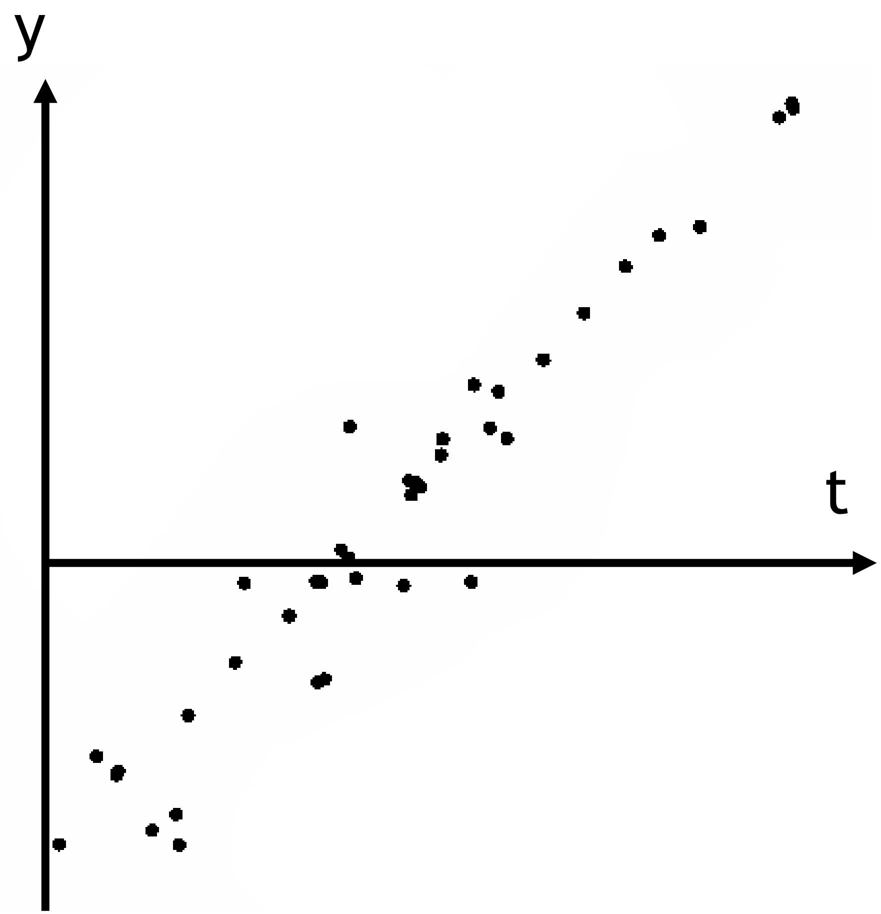
number of data points

number of parameters

Linear least square fitting



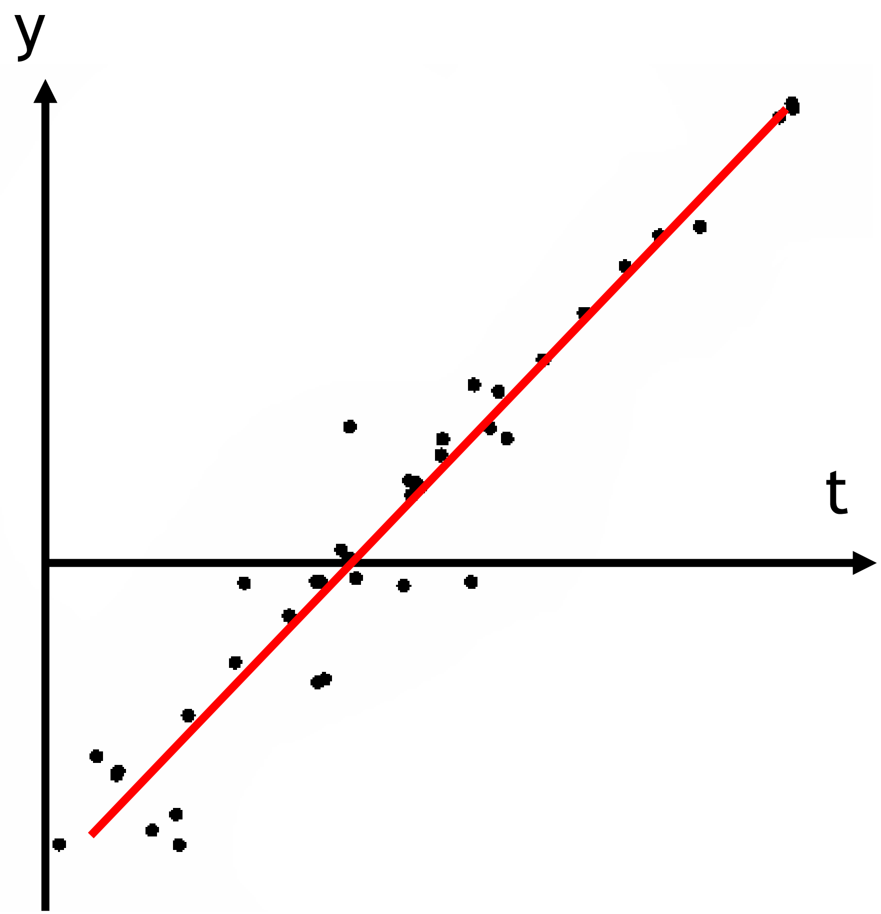
Linear least square fitting



model parameters

$$y(t) = M(t; \mathbf{x})$$

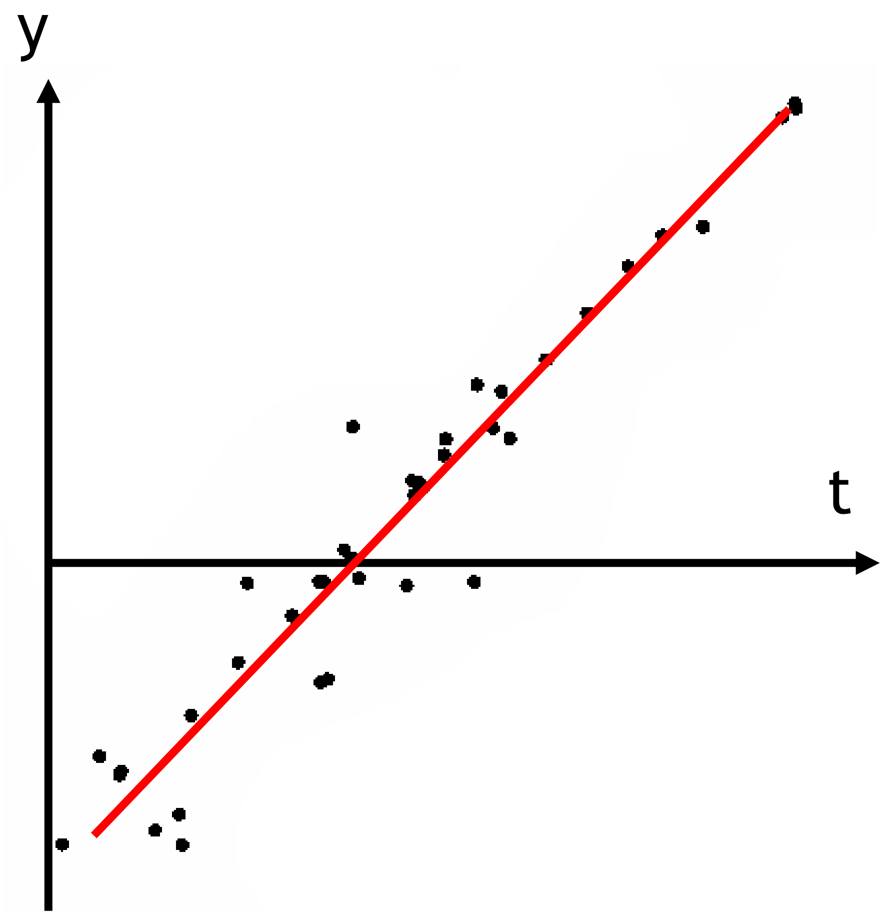
Linear least square fitting



model parameters

$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

Linear least square fitting



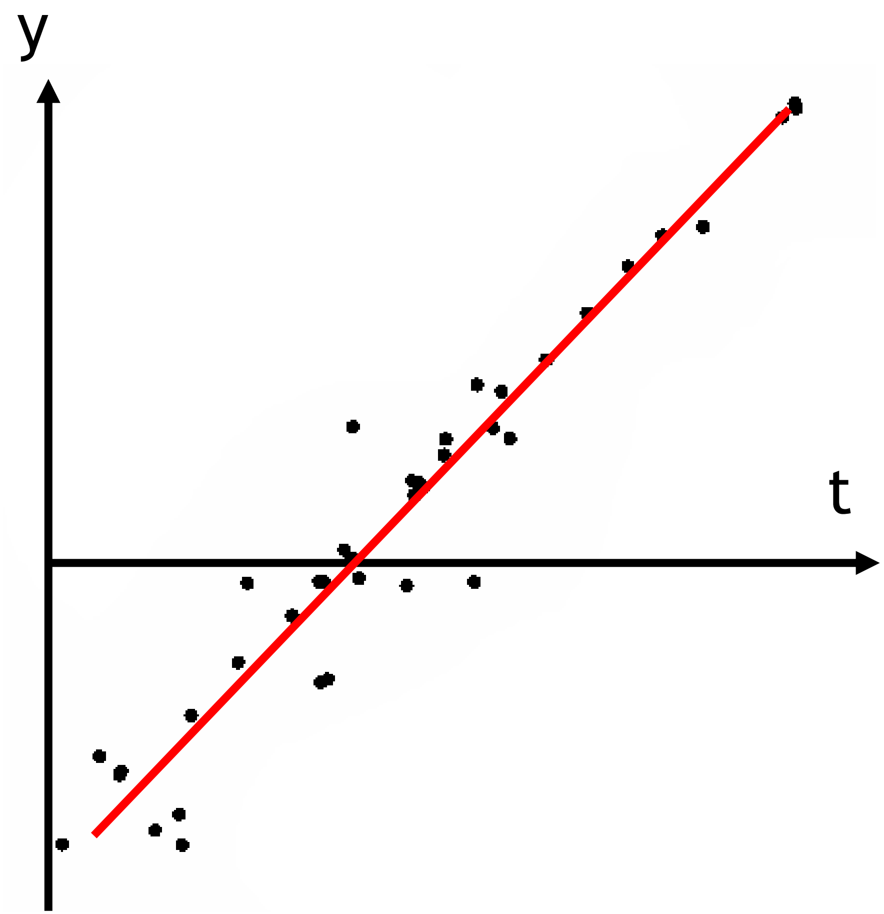
model parameters

$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

$$f_i(x) = y_i - M(t_i;\mathbf{x})$$

residual prediction

Linear least square fitting



model parameters

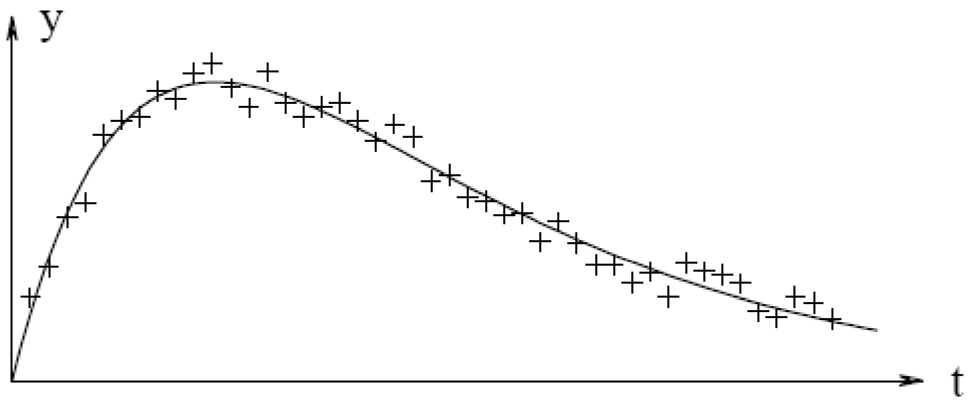
$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

$$f_i(x) = y_i - M(t_i;\mathbf{x})$$

residual prediction

$M(t;\mathbf{x}) = x_0 + x_1 t + x_2 t^3$ is linear, too.

Nonlinear least square fitting



model $M(t;\mathbf{x}) = x_3 e^{x_1 t} + x_4 e^{x_2 t}$

parameters $\mathbf{x} = [x_1, x_2, x_4, x_4]^T$

residuals $f_i(\mathbf{x}) = y_i - M(t_i;\mathbf{x})$

$$= y_i - \left(x_3 e^{x_1 t} + x_4 e^{x_2 t}\right)$$

Function minimization

Least square is related to function minimization.

Global Minimizer

Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find

$$\mathbf{x}^+ = \text{argmin}_{\mathbf{x}} \{F(\mathbf{x})\} .$$

It is very hard to solve in general. Here, we only consider a simpler problem of finding local minimum.

Local Minimizer

Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find $\mathbf{x}^*$ so that

$$F(\mathbf{x}^*) \leq F(\mathbf{x}) \quad \text{for} \quad \|\mathbf{x} - \mathbf{x}^*\| < \delta .$$

Function minimization

We assume that the cost function F is differentiable and so smooth that the following *Taylor expansion* is valid,[2)]

$$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^\top \mathbf{g} + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}\,\mathbf{h} + O(\|\mathbf{h}\|^3) ,$$

where g is the *gradient*,

$$\mathbf{g} \equiv \mathbf{F}'(\mathbf{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial F}{\partial x_n}(\mathbf{x}) \end{bmatrix} ,$$

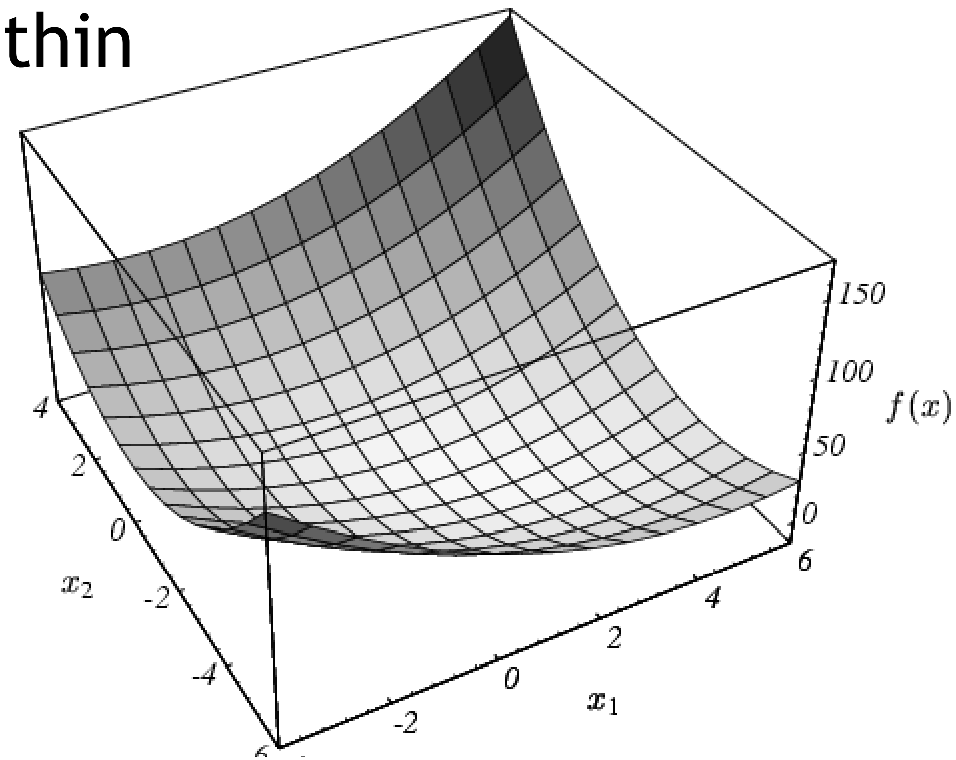
and $\mathbf{H}$ is the *Hessian*,

$$\mathbf{H} \equiv \mathbf{F}''(\mathbf{x}) = \left[\frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{x}) \right] .$$

Quadratic functions

DigiVFX

Approximate the function with
a quadratic function within
a small neighborhood



$$f(x) = \frac{1}{2}x^T A x - b^T x + c$$

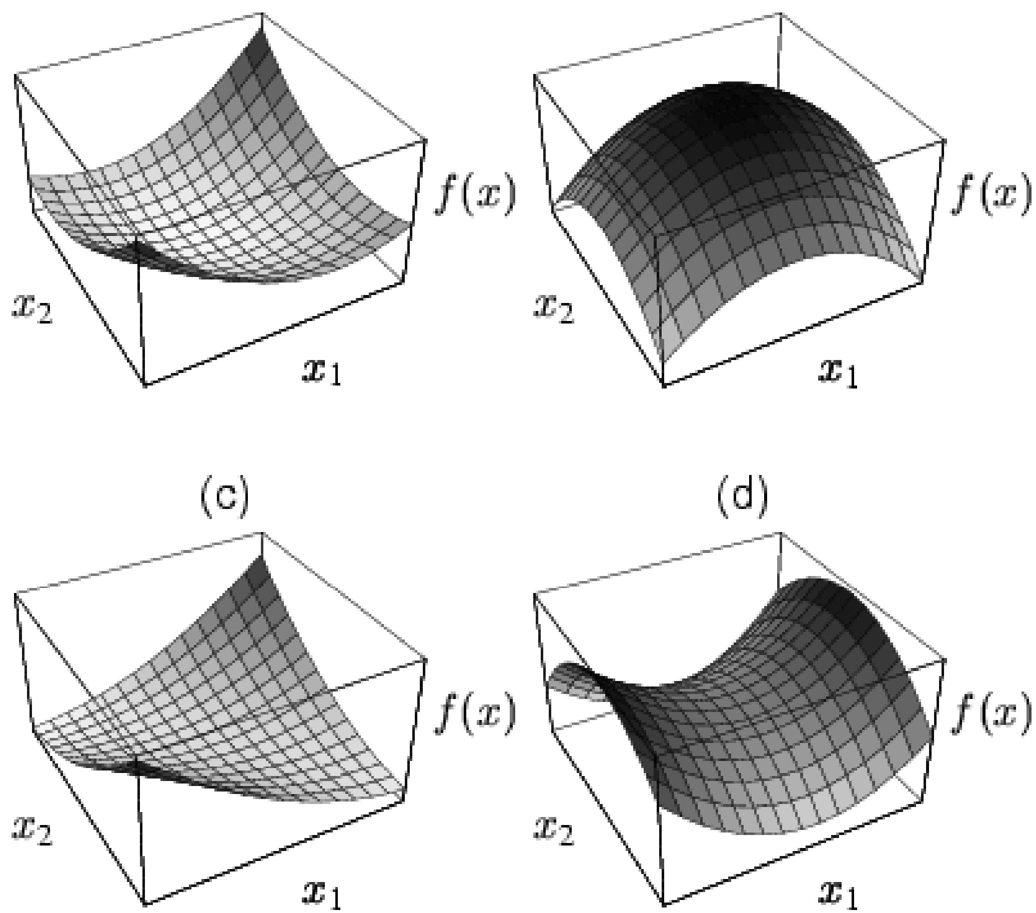
$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \qquad c = 0.$$

Quadratic functions

A is positive definite. (a) (b) negative definite

All eigenvalues are positive.

For all x, $x^T A x > 0$.



A is singular

A is indefinite

Function minimization

Theorem 1.5. Necessary condition for a local minimizer.
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}.$$

Why?

By definition, if $\mathbf{x}^*$ is a local minimizer,

$\|\mathbf{h}\|$ is small enough $\longrightarrow$ $\mathbf{F}(\mathbf{x}^* + \mathbf{h}) > \mathbf{F}(\mathbf{x}^*)$

$$\mathbf{F}(\mathbf{x}^* + \mathbf{h}) = \mathbf{F}(\mathbf{x}^*) + \mathbf{h}^T\mathbf{F}'(\mathbf{x}^*) + \mathbf{O}(\|\mathbf{h}\|^2)$$

Function minimization

Theorem 1.5. Necessary condition for a local minimizer.
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}\,.$$

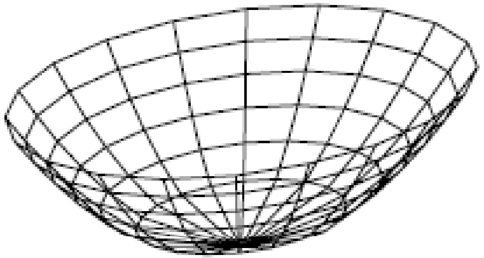
Definition 1.6. Stationary point. If

$$\mathbf{g}_s \equiv \mathbf{F}'(\mathbf{x}_s) = \mathbf{0}\,,$$

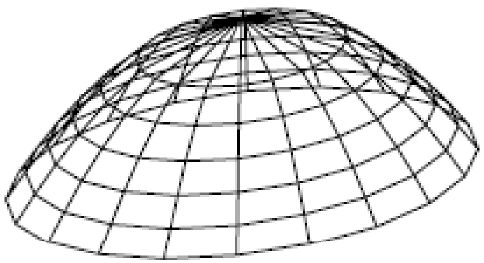
then $\mathbf{x}_s$ is said to be a *stationary point* for F.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$

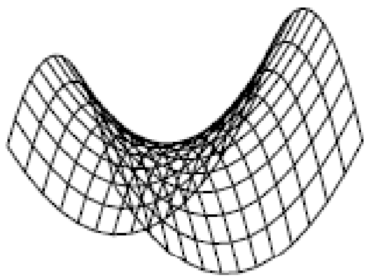
$\mathbf{H}_s$ is *positive definite*



a) minimum



b) maximum



c) saddle point

Function minimization

Theorem 1.8. Sufficient condition for a local minimizer.
Assume that $\mathbf{x}_s$ is a stationary point and that $\mathbf{F}''(\mathbf{x}_s)$ is positive definite.
Then $\mathbf{x}_s$ is a local minimizer.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$

$$\text{with } \mathbf{H}_s = \mathbf{F}''(\mathbf{x}_s)$$

If we request that $\mathbf{H}_s$ is *positive definite*, then its eigenvalues are greater than some number $\delta > 0$

$$\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} > \delta\,\|\mathbf{h}\|^2$$

Descent methods

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \rightarrow \mathbf{x}^* \quad \text{for} \quad k \rightarrow \infty$$

1. Find a descent direction $\mathbf{h}_d$
2. find a step length giving a good decrease in the F-value.

Algorithm Descent method

begin
  $k := 0$; $\mathbf{x} := \mathbf{x}_0$; *found* := **false** {Starting point}
  while (**not** *found*) **and** $(k < k_{max})$
    $\mathbf{h}_d :=$ search_direction($\mathbf{x}$) {From $\mathbf{x}$ and downhill}
    if (no such $\mathbf{h}$ exists)
      found := **true** {$\mathbf{x}$ is stationary}
    else
      $\alpha :=$ step_length($\mathbf{x}, \mathbf{h}_d$) {from $\mathbf{x}$ in direction $\mathbf{h}_d$}
      $\mathbf{x} := \mathbf{x} + \alpha\mathbf{h}_d$; $k := k+1$ {next iterate}
end

Descent direction

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

Definition Descent direction.

$\mathbf{h}$ is a descent direction for F at $\mathbf{x}$ if $\quad \mathbf{h}^\top\mathbf{F}'(\mathbf{x}) < 0$.

Steepest descent method

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

$$\frac{F(\mathbf{x}) - F(\mathbf{x}+\alpha\mathbf{h})}{\alpha\|\mathbf{h}\|} = -\frac{1}{\|\mathbf{h}\|}\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) = -\|\mathbf{F}'(\mathbf{x})\|\cos\theta$$

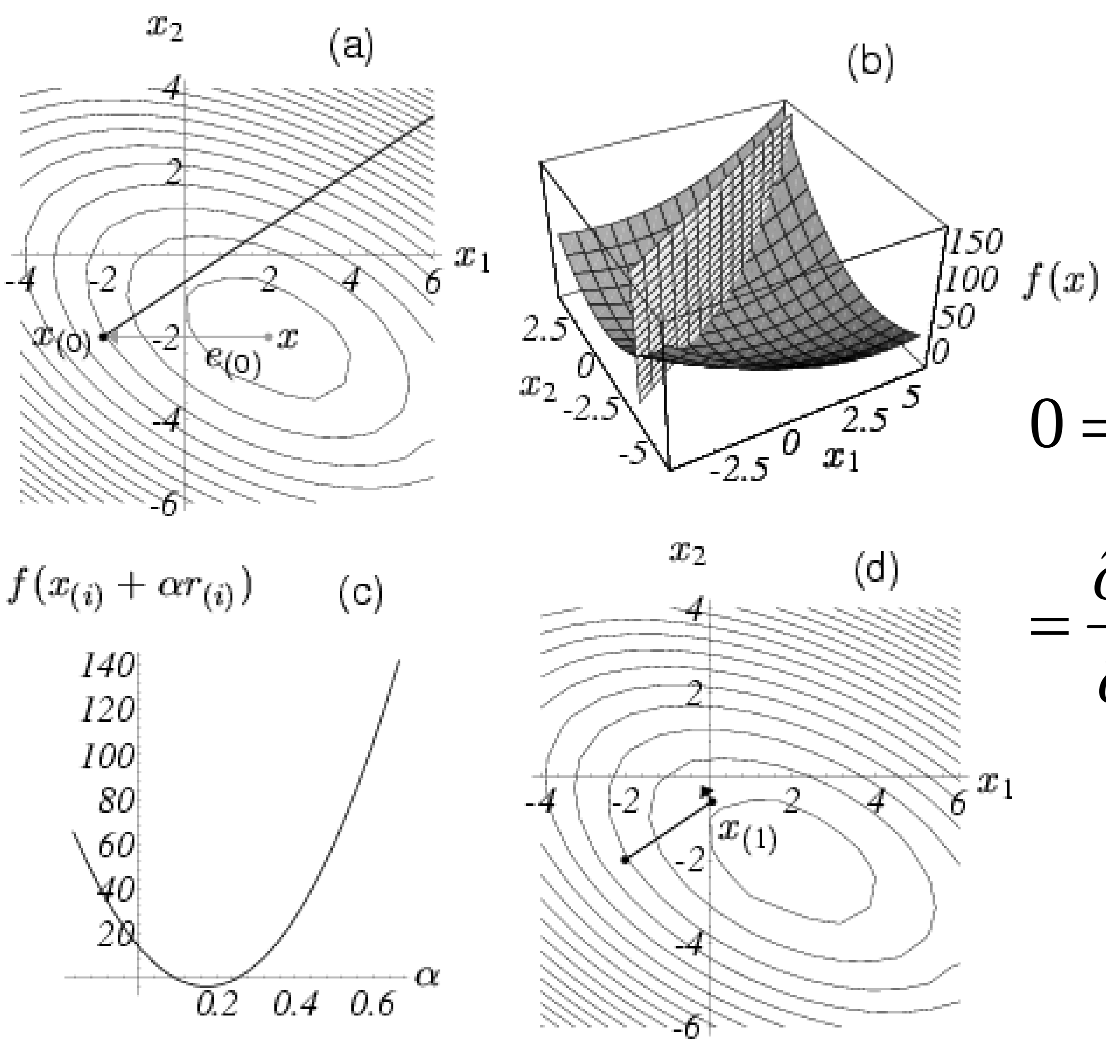
the decrease of ***F(x)*** per
unit along h direction

$$\text{greatest gain rate if } \theta = \pi \;\longrightarrow\; \mathbf{h}_{\text{sd}} = -\mathbf{F}'(\mathbf{x})$$

h_{sd} is a descent direction because $h^T_{sd} F'(x) = -F'(x)^2 < 0$

Line search

$\varphi(\alpha) = F(\mathbf{x}+\alpha\mathbf{h})$, $\mathbf{x}$ and $\mathbf{h}$ fixed, $\alpha \geq 0$.



Find α so that

$$\varphi(\alpha) = \mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})$$

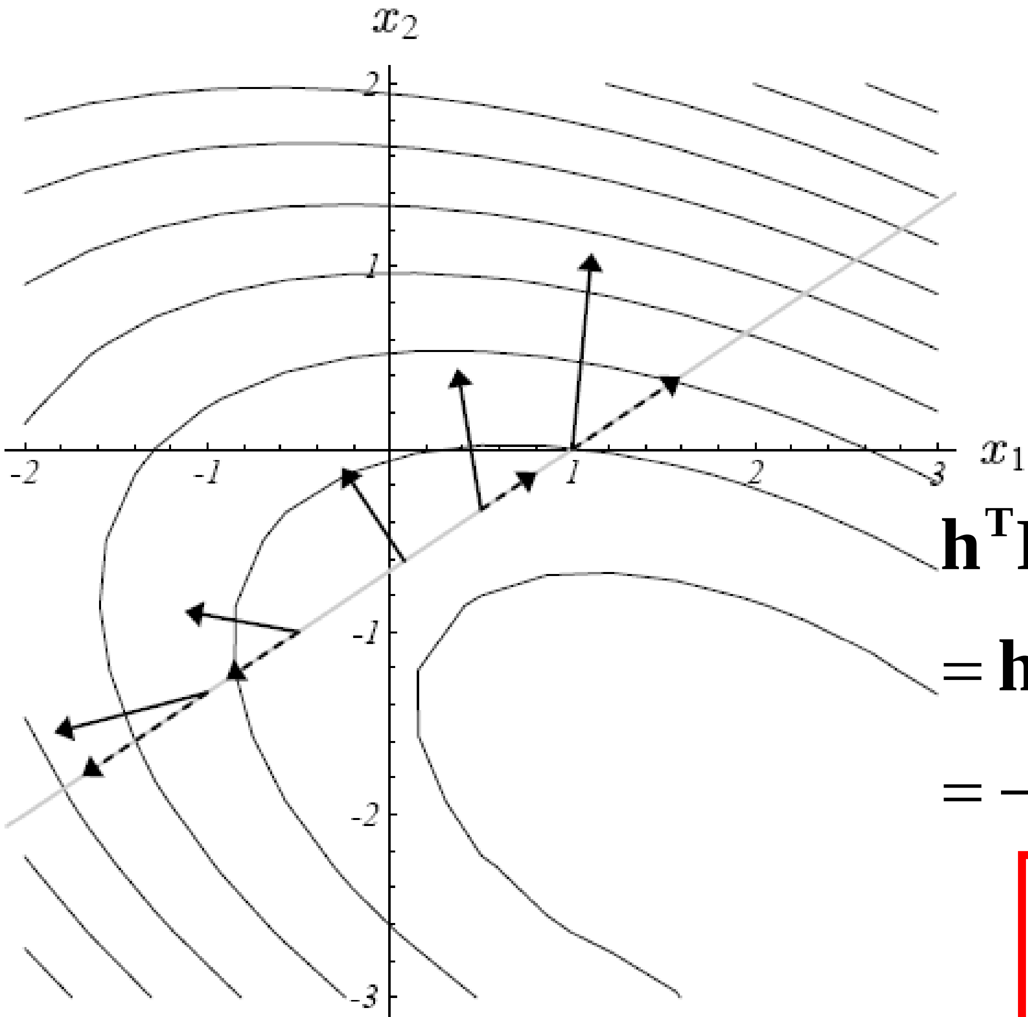
is minimum

$$0 = \frac{\partial\varphi(\alpha)}{\partial\alpha} = \frac{\partial\mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})}{\partial\alpha}$$

$$= \frac{\partial\mathbf{F}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\alpha} = \mathbf{h}^T\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

Line search



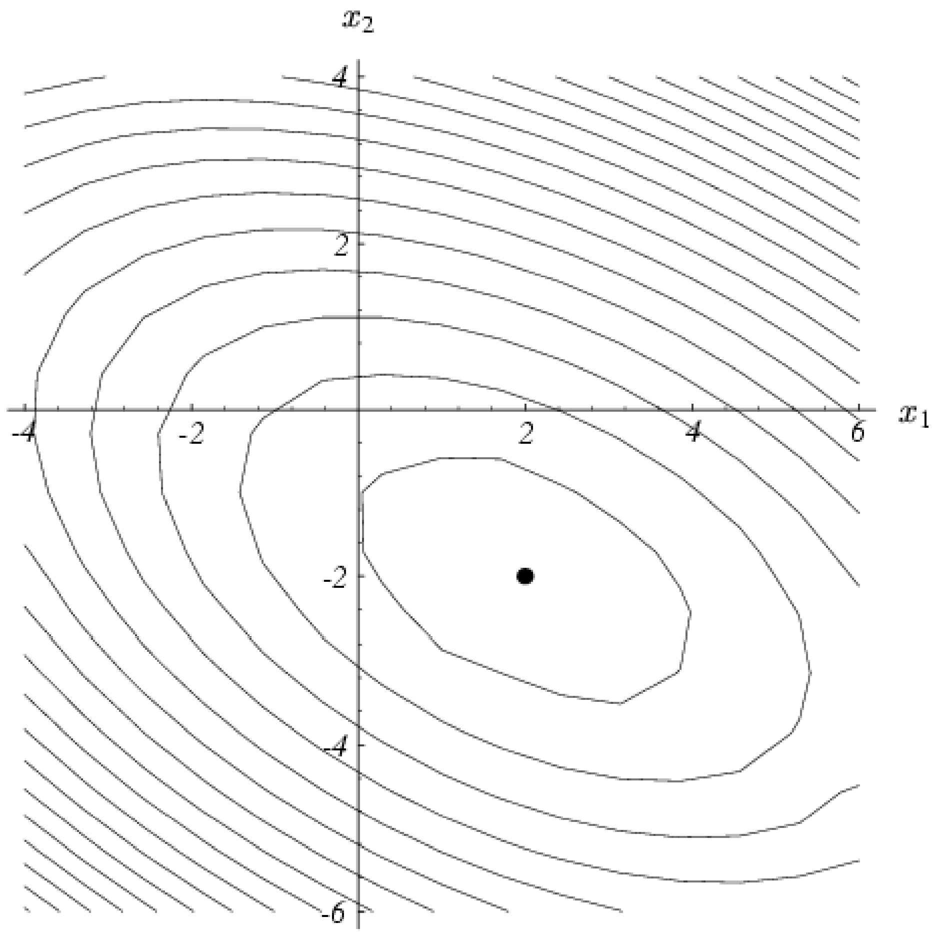
$$\mathbf{h}^T\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h}) = 0$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

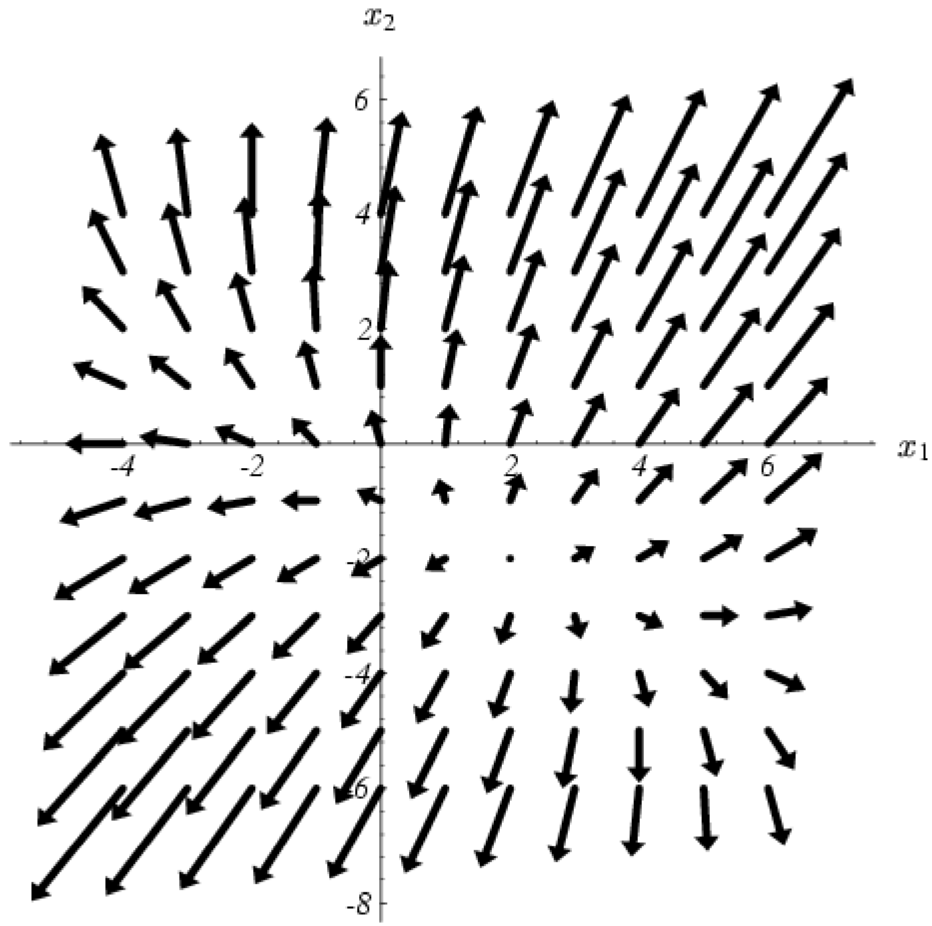
$$\mathbf{h}^T\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$
$$= \mathbf{h}^T\left(\mathbf{F}'(\mathbf{x}_0) + \alpha\mathbf{F}''(\mathbf{x}_0)^T\mathbf{h}\right)$$
$$= -\mathbf{h}^T\mathbf{h} + \alpha\mathbf{h}^T\mathbf{H}\mathbf{h} = 0$$

$$\alpha = \frac{\mathbf{h}^T\mathbf{h}}{\mathbf{h}^T\mathbf{H}\mathbf{h}}$$

Steepest descent method

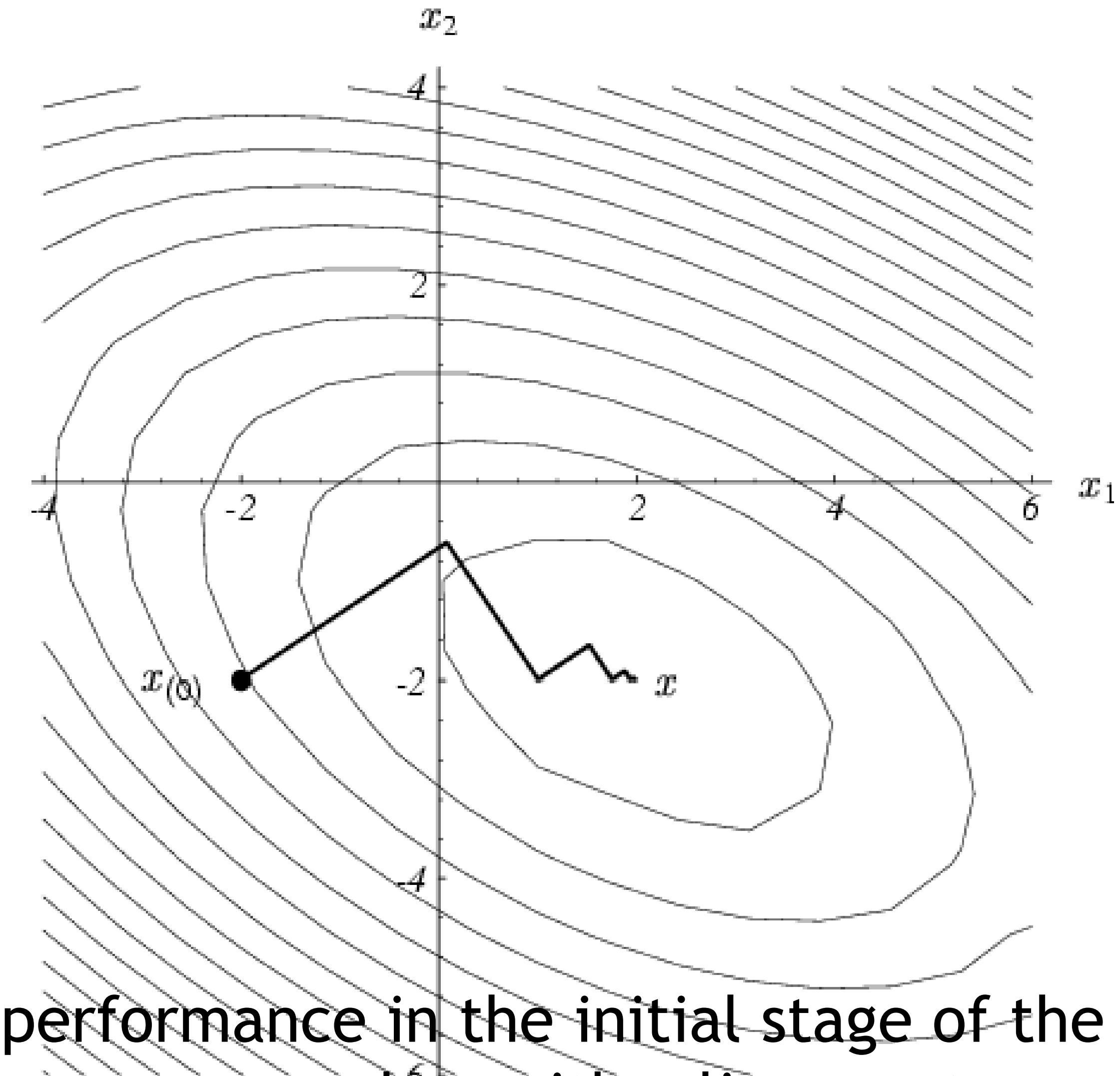


isocontour



gradient

Steepest descent method



It has good performance in the initial stage of the iterative process. Converge very slow with a linear rate.

Newton's method

$\mathbf{x}^*$ is a stationary point $\longrightarrow$ it satisfies $\mathbf{F}'(\mathbf{x}^*) = \mathbf{0}$.

$$\begin{aligned}\mathbf{F}'(\mathbf{x}+\mathbf{h}) &= \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &\simeq \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} \quad \text{for } \|\mathbf{h}\| \text{ sufficiently small}\end{aligned}$$

$$\longrightarrow \quad \mathbf{H}\,\mathbf{h}_n = -\mathbf{F}'(\mathbf{x}) \quad \text{with } \mathbf{H} = \mathbf{F}''(\mathbf{x})$$

$$\mathbf{x} := \mathbf{x} + \mathbf{h}_n$$

Suppose that $\mathbf{H}$ is positive definite

$\longrightarrow$ $\mathbf{u}^\top \mathbf{H}\,\mathbf{u} > 0$ for all nonzero $\mathbf{u}$.

$\longrightarrow$ $0 < \mathbf{h}_n^\top \mathbf{H}\,\mathbf{h}_n = -\mathbf{h}_n^\top \mathbf{F}'(\mathbf{x})$ $\;\mathbf{h}_n$ is a descent direction

Newton's method

- Another view

$$E(\mathbf{h}) = F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^{\mathrm{T}}\mathbf{g} + \frac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h}$$

- Minimizer satisfies $E'(\mathbf{h}^*) = 0$

$$E'(\mathbf{h}) = \mathbf{g} + \mathbf{H}\mathbf{h} = 0$$

$$\mathbf{h} = -\mathbf{H}^{-1}\mathbf{g}$$

Newton's method

$$\mathbf{h} = -\mathbf{H}^{-1}\mathbf{g}$$

- It requires solving a linear system and H is not always positive definite.
- It has good performance in the final stage of the iterative process, where x is close to x*.

Gauss-Newton method

- Use the approximate Hessian

$$\mathbf{H} \approx \mathbf{J}^{\mathbf{T}}\mathbf{J}$$

- No need for second derivative
- H is positive semi-definite

Hybrid method

$$
\begin{aligned}
&\textbf{if } \ \mathbf{F}''(\mathbf{x}) \text{ is positive definite} \\
&\quad \mathbf{h} := \mathbf{h}_{\mathrm{n}} \\
&\textbf{else} \\
&\quad \mathbf{h} := \mathbf{h}_{\mathrm{sd}} \\
&\mathbf{x} := \mathbf{x} + \alpha \mathbf{h}
\end{aligned}
$$

This needs to calculate second-order derivative which might not be available.

Levenberg-Marquardt method

- LM can be thought of as a combination of steepest descent and the Newton method. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method: slow, but guaranteed to converge. When the current solution is close to the correct solution, it becomes a Newton's method.

Nonlinear least square

Given a set of measurements $\mathbf{x}$, try to find the best parameter vector $\mathbf{p}$ so that the squared distance $\varepsilon^T\varepsilon$ is minimal. Here, $\varepsilon = \mathbf{x} - \hat{\mathbf{x}}$, with $\hat{\mathbf{x}} = f(\mathbf{p})$.

Levenberg-Marquardt method

DigiVFX

For a small $||\delta_{\mathbf{p}}||$, $f(\mathbf{p}+\delta_{\mathbf{p}}) \approx f(\mathbf{p}) + \mathbf{J}\delta_{\mathbf{p}}$

$\mathbf{J}$ is the Jacobian matrix $\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}$

it is required to find the $\delta_{\mathbf{p}}$ that minimizes the quantity

$$||\mathbf{x} - f(\mathbf{p}+\delta_{\mathbf{p}})|| \approx ||\mathbf{x} - f(\mathbf{p}) - \mathbf{J}\delta_{\mathbf{p}}|| = ||\epsilon - \mathbf{J}\delta_{\mathbf{p}}||$$

$$\mathbf{J}^T\mathbf{J}\delta_{\mathbf{p}} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}\delta_{\mathbf{p}} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}_{ii} = \mu + \left[\mathbf{J}^T\mathbf{J}\right]_{ii}$$

↑ *damping term*

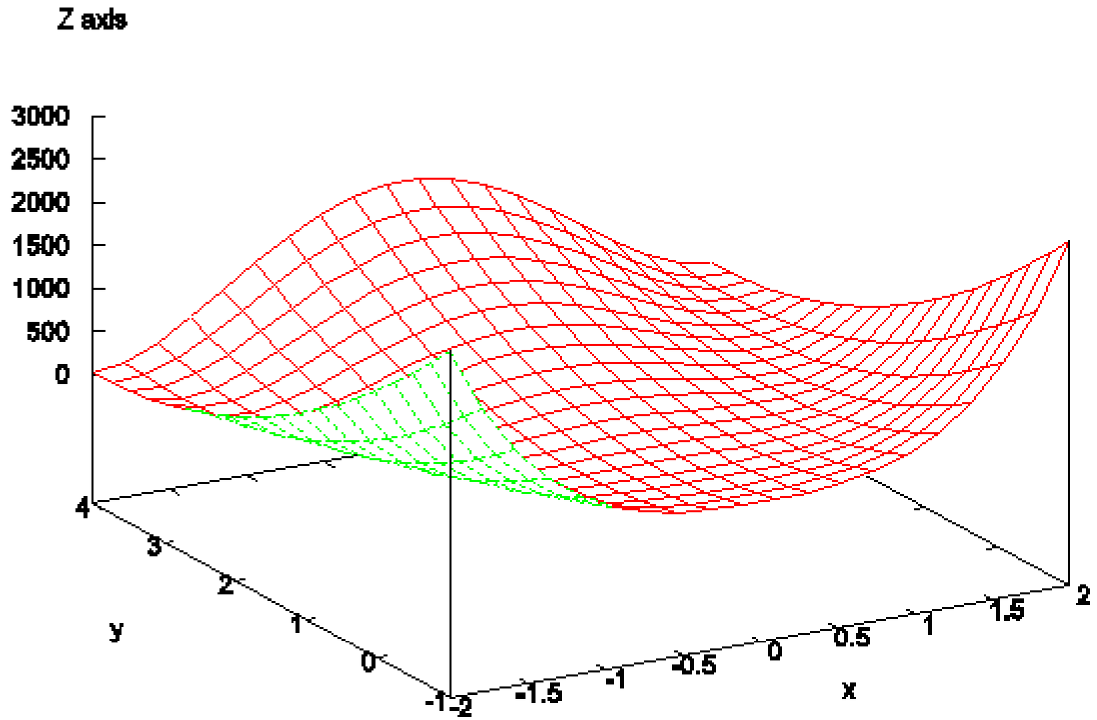
Levenberg-Marquardt method

$$(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\mathbf{h} = -\mathbf{g}$$

- μ=0 → Newton's method
- μ→∞ → steepest descent method

- Strategy for choosing μ
  - Start with some small μ
  - If F is not reduced, keep trying larger μ until it does
  - If F is reduced, accept it and reduce μ for the next iteration

Recap (the Rosenbrock function)



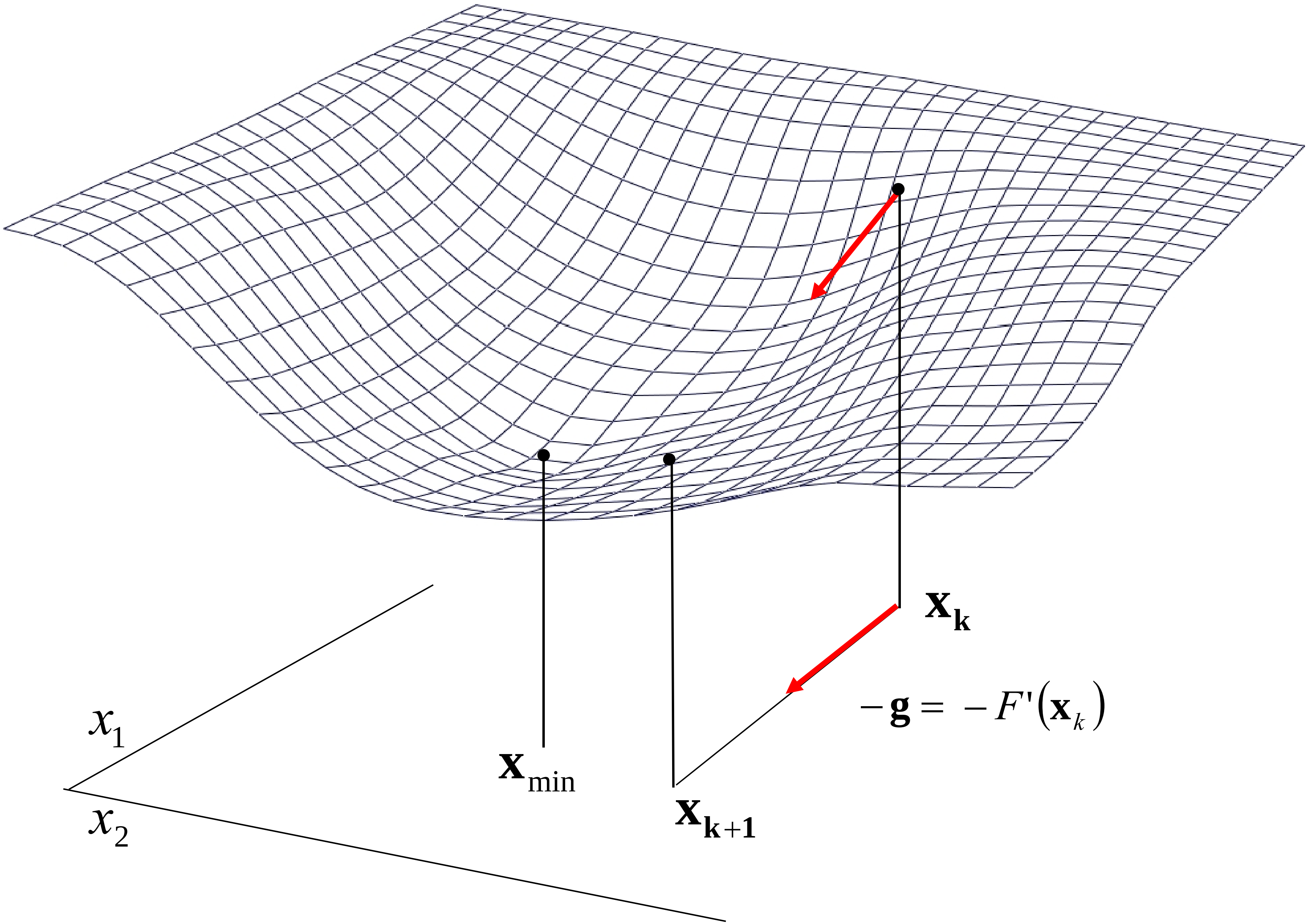
$$z = f(x, y) = (1 - x^2)^2 + 100(y - x^2)^2$$

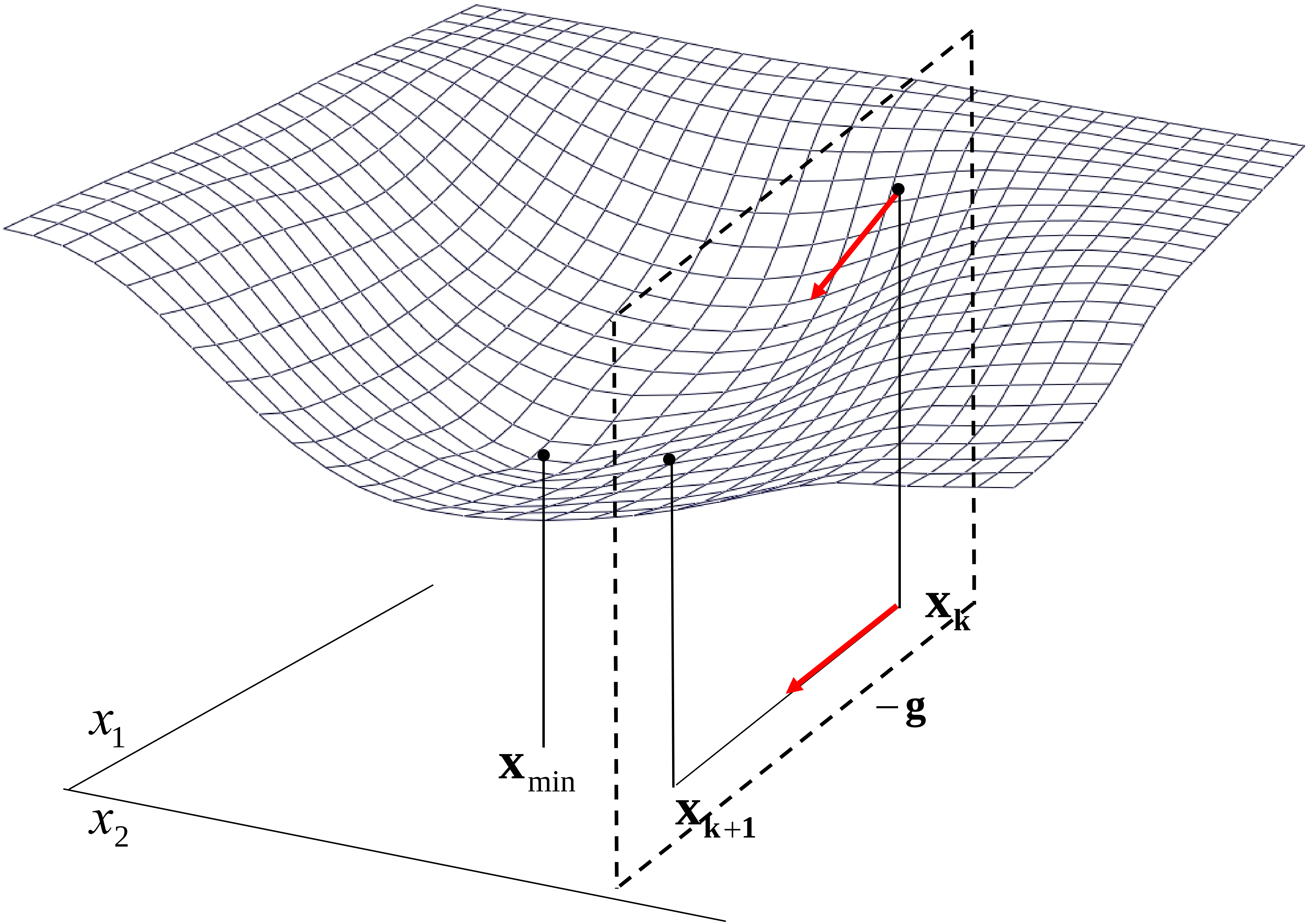
Global minimum at *(1,1)*

Steepest descent

$$\mathbf{x_{k+1}} = \mathbf{x_k} - \alpha \mathbf{g}$$

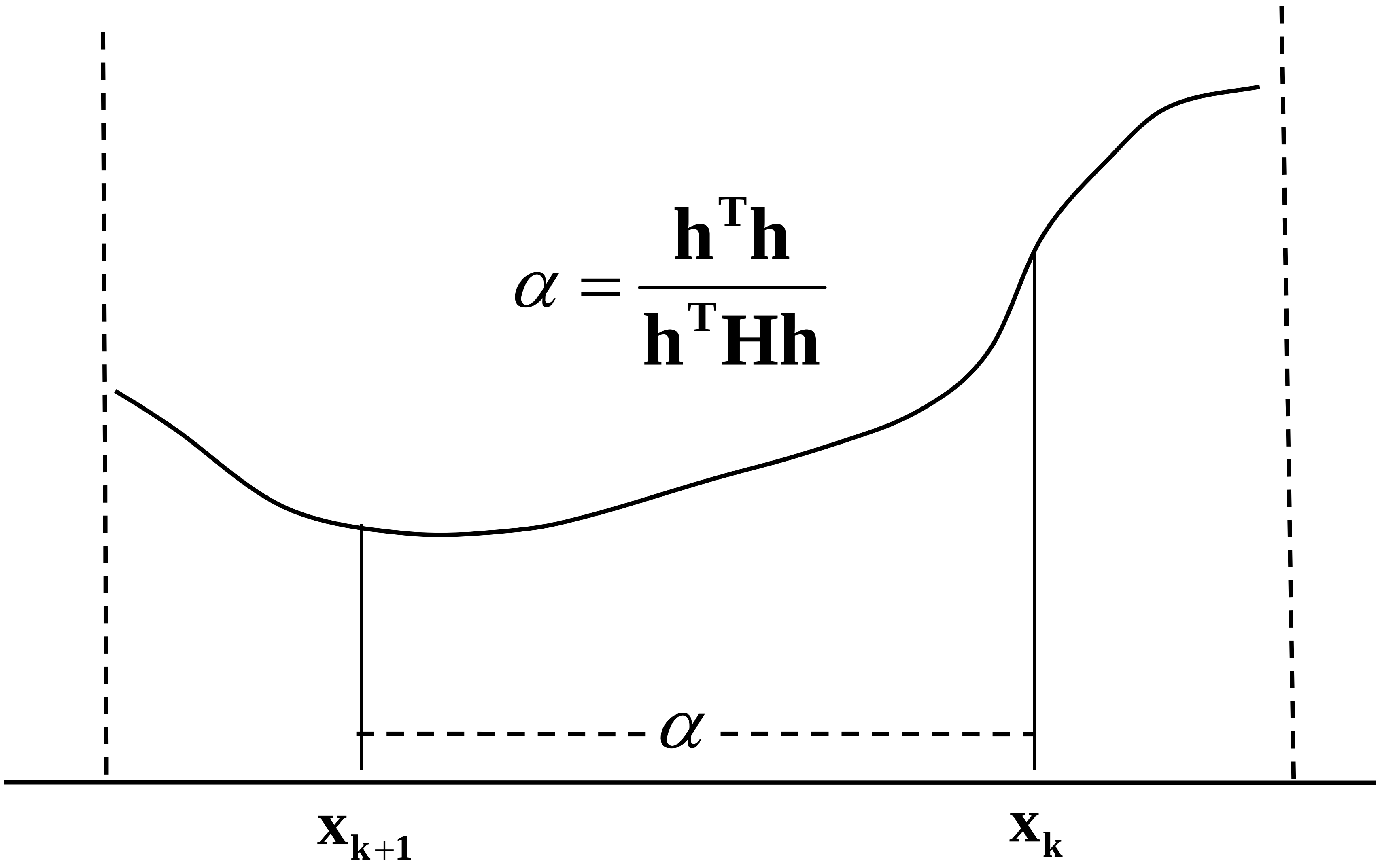
$$\alpha = \frac{\mathbf{h^T h}}{\mathbf{h^T H h}}$$


$\mathbf{x_k}$
$-\mathbf{g} = -F'(\mathbf{x}_k)$
x_1
$\mathbf{x}_{\min}$
x_2
$\mathbf{x_{k+1}}$

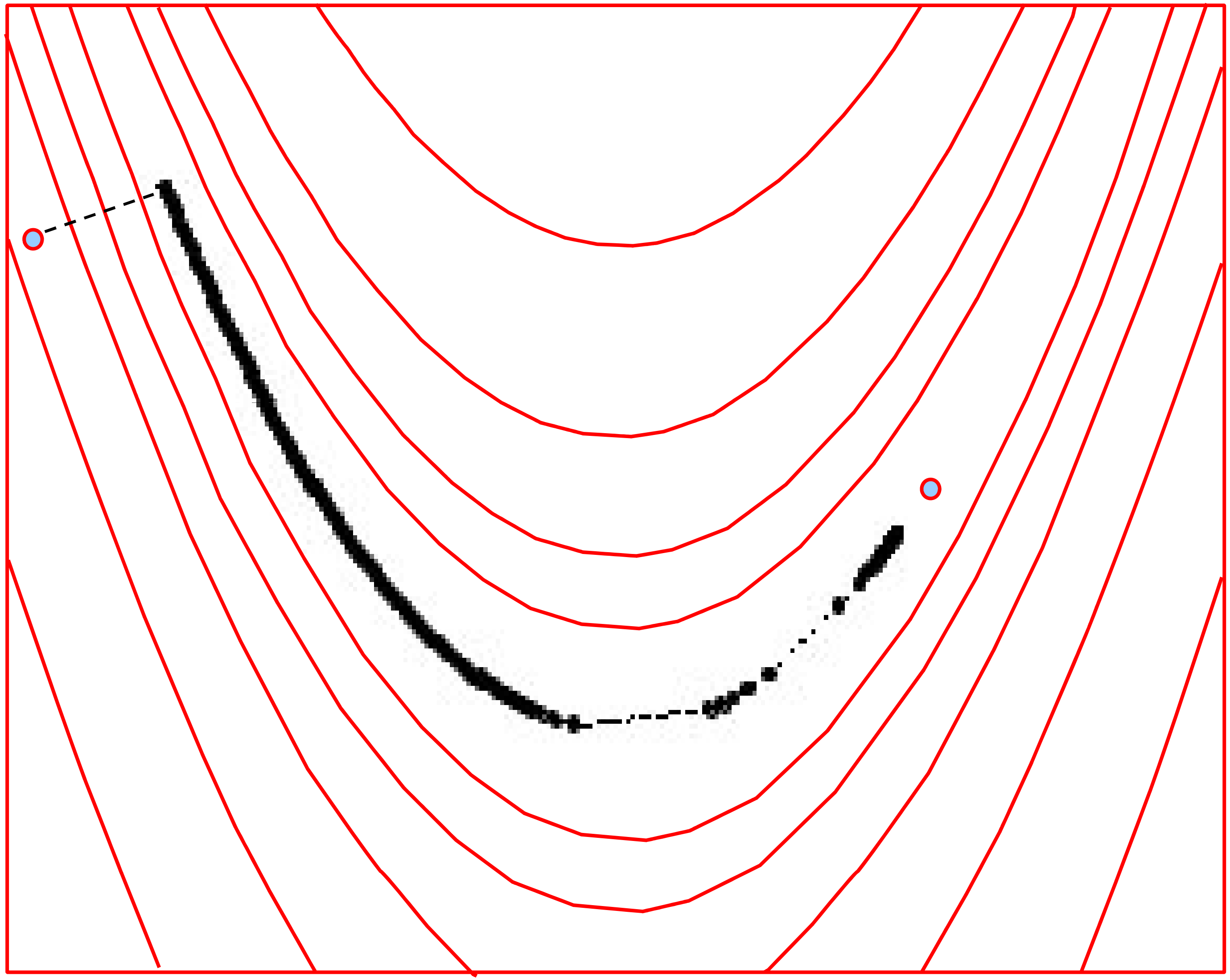


$\mathbf{x_k}$
$-\mathbf{g}$
x_1
x_2
$\mathbf{x}_{\min}$
$\mathbf{x_{k+1}}$

In the plane of the steepest descent direction



Steepest descent (1000 iterations)



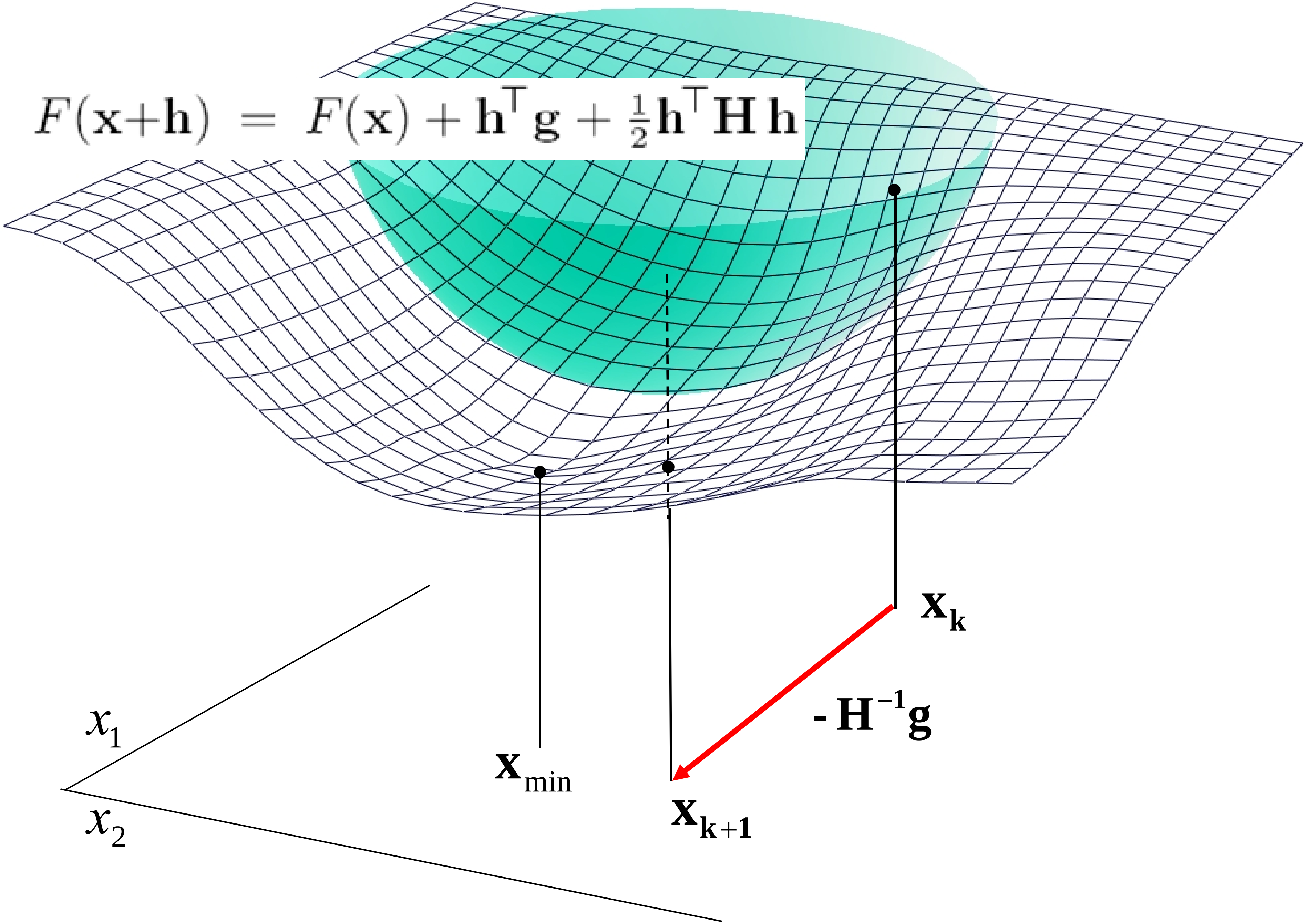
Gauss-Newton method

$$\mathbf{x}_{\mathbf{k+1}} = \mathbf{x}_{\mathbf{k}} - \mathbf{H}^{-1}\mathbf{g}$$

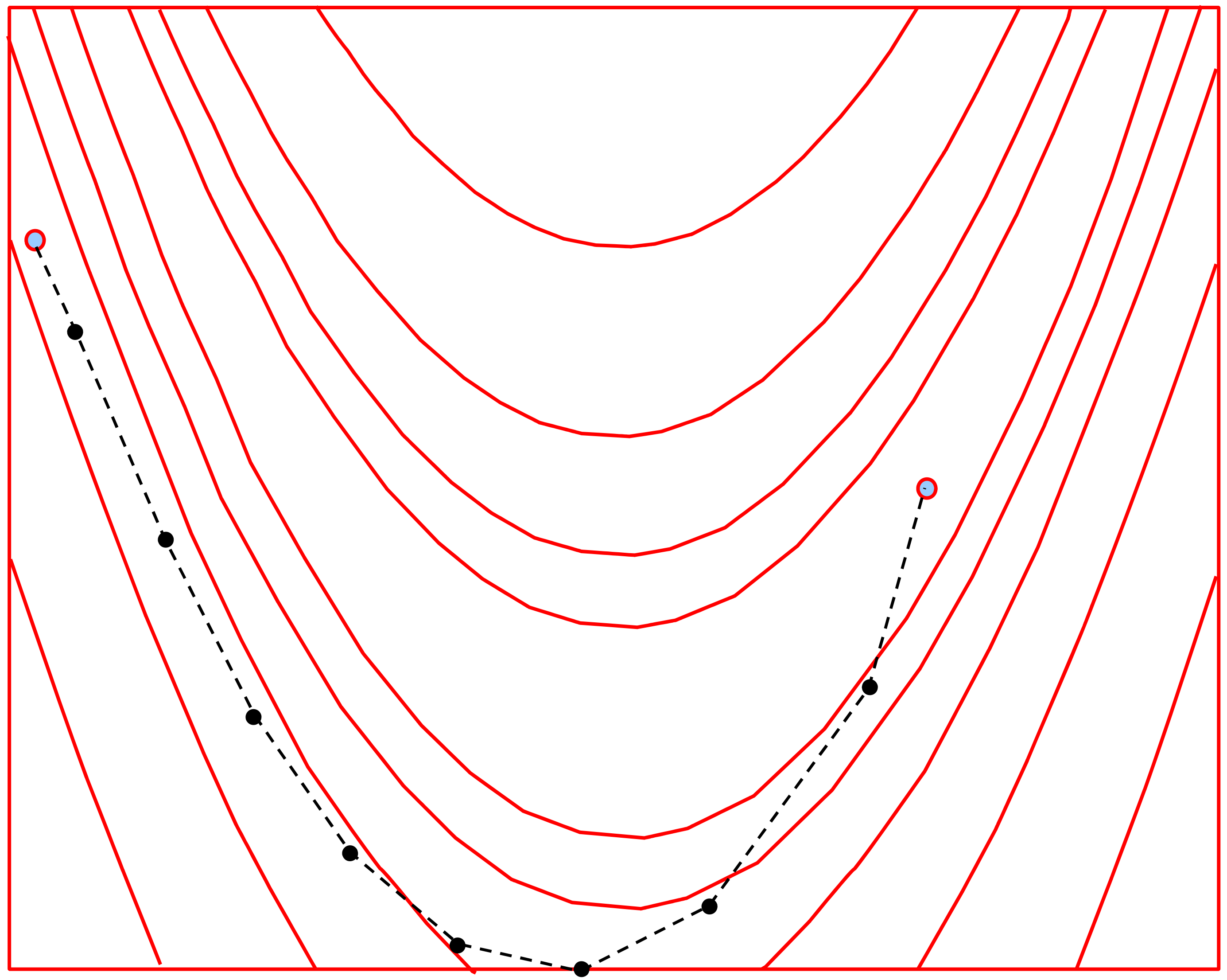
- With the approximate Hessian

$$\mathbf{H} \approx \mathbf{J}^{\mathbf{T}}\mathbf{J}$$

- No need for second derivative
- H is positive semi-definite


$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^\top \mathbf{g} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\,\mathbf{h}$
$\mathbf{x_k}$
$-\mathbf{H}^{-1}\mathbf{g}$
$\mathbf{x}_{\min}$
$\mathbf{x_{k+1}}$
x_1
x_2

Newton's method (48 evaluations)



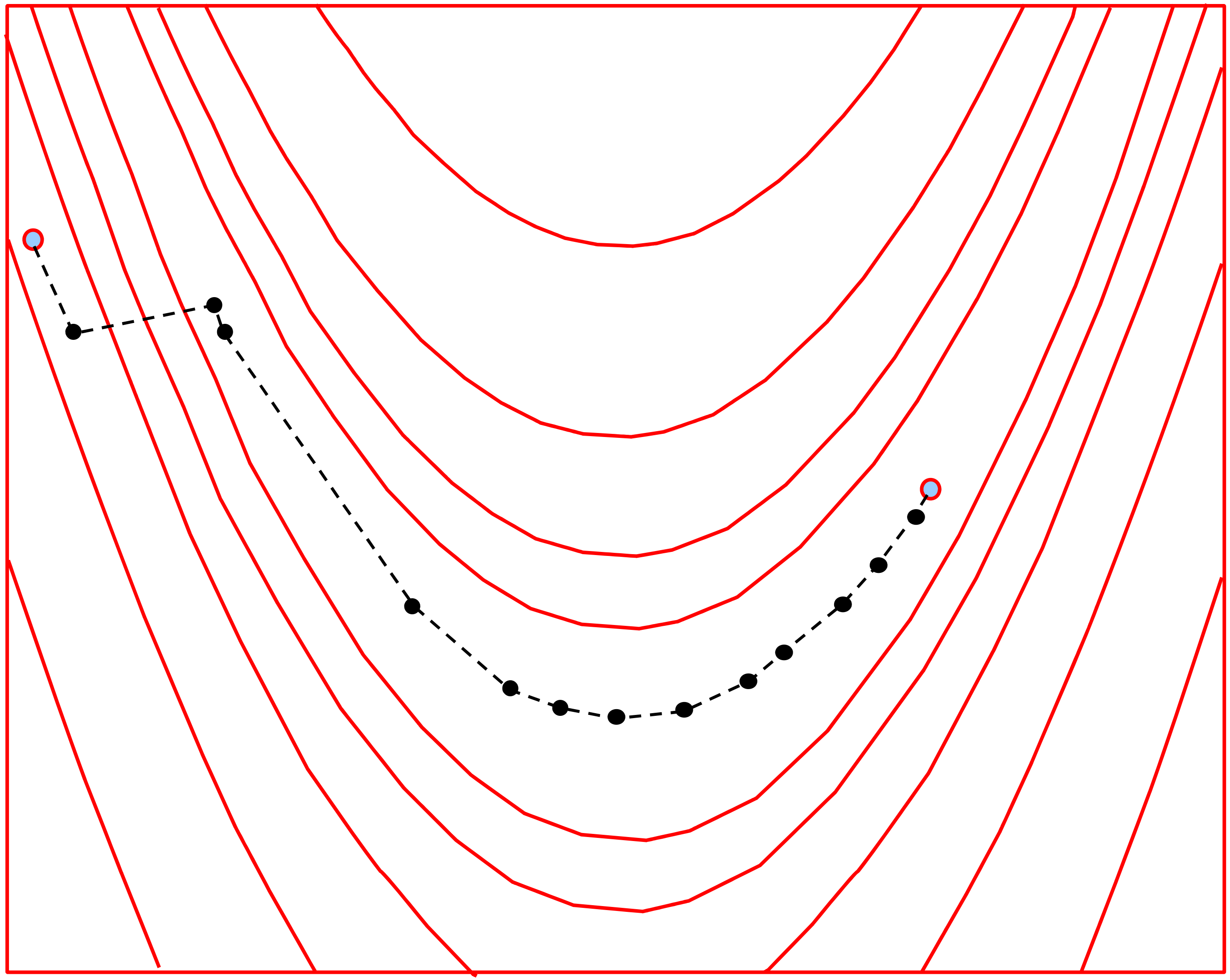
Levenberg-Marquardt

- Blends steepest descent and Gauss-Newton
- At each step, solve for the descent direction h

$$(\mathbf{J}^{\mathrm{T}}\mathbf{J} + \mu\mathbf{I})\mathbf{h} = -\mathbf{g}$$

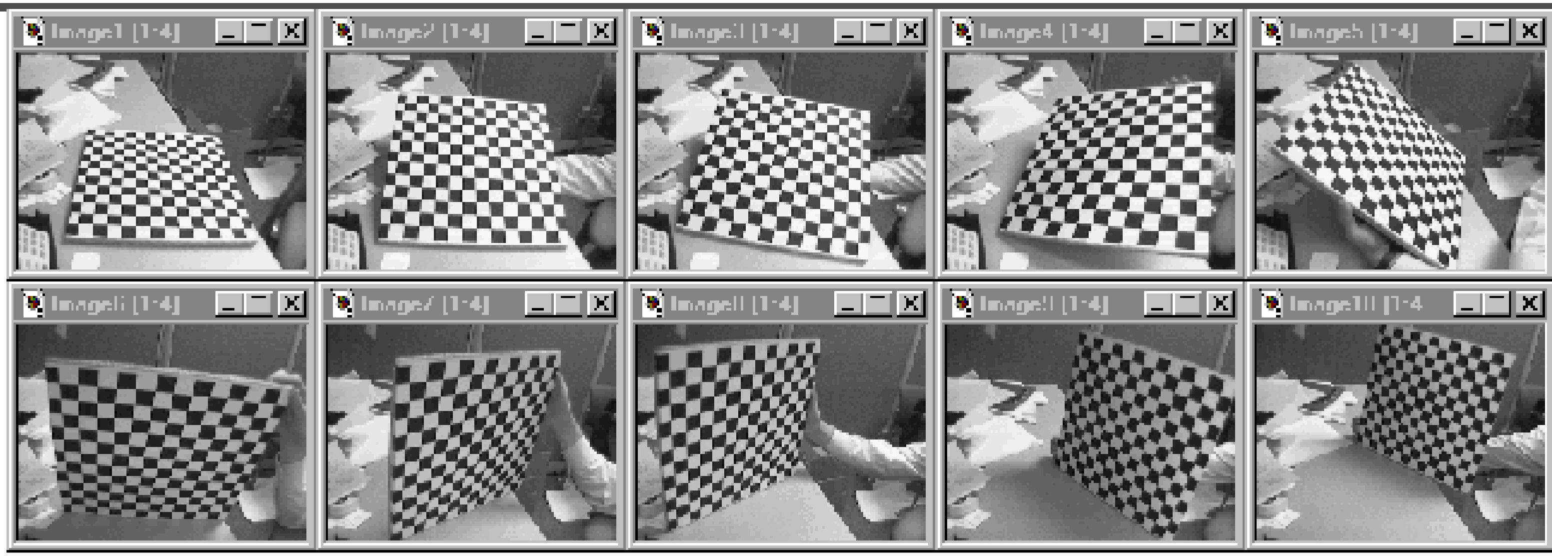
- If μ large, $\mathbf{h} \approx -\mathbf{g}$, steepest descent
- If μ small, $\mathbf{h} \approx -(\mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}\mathbf{g}$, Gauss-Newton

Levenberg-Marquardt (90 evaluations)



A popular calibration tool

Multi-plane calibration

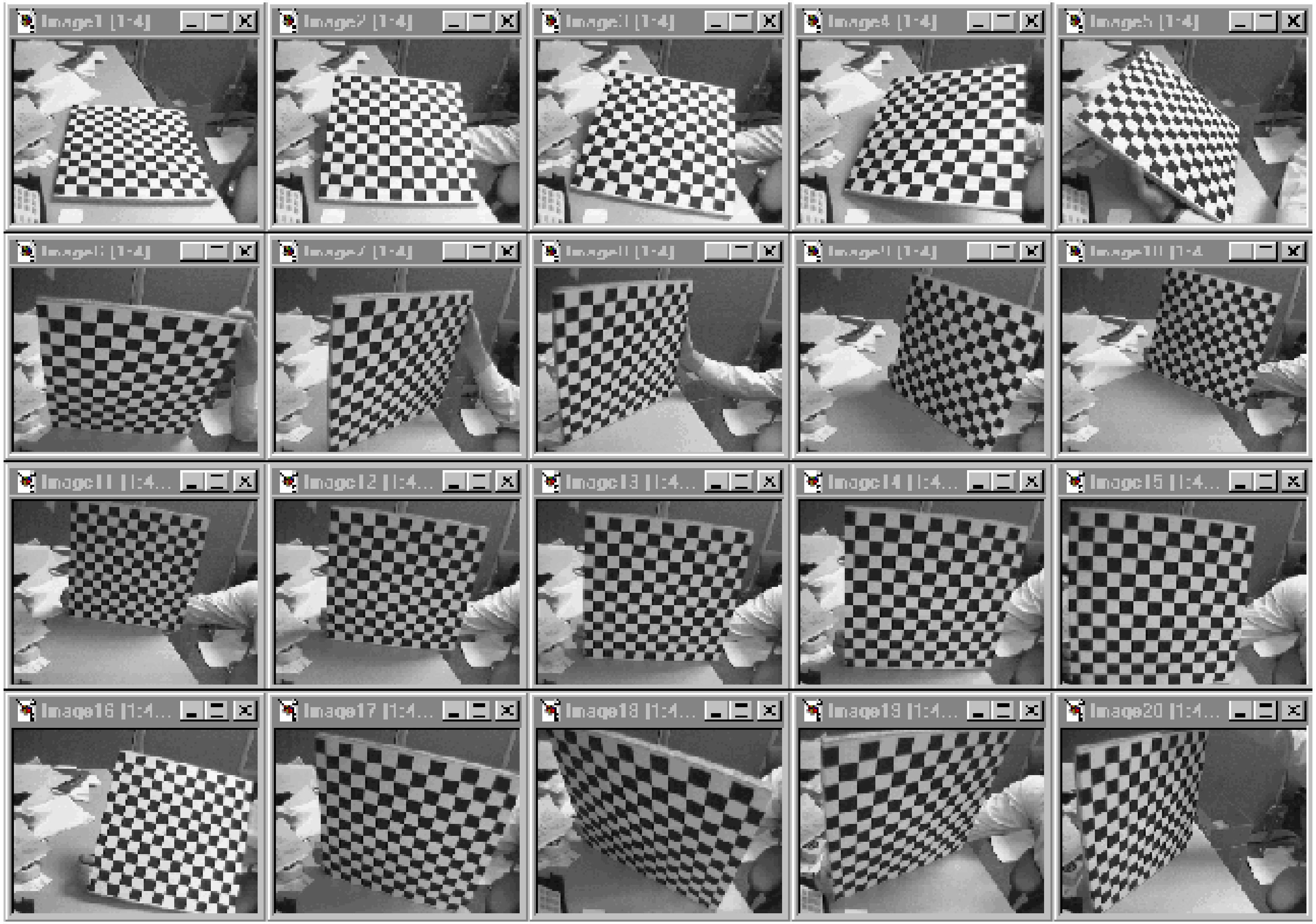


Images courtesy Jean-Yves Bouguet, Intel Corp.

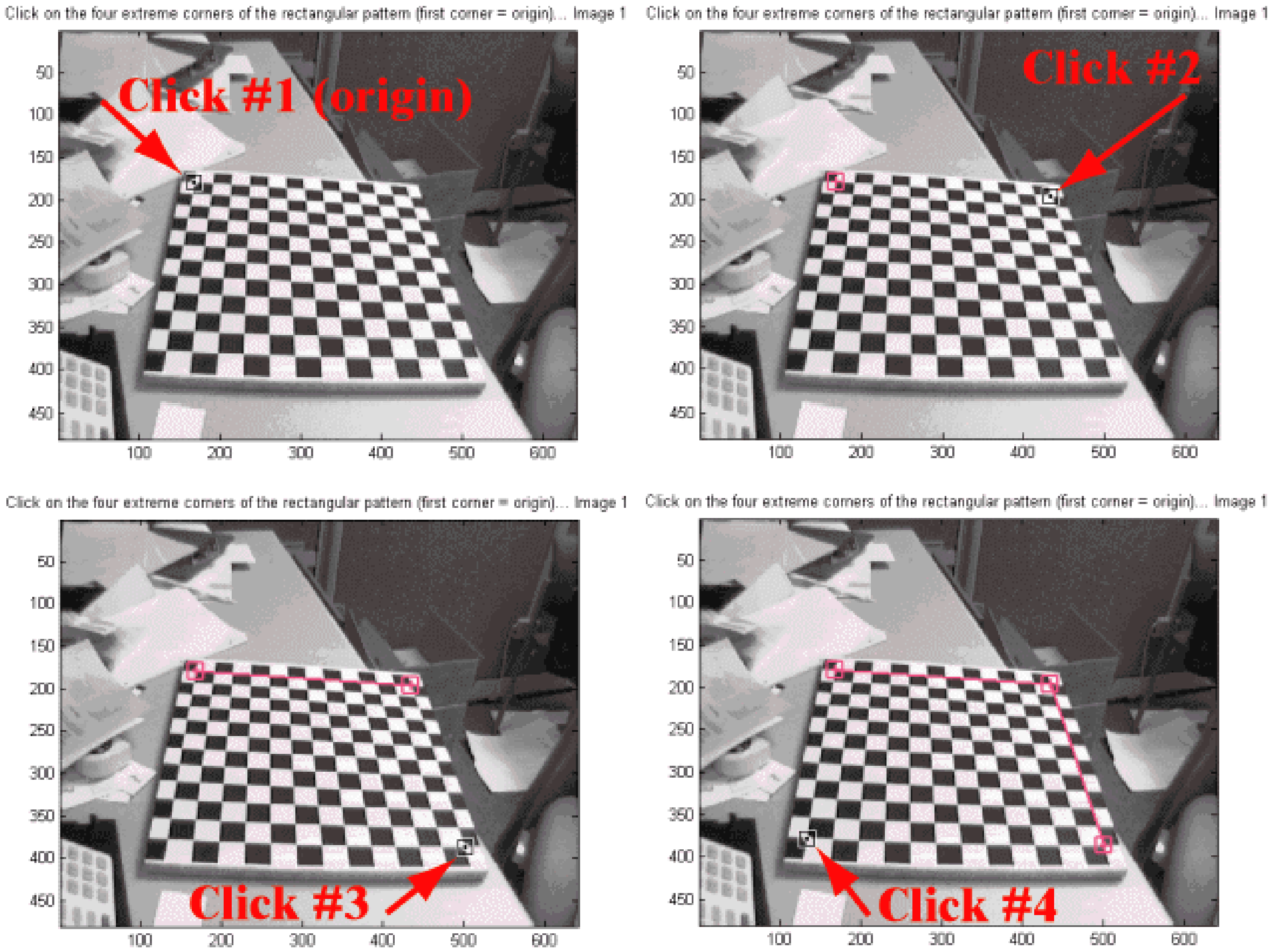
Advantage

- Only requires a plane
- Don't have to know positions/orientations
- Good code available online!
  - Intel's OpenCV library: http://www.intel.com/research/mrl/research/opencv/
  - Matlab version by Jean-Yves Bouget: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Zhengyou Zhang's web site: http://research.microsoft.com/~zhang/Calib/

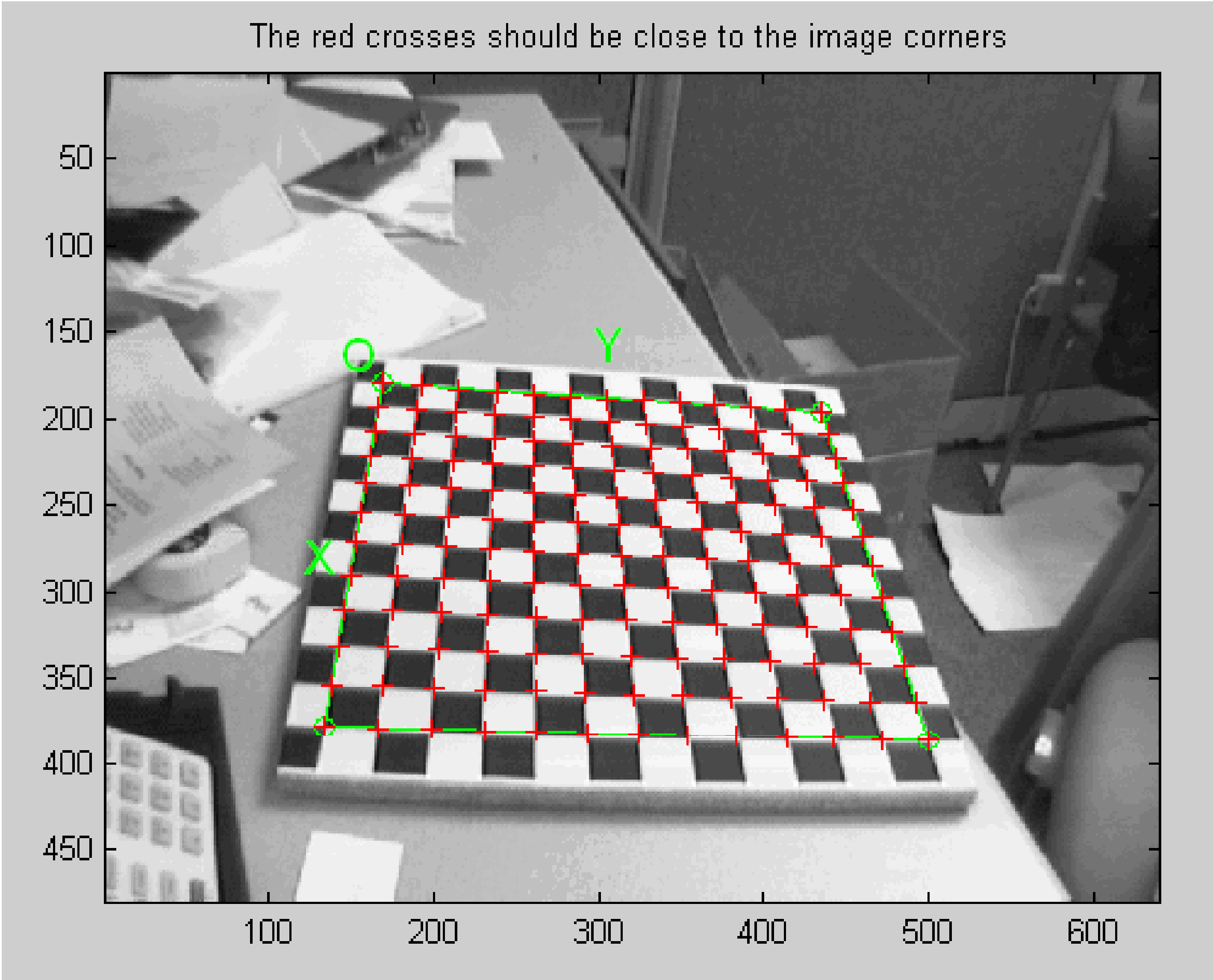
Step 1: data acquisition



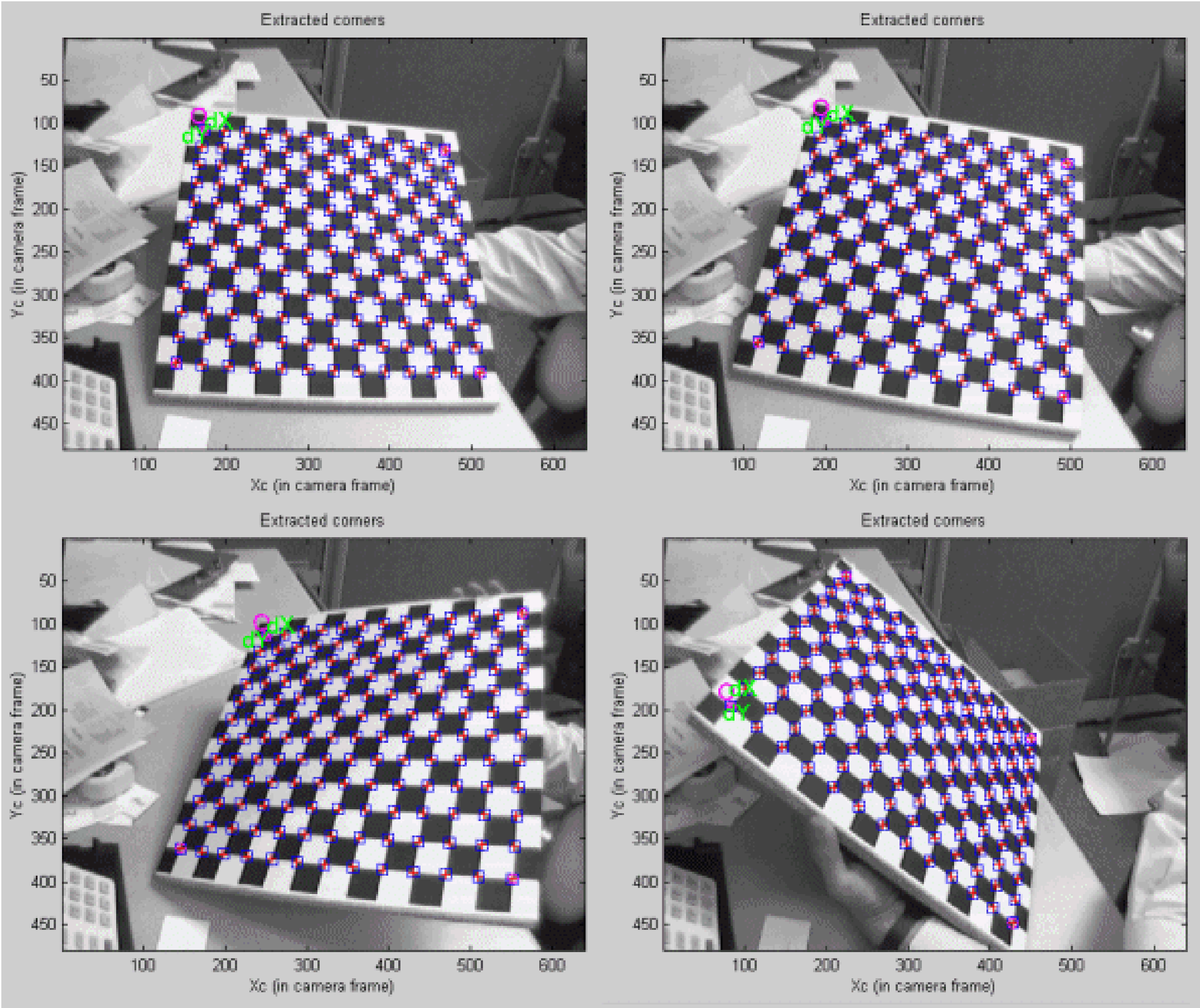
Step 2: specify corner order



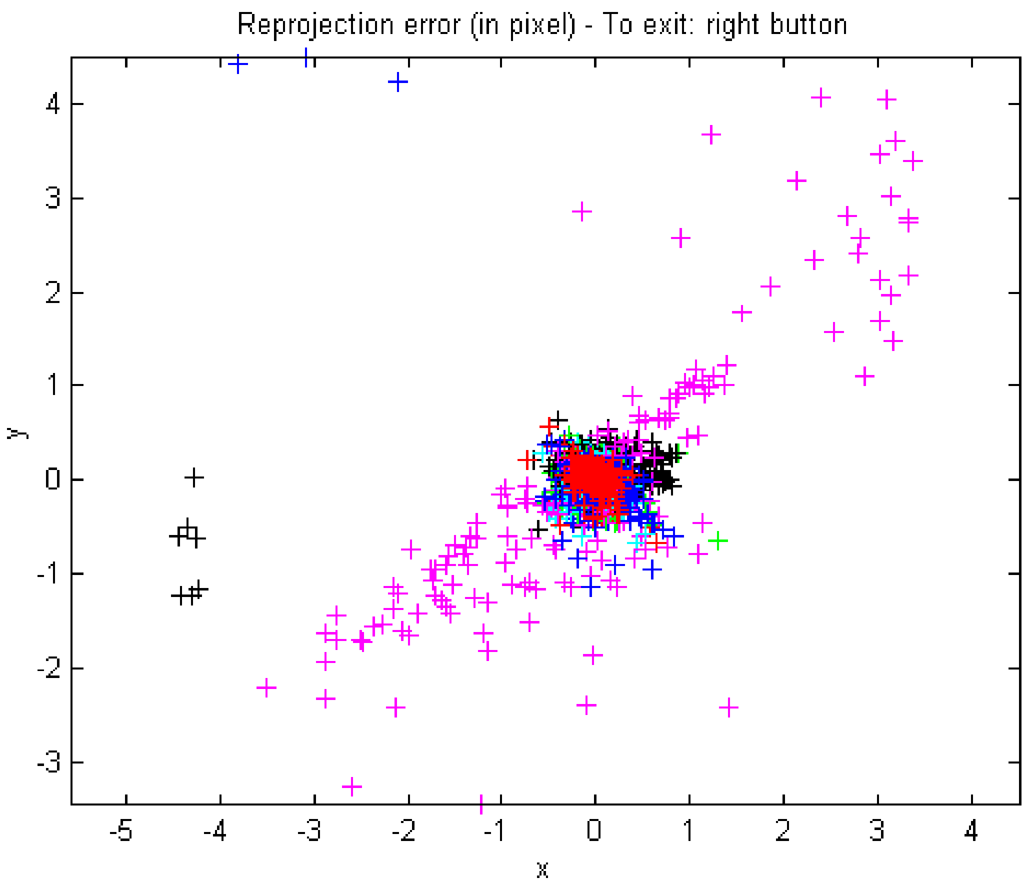
Step 3: corner extraction



Step 3: corner extraction



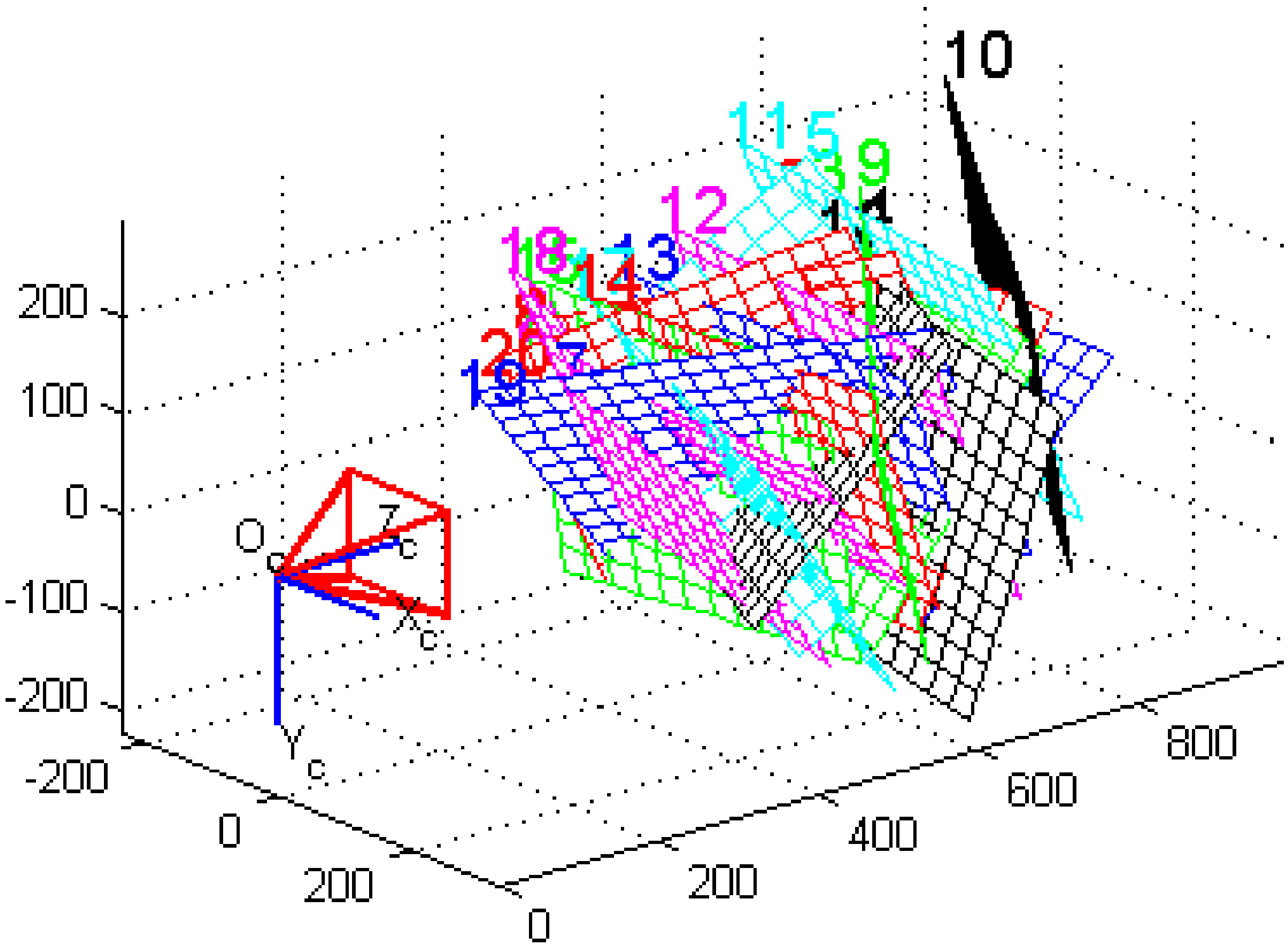
Step 4: minimize projection error



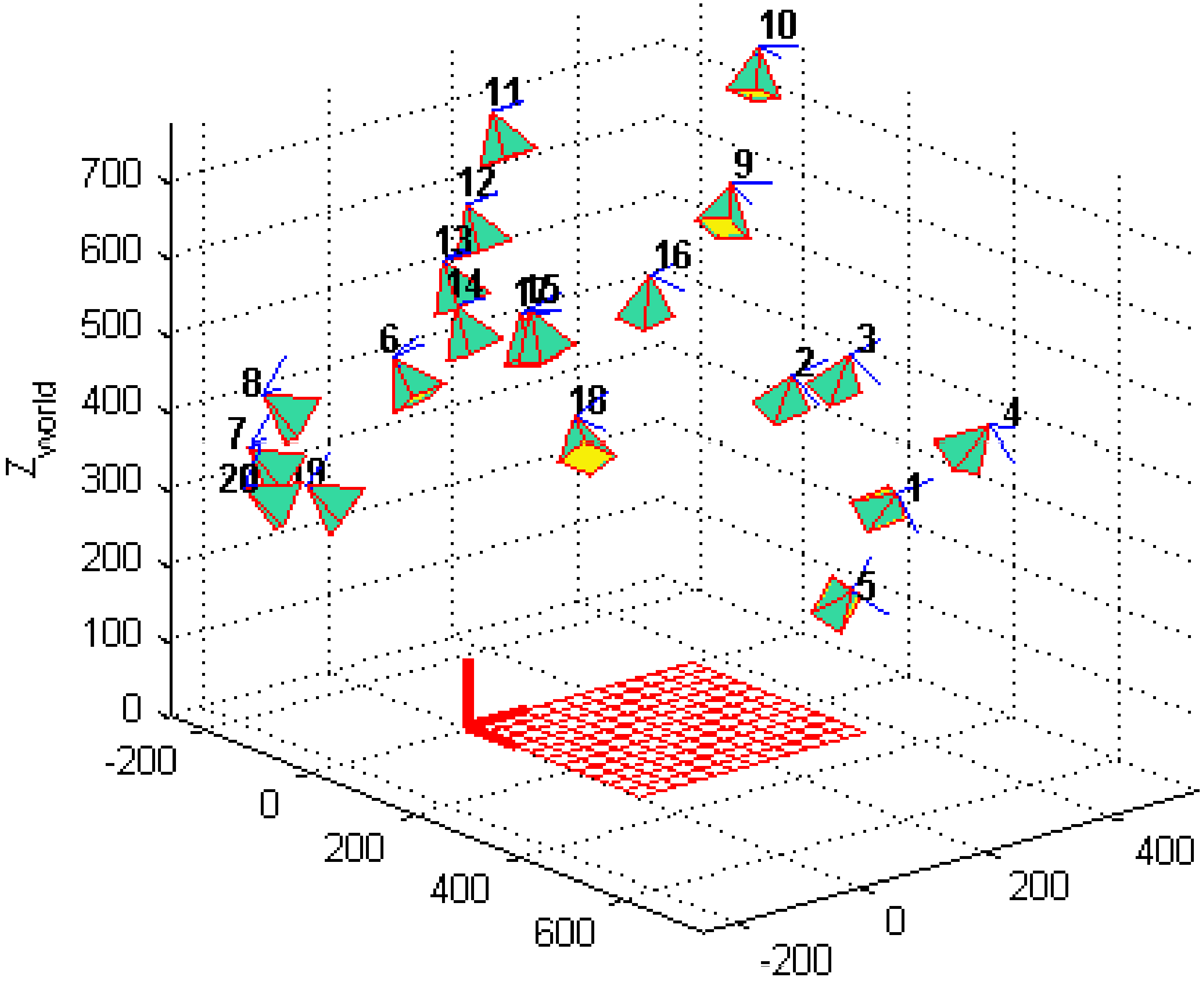

```
Calibration res

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes =
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002   0.00000 ]
Pixel error:          err = [ 0.11689   0.11500 ]
```

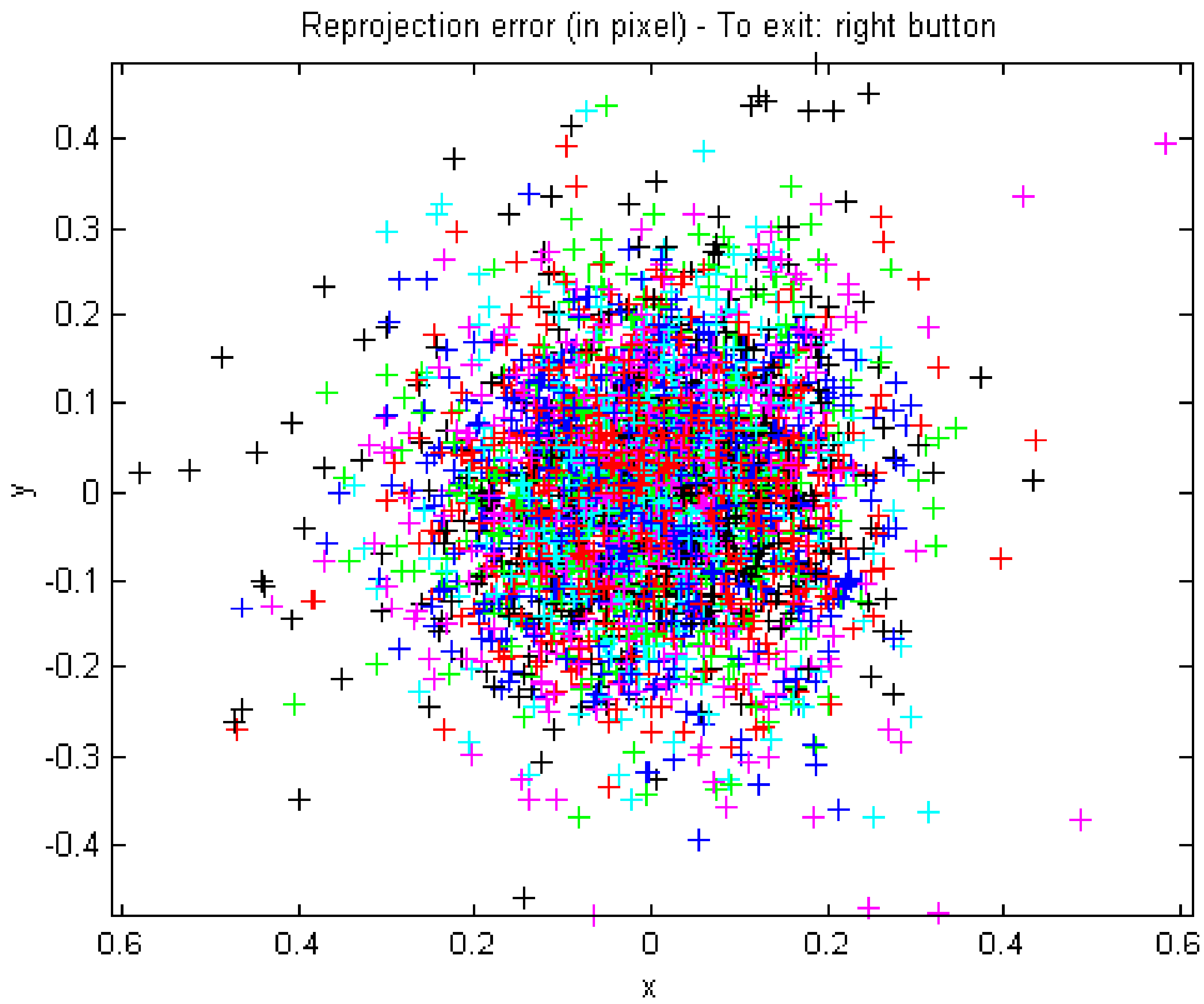
Step 4: camera calibration



Step 4: camera calibration



Step 5: refinement



Optimized parameters

```
Aspect ratio optimized (est_aspect_ratio = 1) -> both components of fc are estimated (DE
Principal point optimized (center_optim=1) - (DEFAULT). To reject principal point, set c
Skew not optimized (est_alpha=0) - (DEFAULT)
Distortion not fully estimated (defined by the variable est_dist):
     Sixth order distortion not estimated (est_dist(5)=0) - (DEFAULT) .

Main calibration optimization procedure - Number of images: 20
Gradient descent iterations: 1...2...3...4...5...done
Estimation of uncertainties...done


Calibration results after optimization (with uncertainties):

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes = 90.000
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002  0.00000 ] ± [ 0.0
Pixel error:          err = [ 0.11689   0.11500 ]

Note: The numerical errors are approximately three times the standard deviations (for re
```

Applications

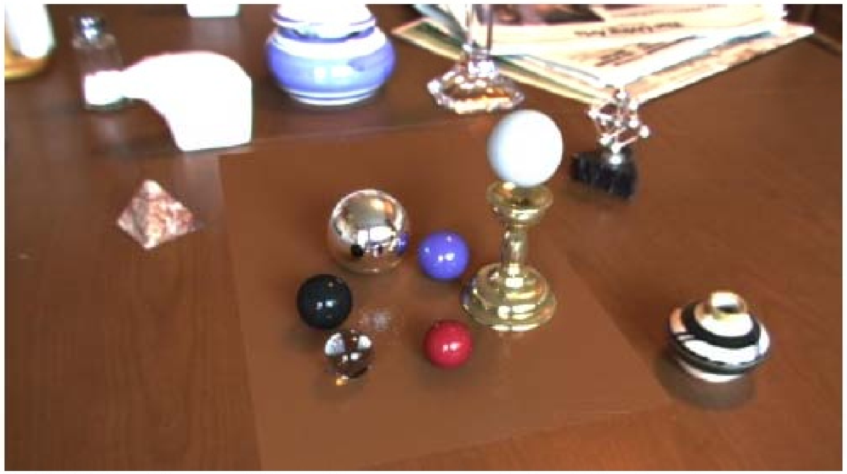
How is calibration used?

- Good for recovering intrinsic parameters; It is thus useful for many vision applications
- Since it requires a calibration pattern, it is often necessary to remove or replace the pattern from the footage or utilize it in some ways…

Example of calibration

(a) Background photograph

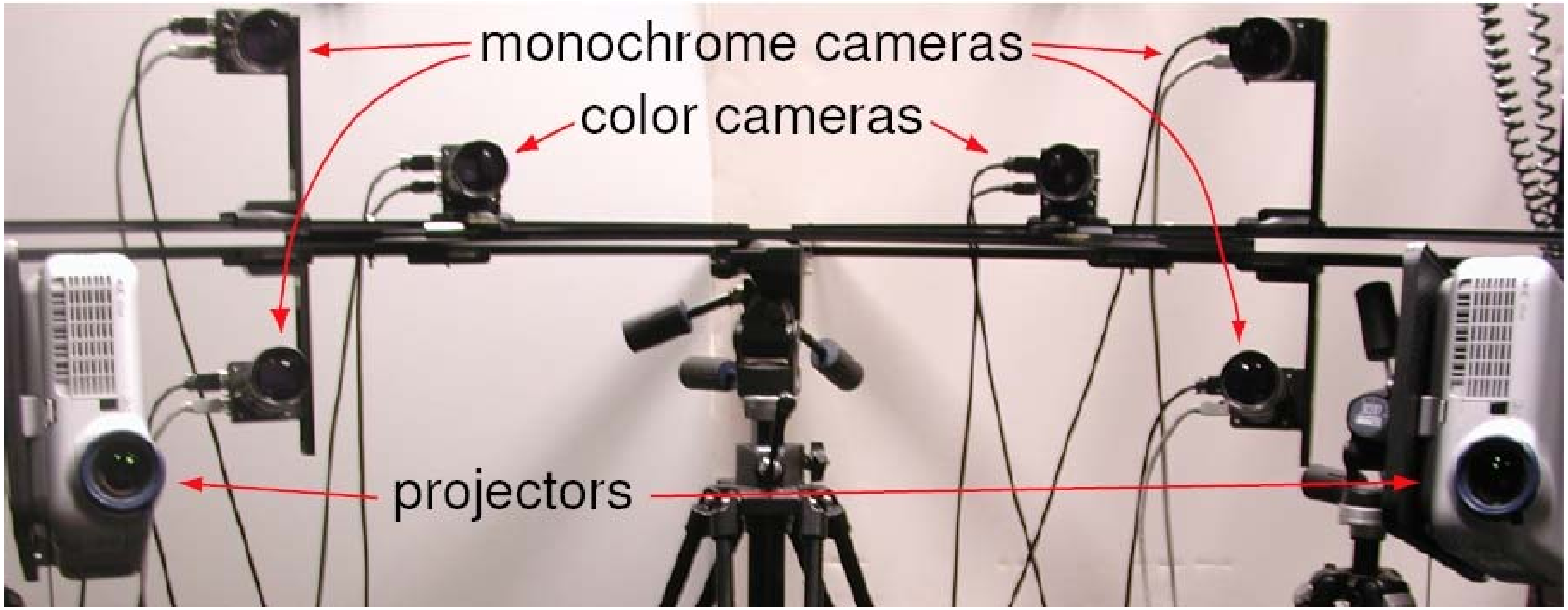
(b) Camera calibration grid and light probe



(c) Objects and local scene matched to background

(g) Final result with differential rendering

Example of calibration



Example of calibration

- Videos from GaTech
- DasTatoo, MakeOf
- P!NG, MakeOf
- Work, MakeOf
- LifeInPaints, MakeOf

PhotoBook

PhotoBook

MakeOf