Camera calibration

Digital Visual Effects

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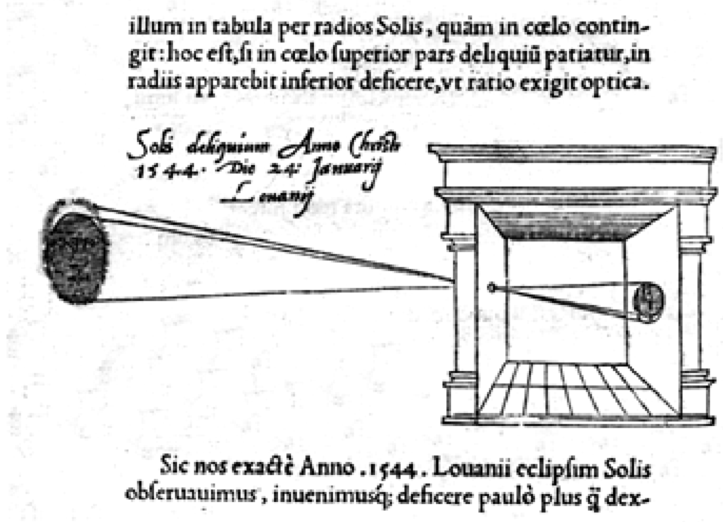
with slides by Richard Szeliski, Steve Seitz,, Fred Pighin and Marc Pollefyes

Outline

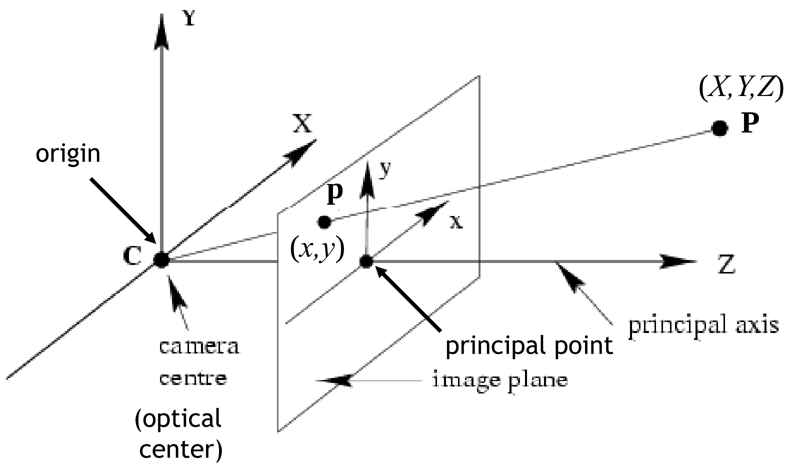
- Camera projection models
- Camera calibration
- Nonlinear least square methods
- A camera calibration tool
- Applications

Camera projection models

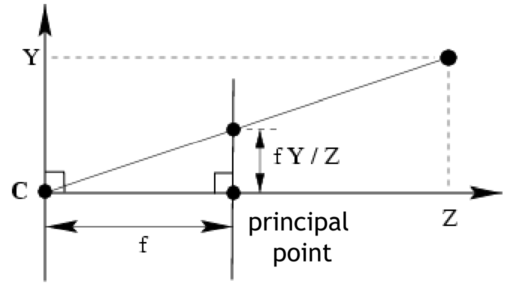
Pinhole camera



Pinhole camera model



Pinhole camera model

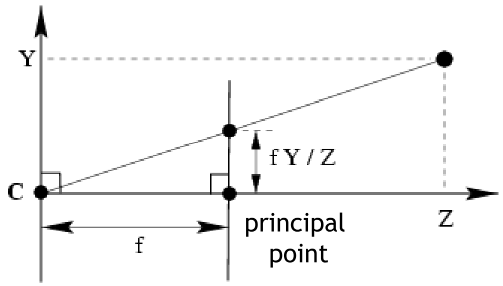


$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

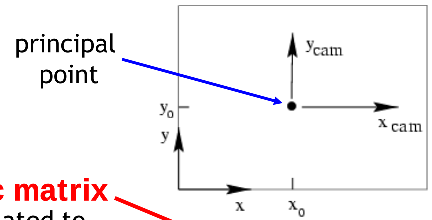
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole camera model



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset



intrinsic matrix
only related to
camera projection

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Intrinsic matrix

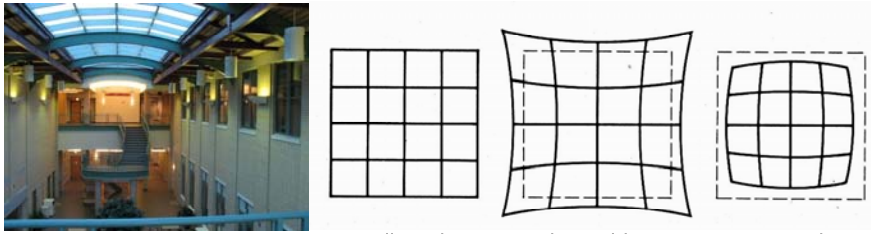
Is this form of $\mathbf{K}$ good enough?

$$\mathbf{K} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

- non-square pixels (digital video)
- skew
- radial distortion

$$\mathbf{K} = \begin{bmatrix} fa & s & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

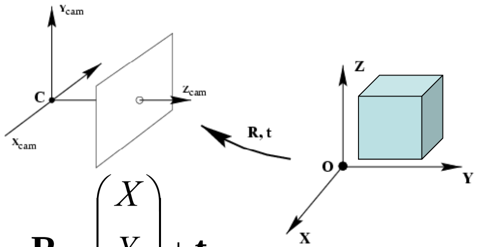
Distortion



No distortion   Pin cushion   Barrel

- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

Camera rotation and translation



$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}_{3\times3} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mathbf{t}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{R}|\mathbf{t}] \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

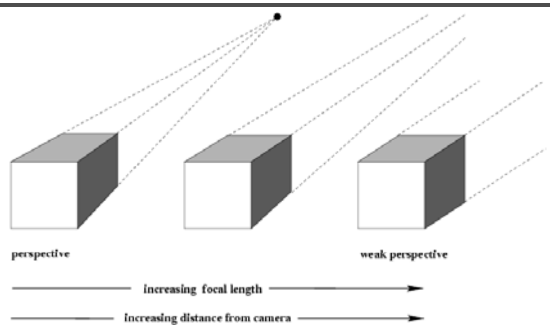
$$\mathbf{x} \sim \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}$$

extrinsic matrix

Two kinds of parameters

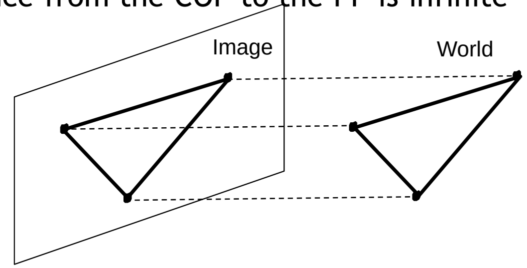
- *internal* or *intrinsic* parameters such as focal length, optical center, aspect ratio: *what kind of camera?*
- *external* or *extrinsic* (pose) parameters including rotation and translation: *where is the camera?*

Other projection models



Orthographic projection

- Special case of perspective projection
  - Distance from the COP to the PP is infinite



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

  - Also called “parallel projection”: (x, y, z) → (x, y)

Other types of projections

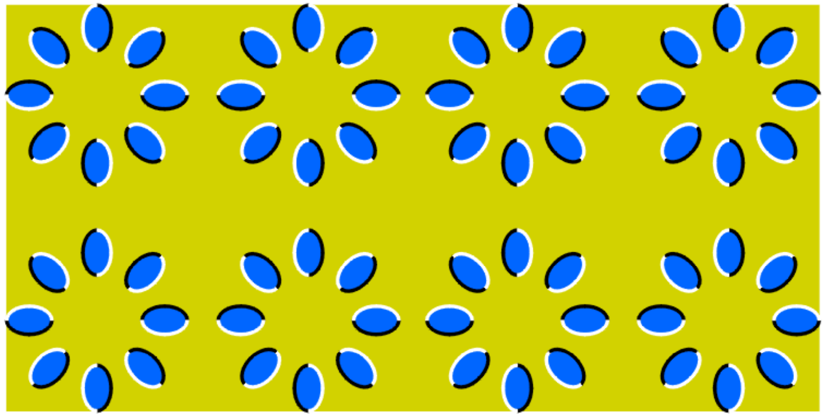
- Scaled orthographic
  - Also called “weak perspective”

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/d \end{bmatrix} \Rightarrow (dx, dy)$$

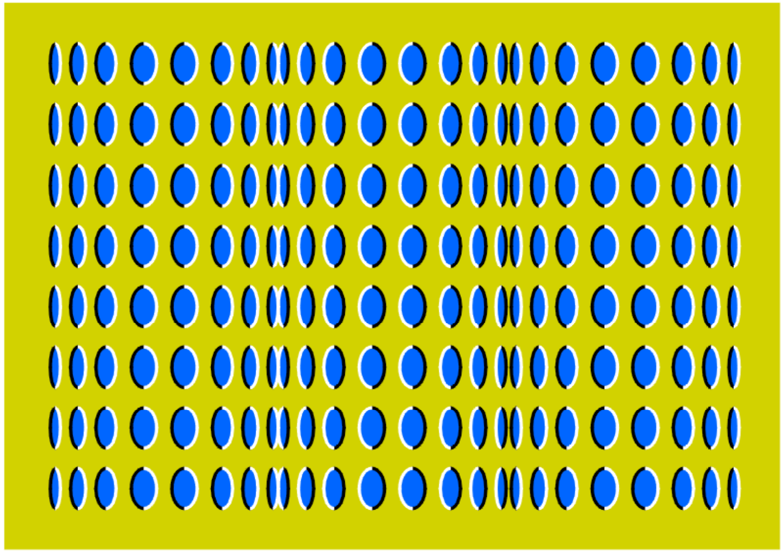
- Affine projection
  - Also called “paraperspective”

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

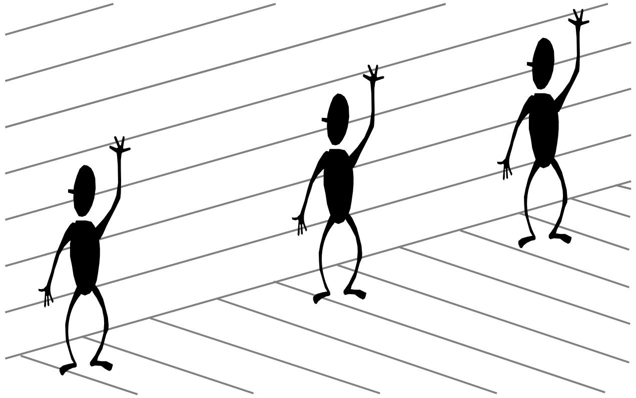
Illusion



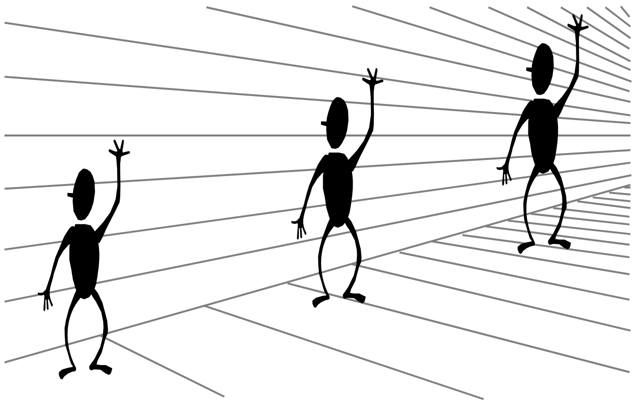
Illusion



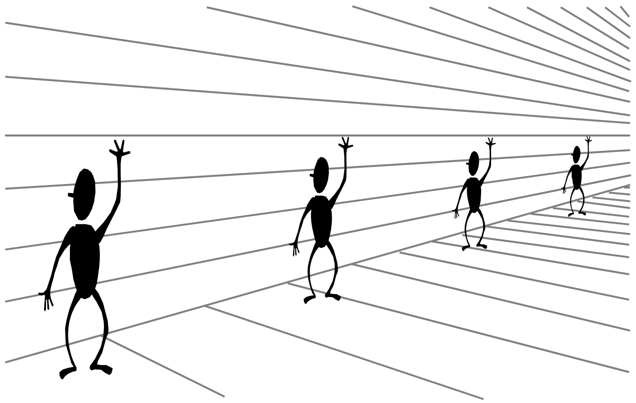
Fun with perspective



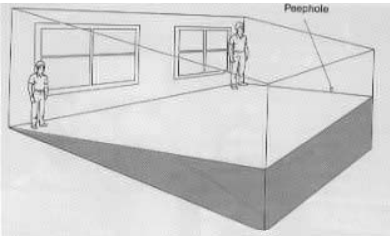
Perspective cues



Perspective cues

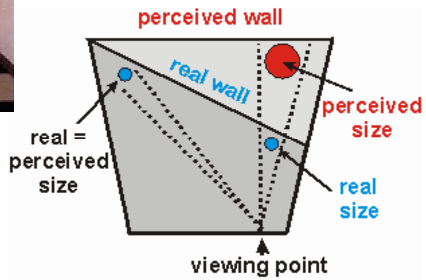


Fun with perspective



Ames room

Ames video BBC story



Forced perspective in LOTR

Camera calibration

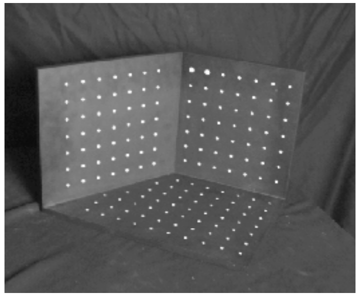
Camera calibration

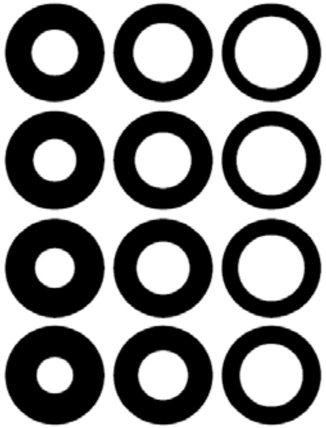
- Estimate both intrinsic and extrinsic parameters. Two main categories:

1. Photometric calibration: uses reference objects with known geometry
2. Self calibration: only assumes static scene, e.g. structure from motion

Camera calibration approaches

1. linear regression (least squares)
2. nonlinear optimization





Chromaglyphs (HP research)

Camera calibration

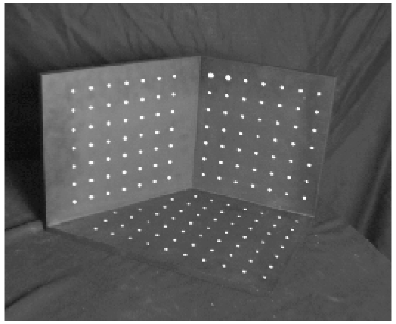
Linear regression

$$\mathbf{x} \sim \mathbf{K}\left[\mathbf{R}\middle|\mathbf{t}\right]\mathbf{X} = \mathbf{M}\mathbf{X}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear regression

- Directly estimate 11 unknowns in the **M** matrix using known 3D points (X_i,Y_i,Z_i) and measured feature positions (u_i,v_i)



Linear regression

$$u_i = \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$v_i = \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$u_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}$$

$$v_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}$$

Linear regression

$$\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i & -u_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i & -v_i \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear regression

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Solve for Projection Matrix M using least-square techniques

Normal equation

Given an overdetermined system

$$\mathbf{Ax} = \mathbf{b}$$

the normal equation is that which minimizes the sum of the square differences between left and right sides

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

Linear regression

- Advantages:
  - All specifics of the camera summarized in one matrix
  - Can predict where any world point will map to in the image
- Disadvantages:
  - Doesn't tell us about particular parameters
  - Mixes up internal and external parameters
    - pose specific: move the camera and everything breaks
  - More unknowns than true degrees of freedom

Nonlinear optimization

- A probabilistic view of least square
- Feature measurement equations

$$\begin{aligned} u_i &= f(\mathbf{M}, \mathbf{x}_i) + n_i = \hat{u}_i + n_i, \quad n_i \sim N(0, \sigma) \\ v_i &= g(\mathbf{M}, \mathbf{x}_i) + m_i = \hat{v}_i + m_i, \quad m_i \sim N(0, \sigma) \end{aligned}$$

- Probability of $\mathbf{M}$ given $\{(u_i, v_i)\}$

$$\begin{aligned} P &= \prod_i p(u_i|\hat{u}_i)p(v_i|\hat{v}_i) \\ &= \prod_i e^{-(u_i-\hat{u}_i)^2/\sigma^2} e^{-(v_i-\hat{v}_i)^2/\sigma^2} \end{aligned}$$

Optimal estimation

- Likelihood of $\mathbf{M}$ given $\{(u_i, v_i)\}$

$$L = -\log P = \sum_i (u_i - \hat{u}_i)^2/\sigma_i^2 + (v_i - \hat{v}_i)^2/\sigma_i^2$$

- It is a least square problem (but not necessarily linear least square)
- How do we minimize L?

Optimal estimation

DigiVFX

- Non-linear regression (least squares), because the relations between $\hat{u}_i$ and u_i are non-linear functions of $\mathbf{M}$

unknown parameters

We could have terms like $f \cos\theta$ in this

$$\mathbf{u} - \hat{\mathbf{u}} \sim \mathbf{u} - \mathbf{K}\left[\mathbf{R} \middle| \mathbf{t}\right]\mathbf{X}$$

known constant

- We can use Levenberg-Marquardt method to minimize it

Nonlinear least square methods

Least square fitting

DigiVFX

Least Squares Problem

Find $\mathbf{x}^*$, a local minimizer for

$$F(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}(f_i(\mathbf{x}))^2 ,$$

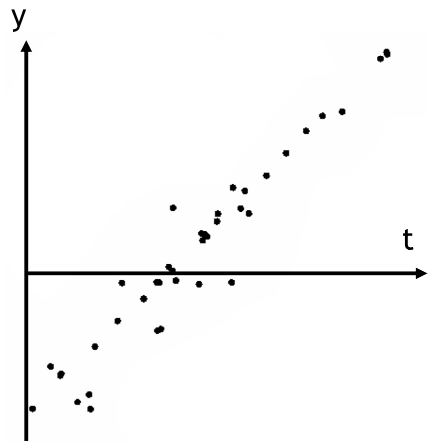
where $f_i : \mathbb{R}^n \mapsto \mathbb{R},\ i = 1, \ldots, m$ are given functions, and $m \geq n$.

number of data points

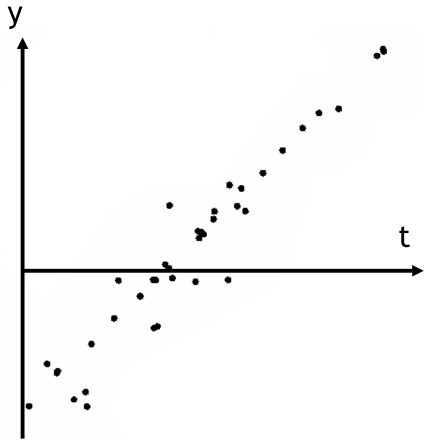
number of parameters

Linear least square fitting

DigiVFX



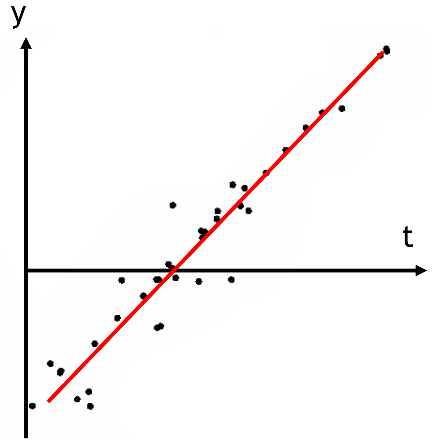
Linear least square fitting



model parameters

$$y(t) = M(t; \mathbf{x})$$

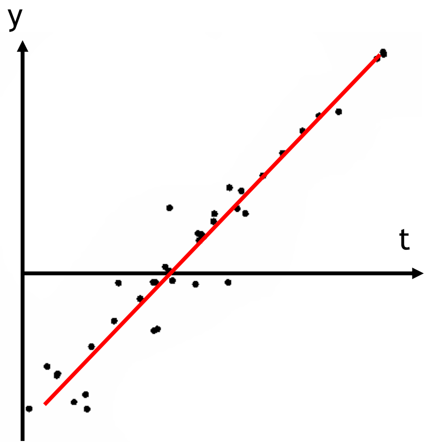
Linear least square fitting



model parameters

$$y(t) = M(t; \mathbf{x}) = x_0 + x_1 t$$

Linear least square fitting



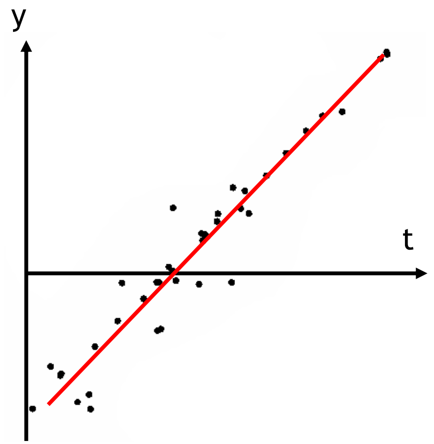
model parameters

$$y(t) = M(t; \mathbf{x}) = x_0 + x_1 t$$

$$f_i(x) = y_i - \boxed{M(t_i; \mathbf{x})}$$

residual prediction

Linear least square fitting



model parameters

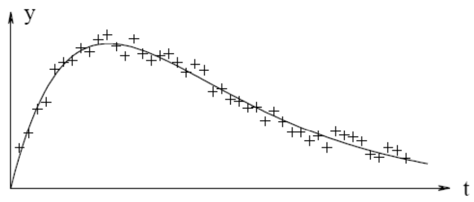
$$y(t) = M(t; \mathbf{x}) = x_0 + x_1 t$$

$$f_i(x) = y_i - \boxed{M(t_i; \mathbf{x})}$$

residual prediction

$M(t; \mathbf{x}) = x_0 + x_1 t + x_2 t^3$ is linear, too.

Nonlinear least square fitting



model $M(t;\mathbf{x}) = x_3 e^{x_1 t} + x_4 e^{x_2 t}$

parameters $\mathbf{x} = [x_1, x_2, x_4, x_4]^T$

residuals $f_i(\mathbf{x}) = y_i - M(t_i;\mathbf{x})$

$$= y_i - \left(x_3 e^{x_1 t} + x_4 e^{x_2 t}\right)$$

Function minimization

Least square is related to function minimization.

Global Minimizer
Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find

$$\mathbf{x}^+ = \text{argmin}_{\mathbf{x}} \{F(\mathbf{x})\} .$$

It is very hard to solve in general. Here, we only consider a simpler problem of finding local minimum.

Local Minimizer
Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find $\mathbf{x}^*$ so that

$$F(\mathbf{x}^*) \le F(\mathbf{x}) \quad \text{for} \quad \|\mathbf{x} - \mathbf{x}^*\| < \delta .$$

Function minimization

We assume that the cost function F is differentiable and so smooth that the following *Taylor expansion* is valid,[2)]

$$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^\top \mathbf{g} + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}\,\mathbf{h} + O(\|\mathbf{h}\|^3) ,$$

where $\mathbf{g}$ is the *gradient*,

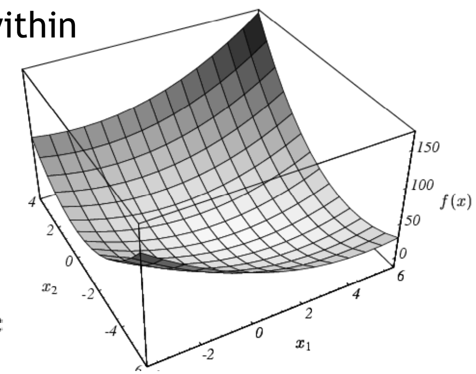
$$\mathbf{g} \equiv \mathbf{F}'(\mathbf{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial F}{\partial x_n}(\mathbf{x}) \end{bmatrix} ,$$

and $\mathbf{H}$ is the *Hessian*,

$$\mathbf{H} \equiv \mathbf{F}''(\mathbf{x}) = \left[\frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{x})\right] .$$

Quadratic functions

Approximate the function with a quadratic function within a small neighborhood

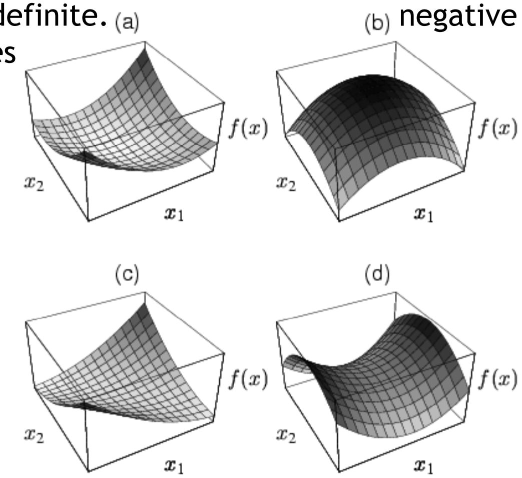


$$f(x) = \frac{1}{2}x^T A x - b^T x + c$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \qquad c = 0.$$

Quadratic functions

A is positive definite.
All eigenvalues are positive.
For all x, $x^TAx>0$.



(b) negative definite

A is singular

A is indefinite

Function minimization

Theorem 1.5. Necessary condition for a local minimizer.
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}.$$

Why?

By definition, if $\mathbf{x}^*$ is a local minimizer,

$\|\mathbf{h}\|$ is small enough $\longrightarrow$ $\mathbf{F(x^*+h)>F(x^*)}$

$$\mathbf{F(x^*+h)=F(x^*)+h^TF'(x^*)+O(\|h\|^2)}$$

Function minimization

Theorem 1.5. Necessary condition for a local minimizer.
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}.$$

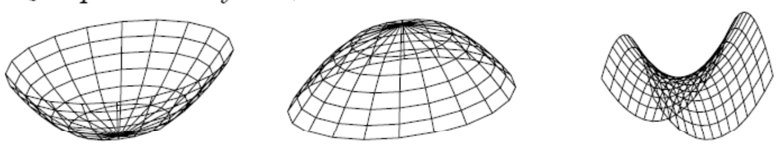
Definition 1.6. Stationary point. If

$$\mathbf{g}_s \equiv \mathbf{F}'(\mathbf{x}_s) = \mathbf{0},$$

then $\mathbf{x}_s$ is said to be a *stationary point* for F.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$

$\mathbf{H}_s$ is *positive definite*



a) minimum *b) maximum* *c) saddle point*

Function minimization

Theorem 1.8. Sufficient condition for a local minimizer.
Assume that $\mathbf{x}_s$ is a stationary point and that $\mathbf{F}''(\mathbf{x}_s)$ is positive definite. Then $\mathbf{x}_s$ is a local minimizer.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$
$$\text{with } \mathbf{H}_s = \mathbf{F}''(\mathbf{x}_s)$$

If we request that $\mathbf{H}_s$ is *positive definite*, then its eigenvalues are greater than some number $\delta > 0$

$$\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} > \delta\,\|\mathbf{h}\|^2$$

Descent methods

$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \to \mathbf{x}^*$ for $k \to \infty$

1. Find a descent direction $\mathbf{h}_d$
2. find a step length giving a good decrease in the F-value.

Algorithm Descent method

```
begin
  k := 0;  x := x0;  found := false                {Starting point}
  while (not found) and (k < kmax)
    hd := search_direction(x)                     {From x and downhill}
    if (no such h exists)
      found := true                               {x is stationary}
    else
      α := step_length(x, hd)                     {from x in direction hd}
      x := x + α hd;   k := k+1                   {next iterate}
end
```

Descent direction

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

Definition Descent direction.

$\mathbf{h}$ is a descent direction for F at $\mathbf{x}$ if $\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) < 0$.

Steepest descent method

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

$$\frac{F(\mathbf{x}) - F(\mathbf{x}+\alpha\mathbf{h})}{\alpha\|\mathbf{h}\|} = -\frac{1}{\|\mathbf{h}\|}\mathbf{h}^\top \mathbf{F}'(\mathbf{x}) = -\|\mathbf{F}'(\mathbf{x})\|\cos\theta$$

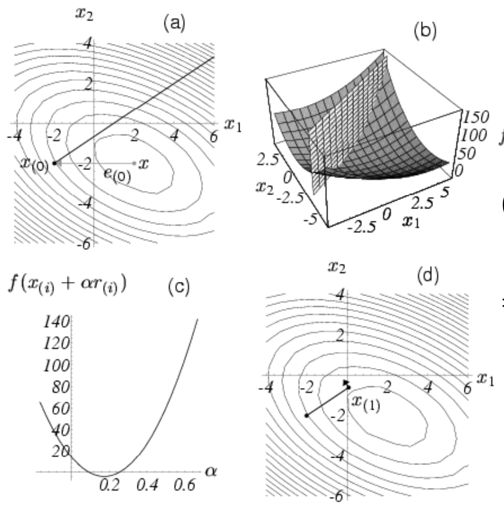
the decrease of $F(x)$ per unit along h direction

greatest gain rate if $\theta = \pi \;\longrightarrow\; \mathbf{h}_{sd} = -\mathbf{F}'(\mathbf{x})$

h_{sd} is a descent direction because $h^T_{sd} F'(x) = -F'(x)^2 < 0$

Line search

$\varphi(\alpha) = F(\mathbf{x}+\alpha\mathbf{h})$, $\mathbf{x}$ and $\mathbf{h}$ fixed, $\alpha \ge 0$.



Find α so that

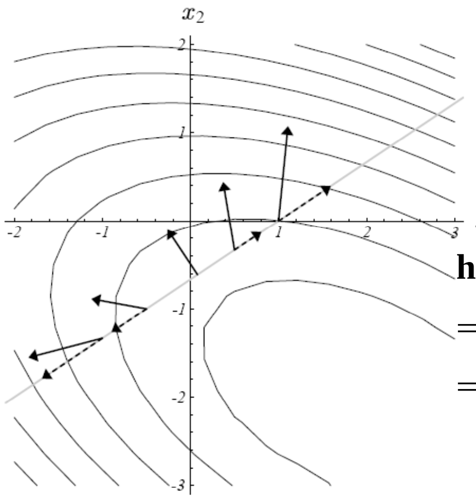
$\varphi(\alpha) = \mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})$

is minimum

$$0 = \frac{\partial \varphi(\alpha)}{\partial \alpha} = \frac{\partial \mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})}{\partial \alpha}$$
$$= \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \alpha} = \mathbf{h}^T\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

Line search



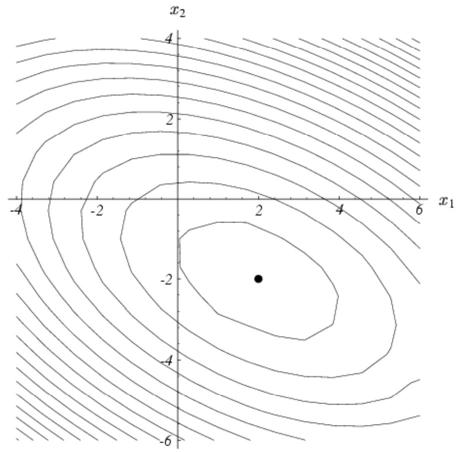
$$\mathbf{h}^T\mathbf{F}'(\mathbf{x}_0+\alpha\mathbf{h})=0$$

$$\mathbf{h}=-\mathbf{F}'(\mathbf{x}_0)$$

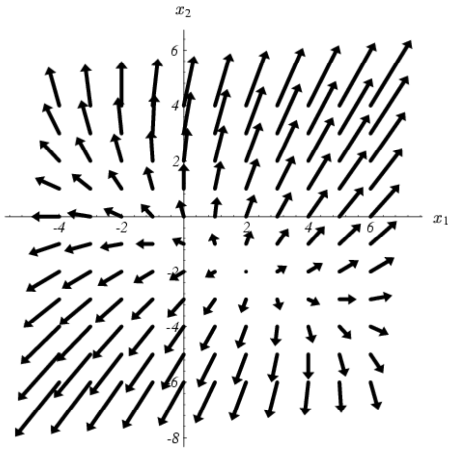
$$\mathbf{h}^T\mathbf{F}'(\mathbf{x}_0+\alpha\mathbf{h})$$
$$=\mathbf{h}^T\left(\mathbf{F}'(\mathbf{x}_0)+\alpha\mathbf{F}''(\mathbf{x}_0)^T\mathbf{h}\right)$$
$$=-\mathbf{h}^T\mathbf{h}+\alpha\mathbf{h}^T\mathbf{H}\mathbf{h}=0$$

$$\boxed{\alpha=\frac{\mathbf{h}^T\mathbf{h}}{\mathbf{h}^T\mathbf{H}\mathbf{h}}}$$

Steepest descent method

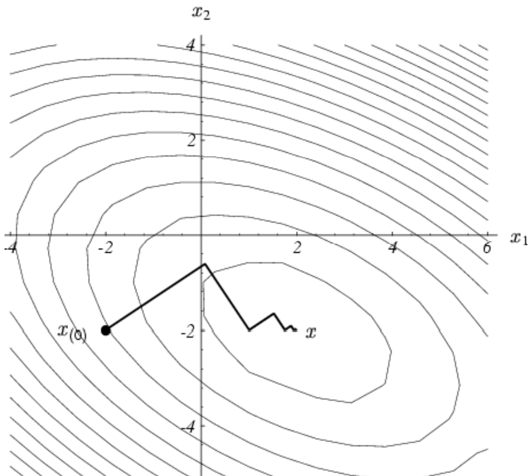


isocontour



gradient

Steepest descent method



It has good performance in the initial stage of the iterative process. Converge very slow with a linear rate.

Newton's method

$\mathbf{x}^*$ is a stationary point $\rightarrow$ it satisfies $\mathbf{F}'(\mathbf{x}^*)=\mathbf{0}$.

$$\begin{aligned}\mathbf{F}'(\mathbf{x}+\mathbf{h}) &= \mathbf{F}'(\mathbf{x})+\mathbf{F}''(\mathbf{x})\mathbf{h}+O(\|\mathbf{h}\|^2)\\ &\simeq \mathbf{F}'(\mathbf{x})+\mathbf{F}''(\mathbf{x})\mathbf{h} \quad \text{for } \|\mathbf{h}\| \text{ sufficiently small}\end{aligned}$$

$$\rightarrow \quad \mathbf{H}\,\mathbf{h}_n=-\mathbf{F}'(\mathbf{x}) \quad \text{with } \mathbf{H}=\mathbf{F}''(\mathbf{x})$$
$$\mathbf{x}:=\mathbf{x}+\mathbf{h}_n$$

Suppose that $\mathbf{H}$ is positive definite

$\rightarrow$ $\mathbf{u}^\top\mathbf{H}\,\mathbf{u}>0$ for all nonzero $\mathbf{u}$.

$\rightarrow$ $0<\mathbf{h}_n^\top\mathbf{H}\,\mathbf{h}_n=-\mathbf{h}_n^\top\mathbf{F}'(\mathbf{x})$ $\mathbf{h}_n$ is a descent direction

Newton's method

- Another view

$$E(\mathbf{h}) = F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^{\mathrm{T}}\mathbf{g} + \frac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h}$$

- Minimizer satisfies $E'(\mathbf{h}^*) = 0$

$$E'(\mathbf{h}) = \mathbf{g} + \mathbf{H}\mathbf{h} = 0$$

$$\mathbf{h} = -\mathbf{H}^{-1}\mathbf{g}$$

Newton's method

$$\mathbf{h} = -\mathbf{H}^{-1}\mathbf{g}$$

- It requires solving a linear system and H is not always positive definite.
- It has good performance in the final stage of the iterative process, where x is close to x*.

Gauss-Newton method

- Use the approximate Hessian

$$\mathbf{H} \approx \mathbf{J}^{\mathrm{T}}\mathbf{J}$$

- No need for second derivative
- H is positive semi-definite

Hybrid method

$$\begin{array}{l} \textbf{if } \ \mathbf{F}''(\mathbf{x}) \text{ is positive definite} \\ \quad \mathbf{h} := \mathbf{h}_{\mathrm{n}} \\ \textbf{else} \\ \quad \mathbf{h} := \mathbf{h}_{\mathrm{sd}} \\ \mathbf{x} := \mathbf{x} + \alpha\mathbf{h} \end{array}$$

This needs to calculate second-order derivative which might not be available.

Levenberg-Marquardt method

- LM can be thought of as a combination of steepest descent and the Newton method. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method: slow, but guaranteed to converge. When the current solution is close to the correct solution, it becomes a Newton's method.

Nonlinear least square

Given a set of measurements $\mathbf{x}$, try to find the best parameter vector $\mathbf{p}$ so that the squared distance $\varepsilon^T \varepsilon$ is minimal. Here, $\varepsilon = \mathbf{x} - \hat{\mathbf{x}}$, with $\hat{\mathbf{x}} = f(\mathbf{p})$.

Levenberg-Marquardt method

For a small $||\delta_\mathbf{p}||$, $f(\mathbf{p}+\delta_\mathbf{p}) \approx f(\mathbf{p}) + \mathbf{J}\delta_\mathbf{p}$

$\mathbf{J}$ is the Jacobian matrix $\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}$

it is required to find the $\delta_\mathbf{p}$ that minimizes the quantity

$$||\mathbf{x} - f(\mathbf{p}+\delta_\mathbf{p})|| \approx ||\mathbf{x} - f(\mathbf{p}) - \mathbf{J}\delta_\mathbf{p}|| = ||\epsilon - \mathbf{J}\delta_\mathbf{p}||$$

$$\mathbf{J}^T\mathbf{J}\delta_\mathbf{p} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}\delta_\mathbf{p} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}_{ii} = \mu + \left[\mathbf{J}^T\mathbf{J}\right]_{ii}$$

↑ *damping term*

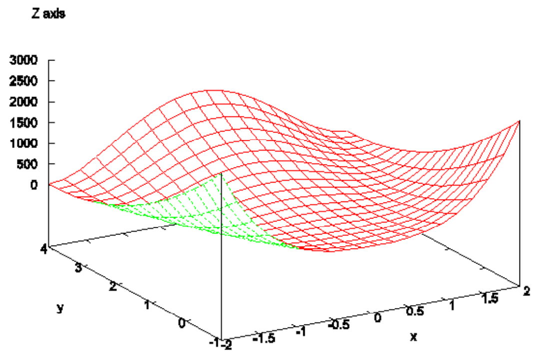
Levenberg-Marquardt method

$$(\mathbf{J}^\mathbf{T}\mathbf{J} + \mu\mathbf{I})\mathbf{h} = -\mathbf{g}$$

- μ=0 → Newton's method
- μ→∞ → steepest descent method

- Strategy for choosing μ
  - Start with some small μ
  - If F is not reduced, keep trying larger μ until it does
  - If F is reduced, accept it and reduce μ for the next iteration

Recap (the Rosenbrock function)



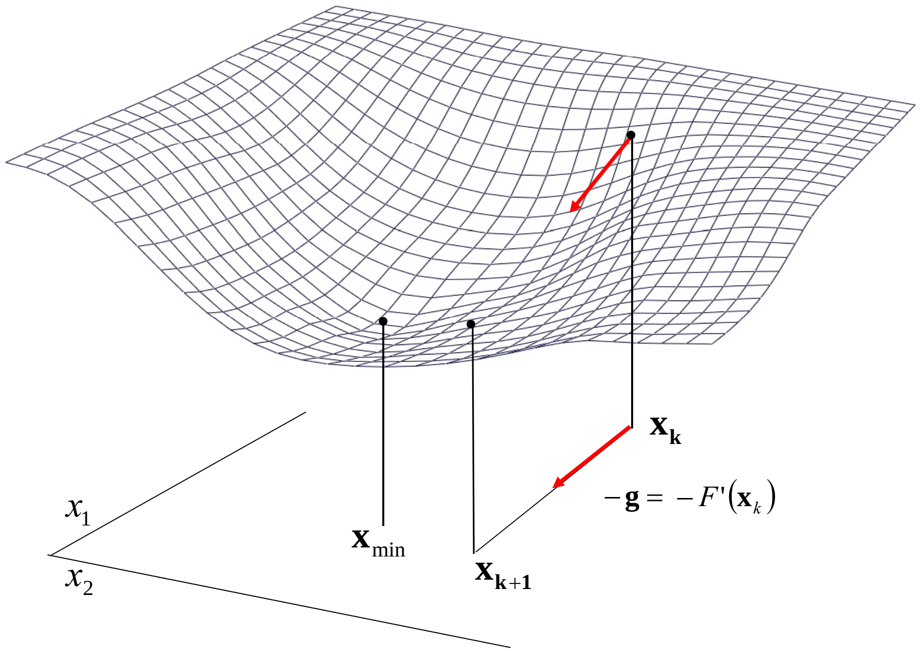
$$z = f(x, y) = (1 - x^2)^2 + 100(y - x^2)^2$$

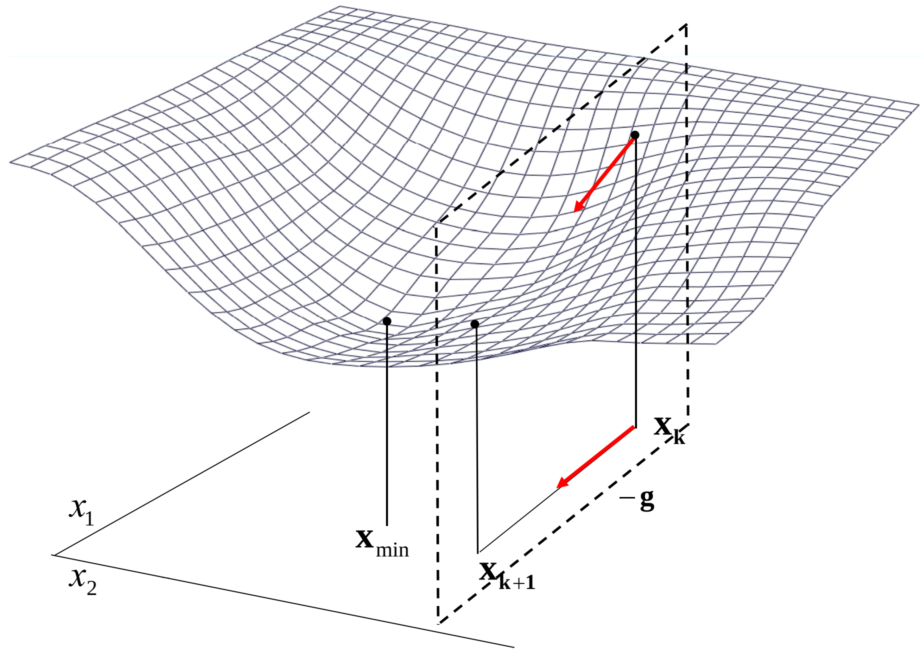
Global minimum at *(1,1)*

Steepest descent

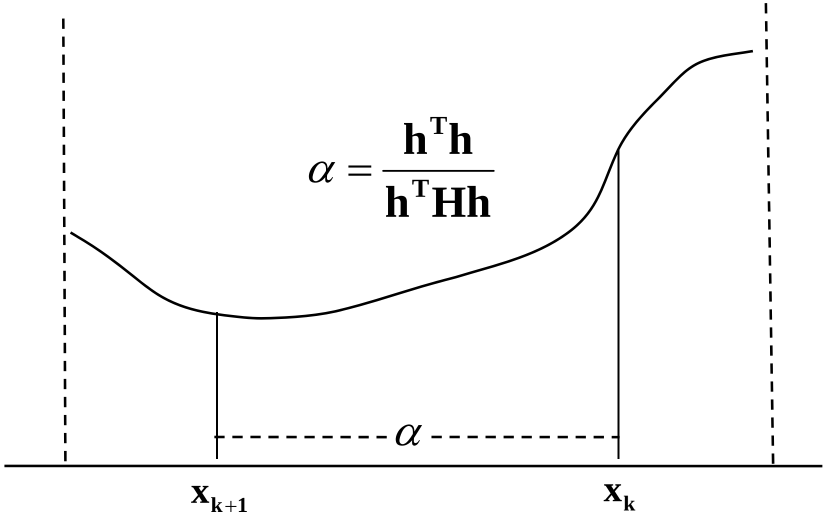
$$\mathbf{x_{k+1}} = \mathbf{x_k} - \alpha \mathbf{g}$$

$$\alpha = \frac{\mathbf{h^T h}}{\mathbf{h^T H h}}$$

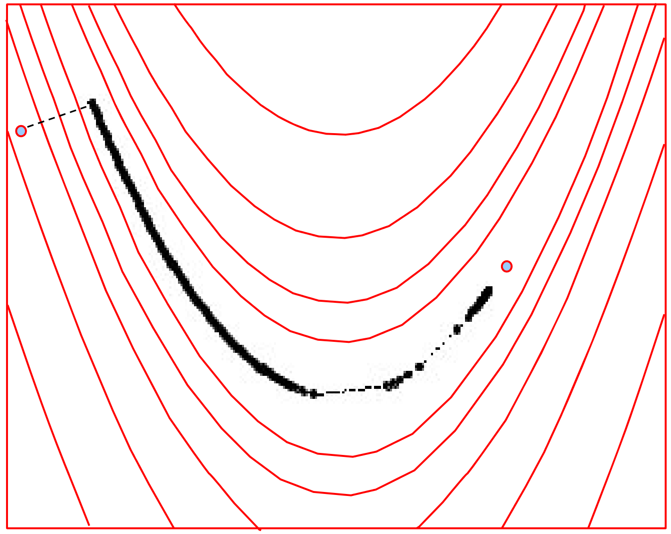




In the plane of the steepest descent direction



Steepest descent (1000 iterations)



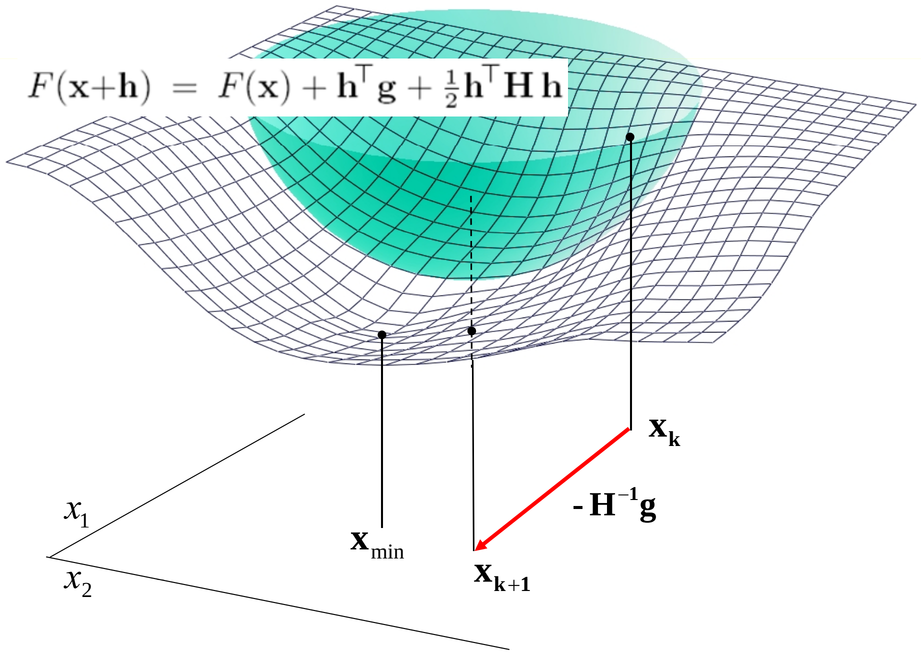
Gauss-Newton method

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}^{-1}\mathbf{g}$$

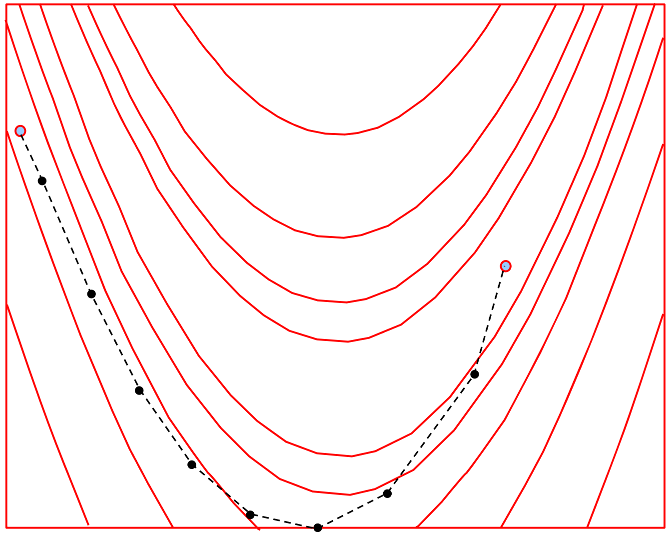
- With the approximate Hessian

$$\mathbf{H} \approx \mathbf{J}^T\mathbf{J}$$

- No need for second derivative
- H is positive semi-definite



Newton's method (48 evaluations)



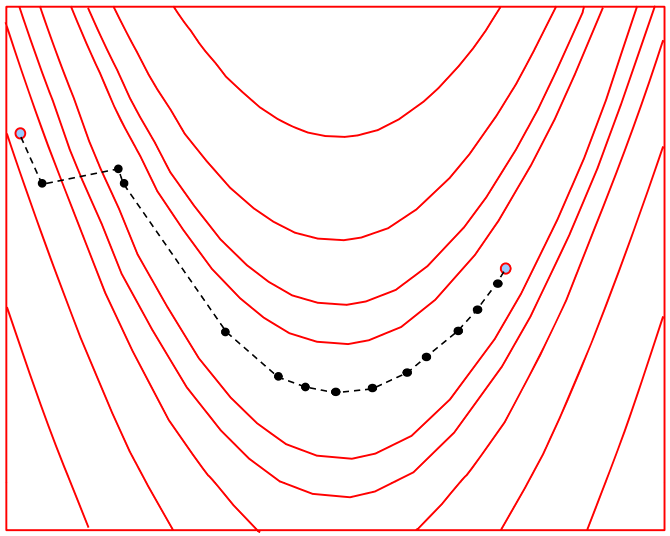
Levenberg-Marquardt

- Blends steepest descent and Gauss-Newton
- At each step, solve for the descent direction h

$$(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\mathbf{h} = -\mathbf{g}$$

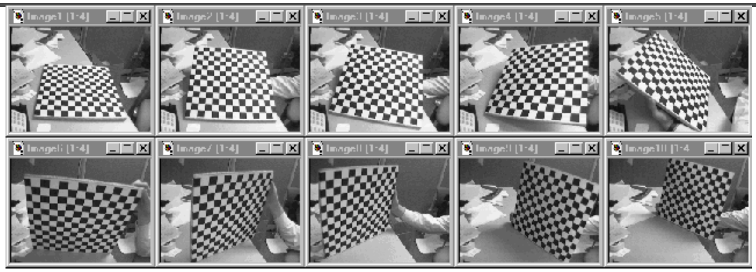
- If μ large, $\mathbf{h} \approx -\mathbf{g}$, steepest descent
- If μ small, $\mathbf{h} \approx -(\mathbf{J}^T\mathbf{J})^{-1}\mathbf{g}$, Gauss-Newton

Levenberg-Marquardt (90 evaluations)



A popular calibration tool

Multi-plane calibration

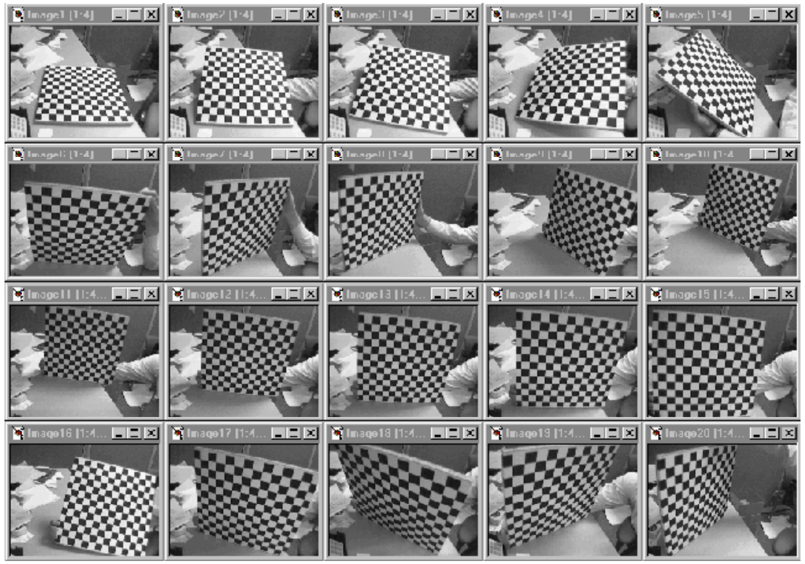


Images courtesy Jean-Yves Bouguet, Intel Corp.

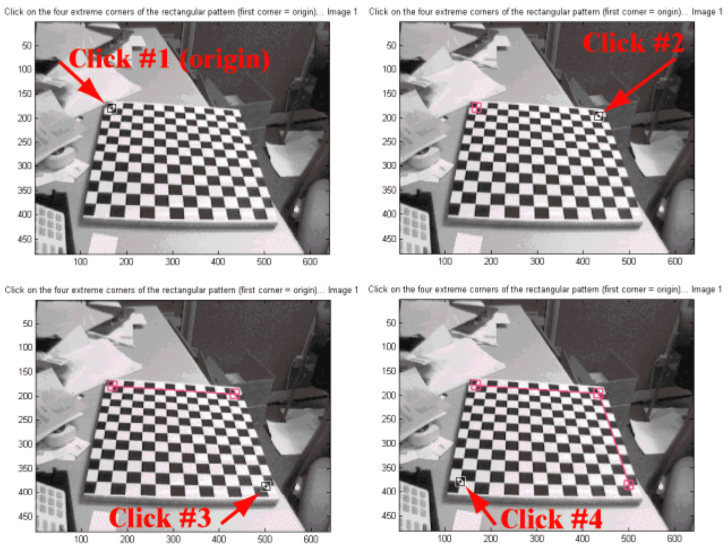
Advantage

- Only requires a plane
- Don't have to know positions/orientations
- Good code available online!
  - Intel's OpenCV library: http://www.intel.com/research/mrl/research/opencv/
  - Matlab version by Jean-Yves Bouget: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Zhengyou Zhang's web site: http://research.microsoft.com/~zhang/Calib/

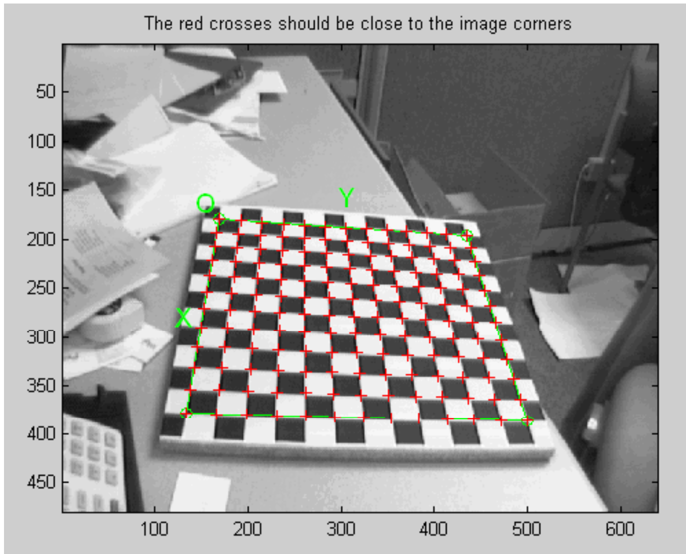
Step 1: data acquisition



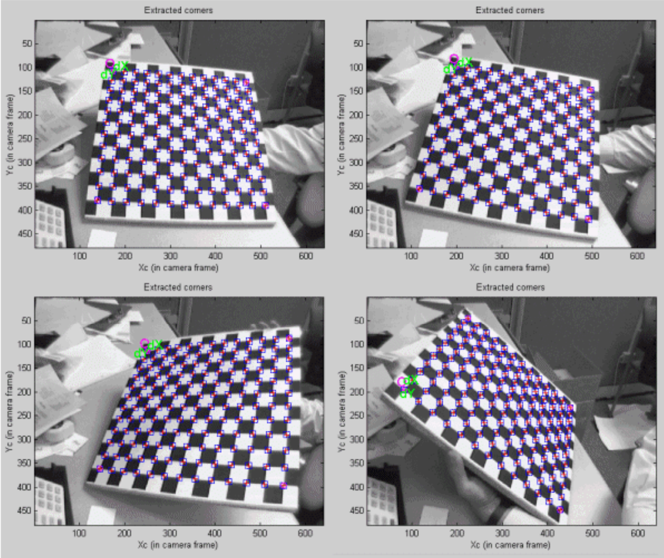
Step 2: specify corner order



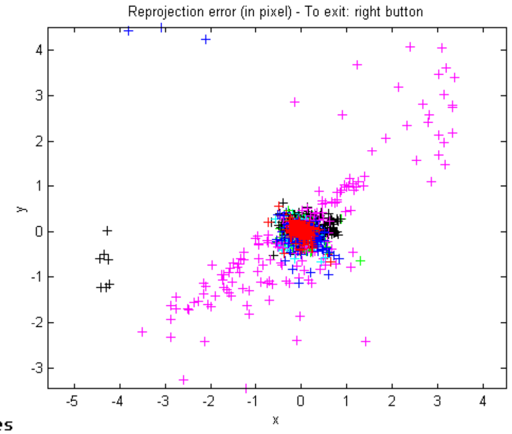
Step 3: corner extraction



Step 3: corner extraction

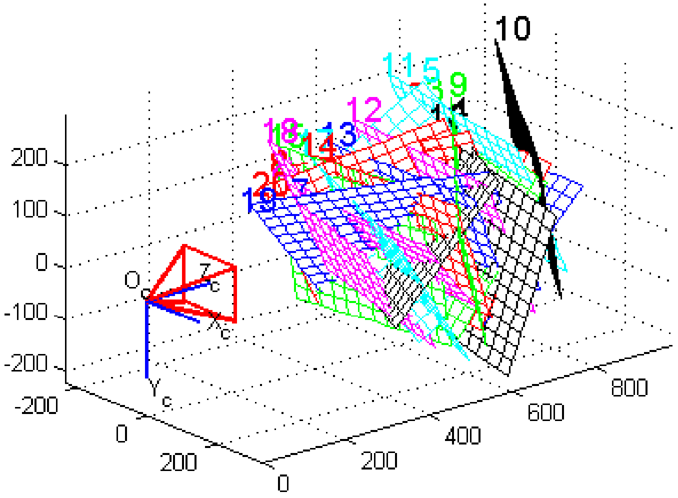


Step 4: minimize projection error

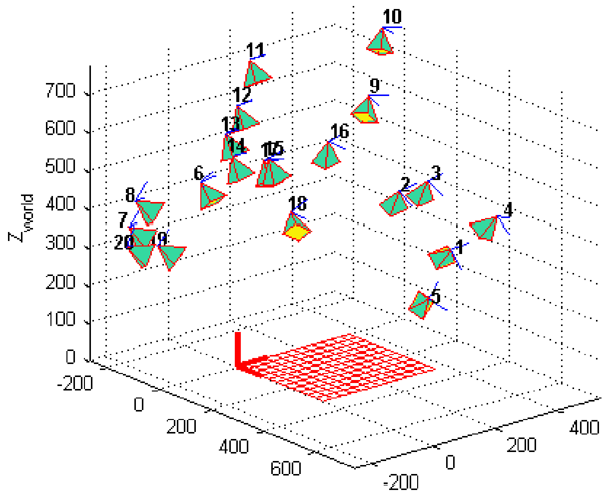



```
Calibration res

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes =
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002  0.00000 ]
Pixel error:          err = [ 0.11689   0.11500 ]
```

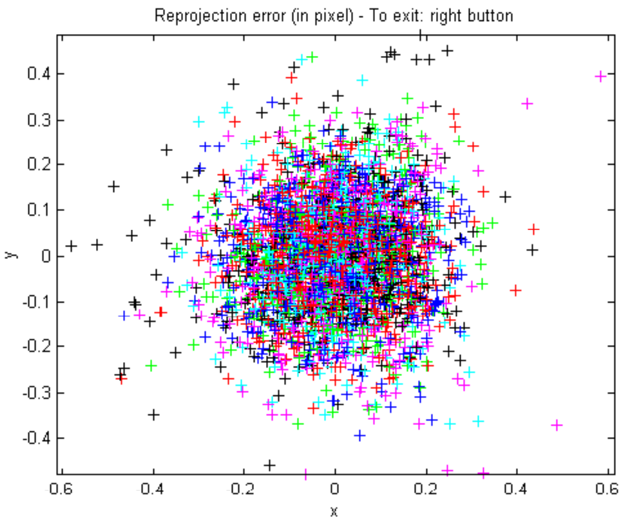
Step 4: camera calibration



Step 4: camera calibration



Step 5: refinement



Optimized parameters

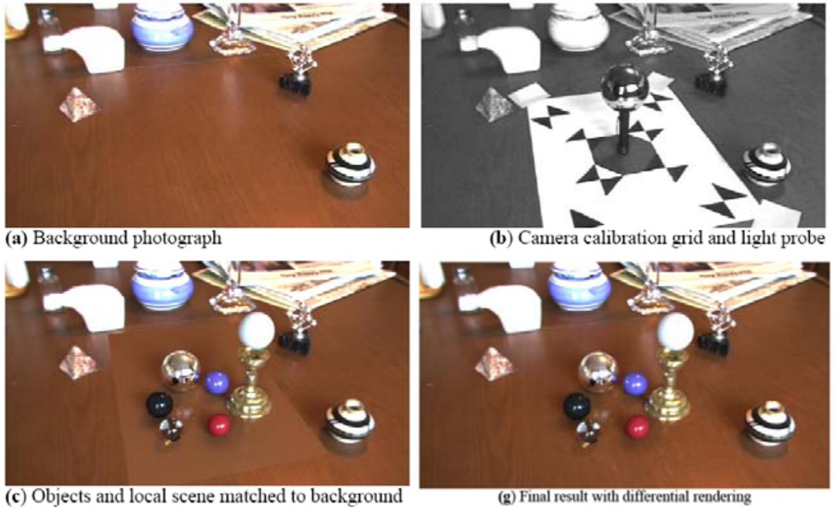
```
Aspect ratio optimized (est_aspect_ratio = 1) -> both components of fc are estimated (DE
Principal point optimized (center_optim=1) - (DEFAULT). To reject principal point, set c
Skew not optimized (est_alpha=0) - (DEFAULT)
Distortion not fully estimated (defined by the variable est_dist):
     Sixth order distortion not estimated (est_dist(5)=0) - (DEFAULT) .

Main calibration optimization procedure - Number of images: 20
Gradient descent iterations: 1...2...3...4...5...done
Estimation of uncertainties...done


Calibration results after optimization (with uncertainties):

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes = 90.000
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002  0.00000 ] ± [ 0.0
Pixel error:          err = [ 0.11689   0.11500 ]

Note: The numerical errors are approximately three times the standard deviations (for re
```

Applications

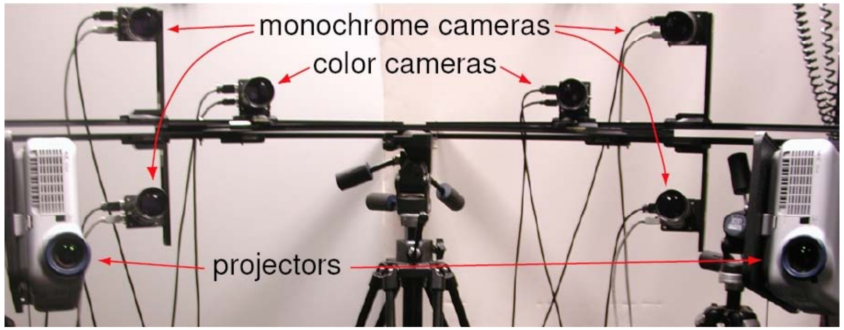
How is calibration used?

- Good for recovering intrinsic parameters; It is thus useful for many vision applications
- Since it requires a calibration pattern, it is often necessary to remove or replace the pattern from the footage or utilize it in some ways…

Example of calibration



Example of calibration



Example of calibration

- Videos from GaTech
- DasTatoo, MakeOf
- P!NG, MakeOf
- Work, MakeOf
- LifeInPaints, MakeOf

PhotoBook

PhotoBook

MakeOf