# Camera calibration

Digital Visual Effects, Spring 2008

*Yung-Yu Chuang*

2008/4/15

*with slides by Richard Szeliski, Steve Seitz, and Marc Pollefyes*

## Announcements

- Project #2 is due midnight next Monday
- Results for project #1 artifacts voting

## Project #1 artifact voting

- Total 141 votes
- 33 of 37 artifacts got votes

## Honorable mention(8): 胡傳甡 高紹航

## Honorable mention(8): 劉俊良

## Honorable mention(10): 周伯相

## Third place (13): 羅聖傑 鄭京恆

## Second place (14): 葉蓉蓉 劉冠廷

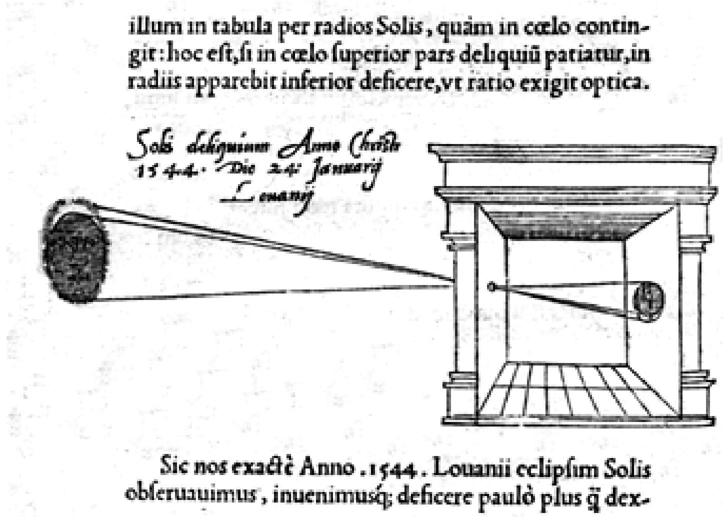
## First place (17): 梁彧 吳孟松

## Outline

- Camera projection models
- Camera calibration
- Nonlinear least square methods

# Camera projection models

## Pinhole camera

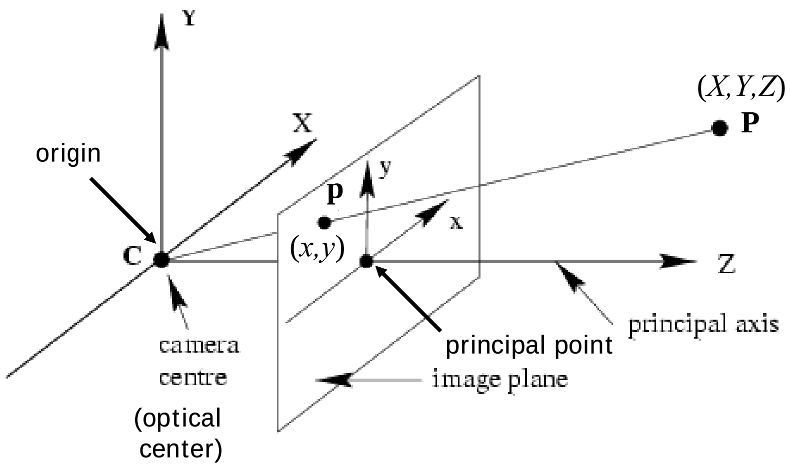


illum in tabula per radios Solis, quàm in cœlo contingit: hoc est, si in cœlo superior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

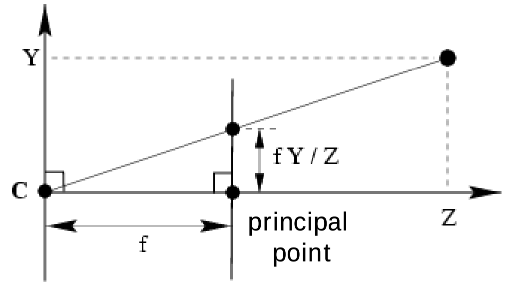


Sic nos exactè Anno .1544. Louanii eclipsim Solis obseruauimus, inuenimusq; deficere paulò plus q̃ dex-

## Pinhole camera model




## Pinhole camera model

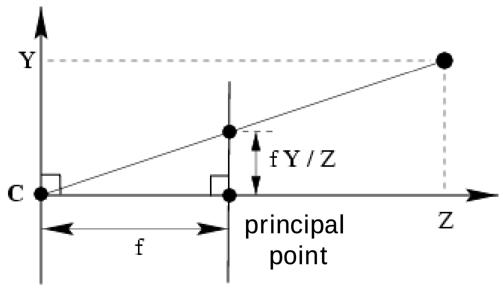



$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

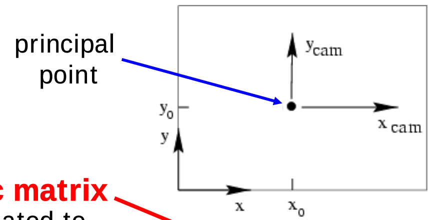
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

## Pinhole camera model




$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

## Principal point offset




**intrinsic matrix** only related to camera projection

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

## Intrinsic matrix

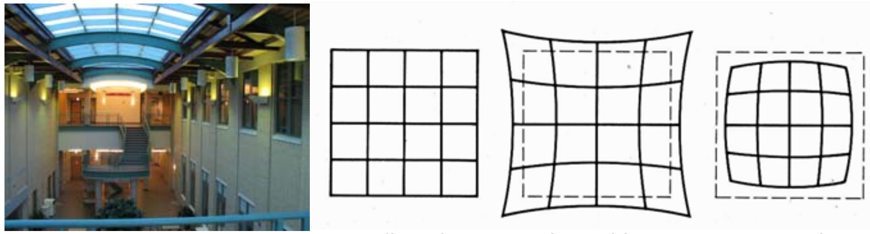
Is this form of $\mathbf{K}$ good enough?

$$\mathbf{K} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

- non-square pixels (digital video)
- skew
- radial distortion

$$\mathbf{K} = \begin{bmatrix} fa & s & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

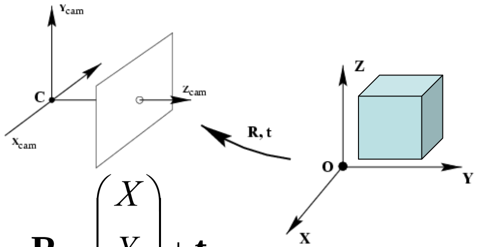
## Distortion



No distortion     Pin cushion     Barrel

- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

## Camera rotation and translation




$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}_{3\times3} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mathbf{t}$$

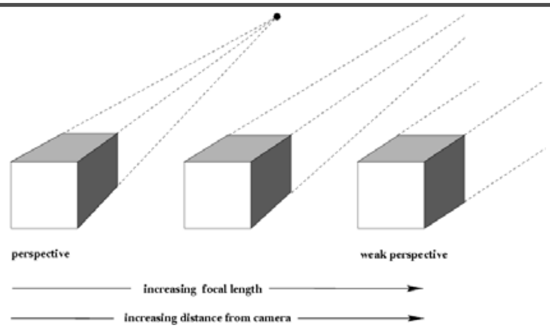
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{R}|\mathbf{t}] \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} \sim \mathbf{K} \underbrace{[\mathbf{R}|\mathbf{t}]}_{\text{extrinsic matrix}} \mathbf{X}$$

## Two kinds of parameters

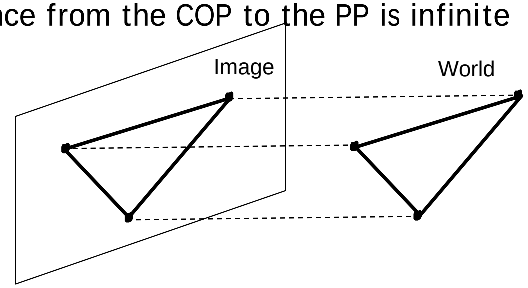
- *internal* or *intrinsic* parameters such as focal length, optical center, aspect ratio: *what kind of camera?*
- *external* or *extrinsic* (pose) parameters including rotation and translation: *where is the camera?*

## Other projection models




## Orthographic projection

- Special case of perspective projection
  - Distance from the COP to the PP is infinite




$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

  - Also called “parallel projection”: (x, y, z) → (x, y)

## Other types of projections

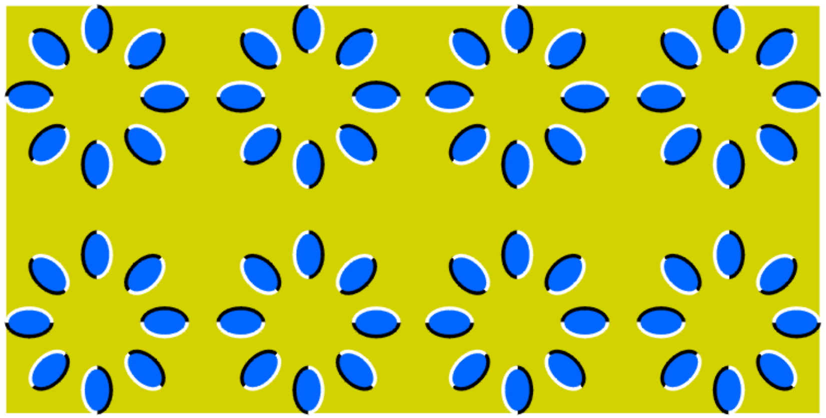
- Scaled orthographic
  - Also called “weak perspective”

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/d \end{bmatrix} \Rightarrow (dx, dy)$$

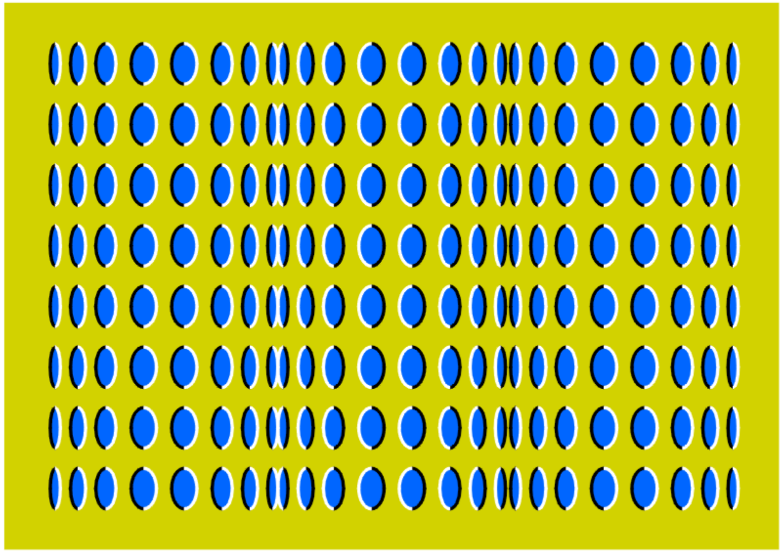
- Affine projection
  - Also called “paraperspective”

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

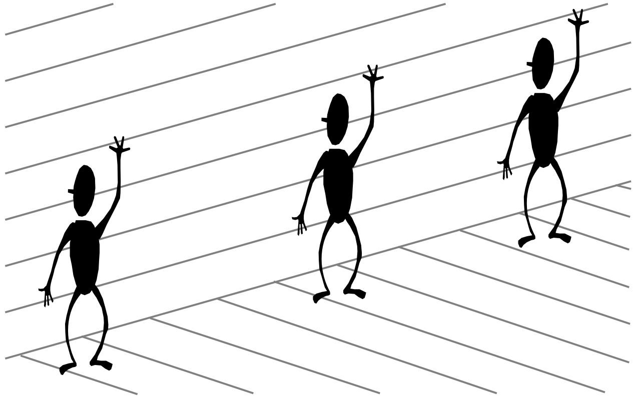
## Illusion



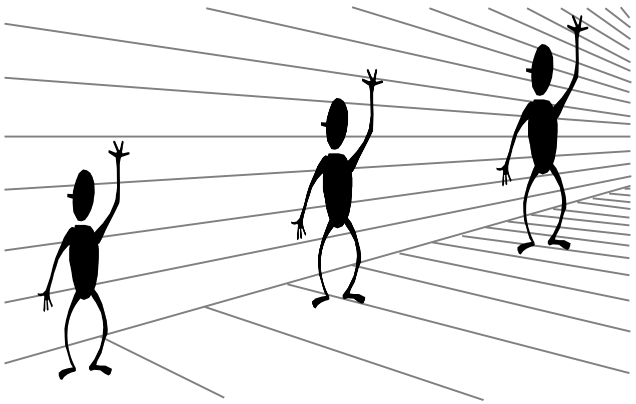
## Illusion



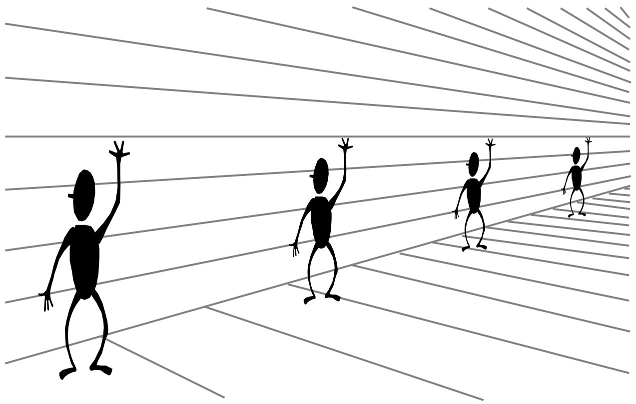
## Fun with perspective



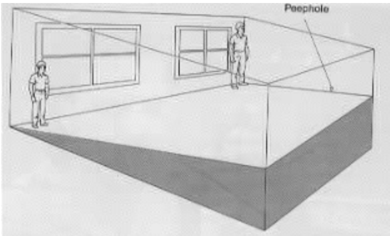
## Perspective cues

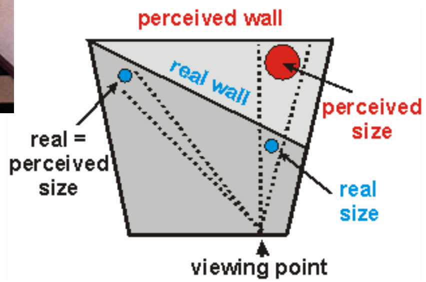


## Perspective cues



## Fun with perspective




Ames room

Ames video BBC story

## Forced perspective in LOTR

# Camera calibration

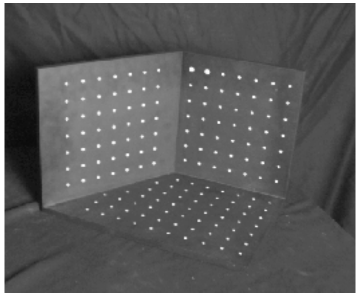
## Camera calibration

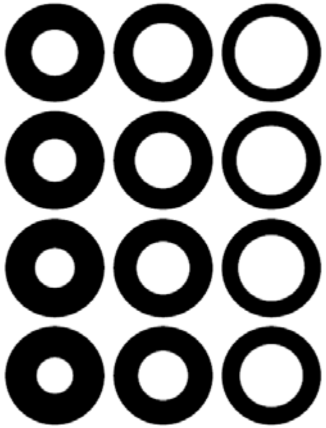
- Estimate both intrinsic and extrinsic parameters
- Mainly, two categories:

1. Photometric calibration: uses reference objects with known geometry
2. Self calibration: only assumes static scene, e.g. structure from motion

## Camera calibration approaches

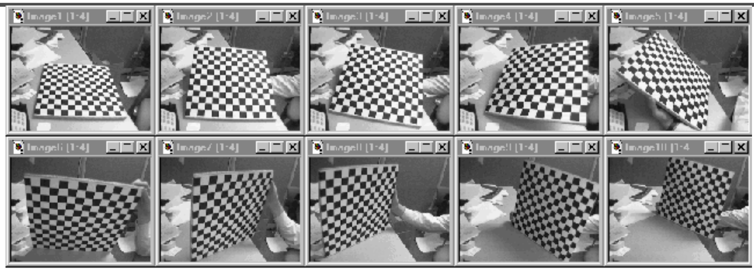
1. linear regression (least squares)
2. nonlinear optimization





## Chromaglyphs (HP research)

## Multi-plane calibration

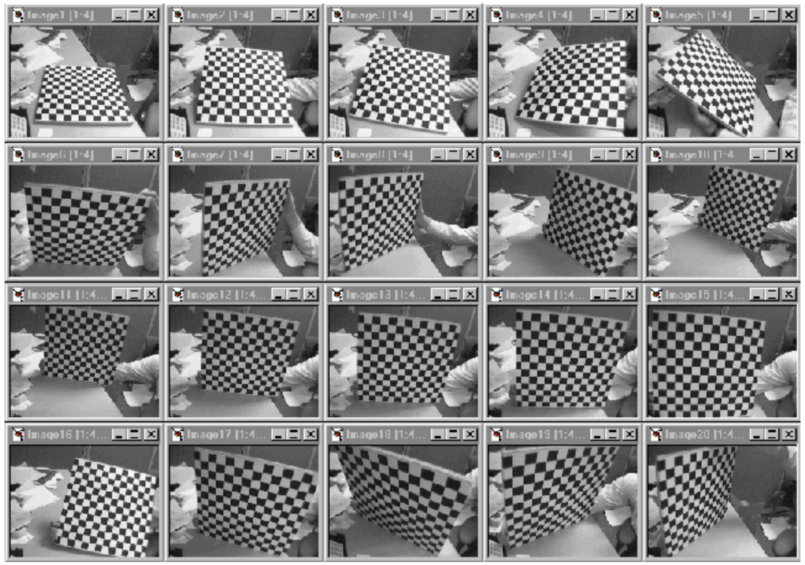


Images courtesy Jean-Yves Bouguet, Intel Corp.

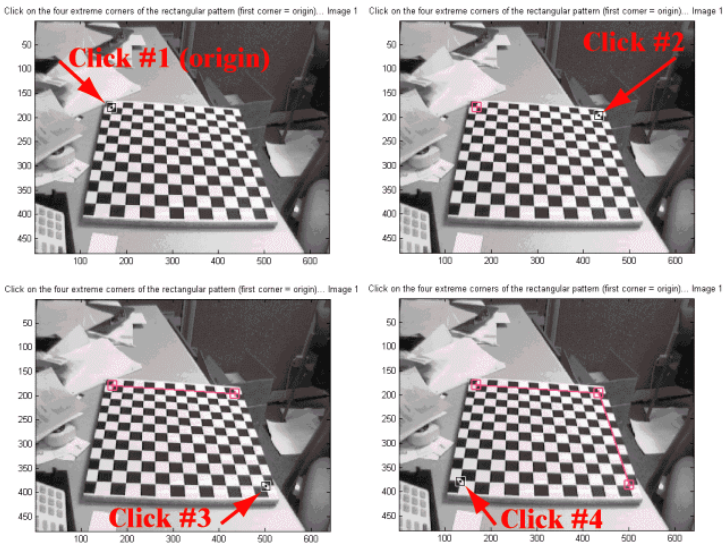
Advantage

- Only requires a plane
- Don't have to know positions/orientations
- Good code available online!
  - Intel's OpenCV library: http://www.intel.com/research/mrl/research/opencv/
  - Matlab version by Jean-Yves Bouget: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Zhengyou Zhang's web site: http://research.microsoft.com/~zhang/Calib/

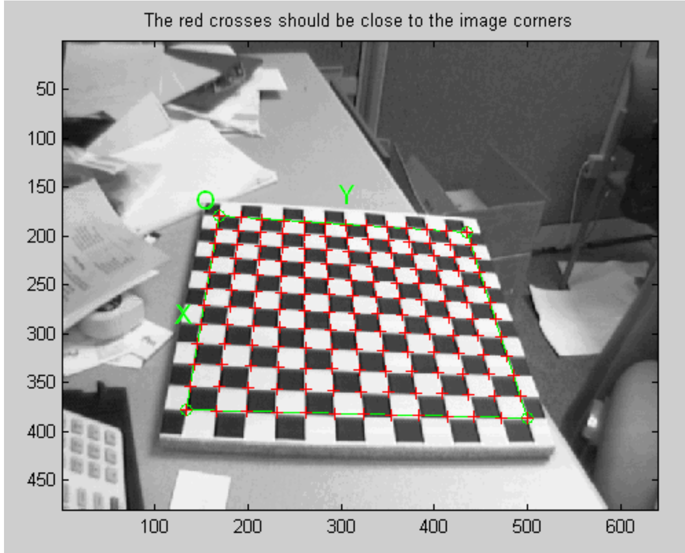
## Step 1: data acquisition



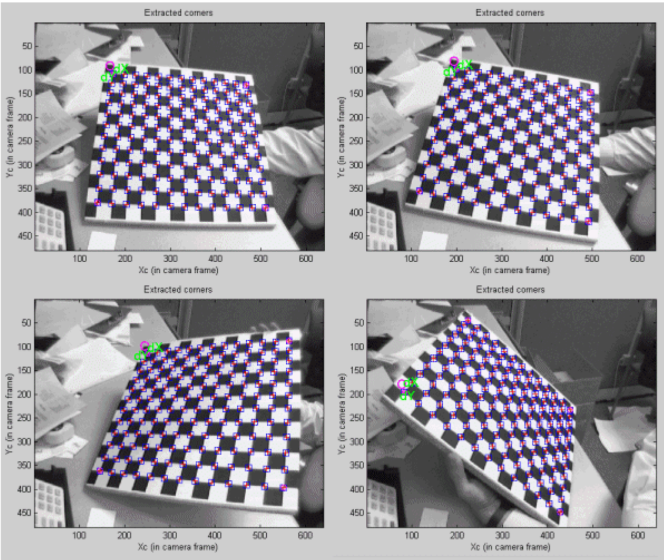
## Step 2: specify corner order



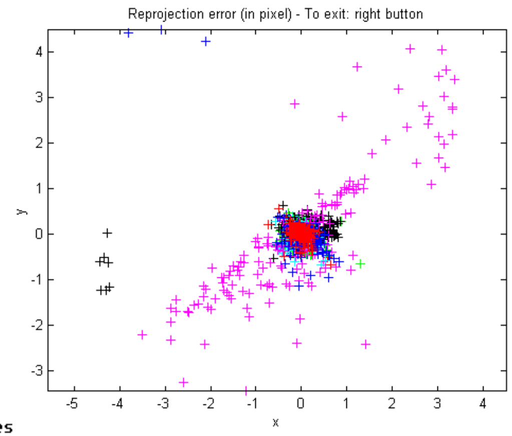

## Step 3: corner extraction




## Step 3: corner extraction



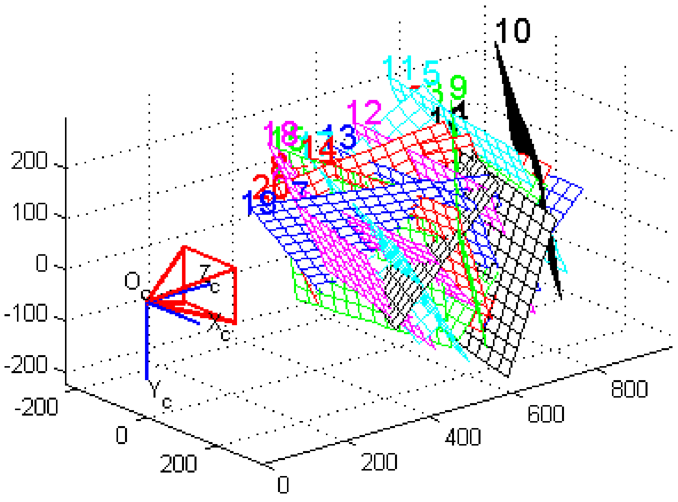

## Step 4: minimize projection error



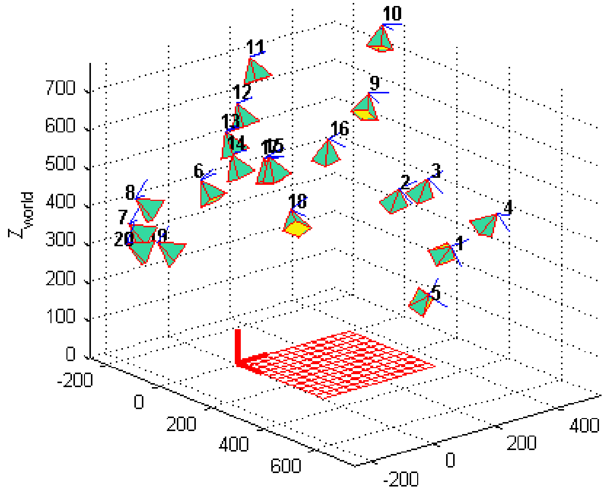

```
Calibration res

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes =
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002  0.00000 ]
Pixel error:          err = [ 0.11689   0.11500 ]
```

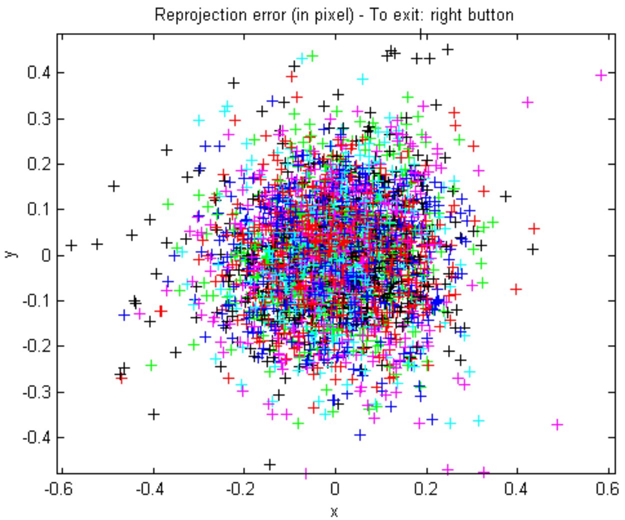
## Step 4: camera calibration



## Step 4: camera calibration




## Step 5: refinement




## Optimized parameters

```
Aspect ratio optimized (est_aspect_ratio = 1) -> both components of fc are estimated (DE
Principal point optimized (center_optim=1) - (DEFAULT). To reject principal point, set c
Skew not optimized (est_alpha=0) - (DEFAULT)
Distortion not fully estimated (defined by the variable est_dist):
     Sixth order distortion not estimated (est_dist(5)=0) - (DEFAULT) .

Main calibration optimization procedure - Number of images: 20
Gradient descent iterations: 1...2...3...4...5...done
Estimation of uncertainties...done


Calibration results after optimization (with uncertainties):

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes = 90.000
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002  0.00000 ] ± [ 0.0
Pixel error:          err = [ 0.11689   0.11500 ]

Note: The numerical errors are approximately three times the standard deviations (for re
```

# Camera calibration

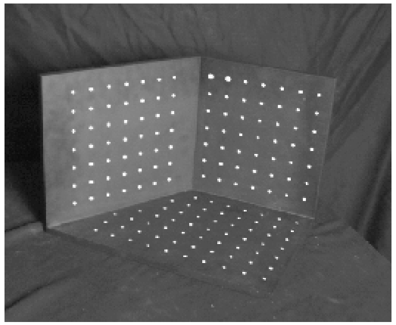
## Linear regression

$$\mathbf{x} \sim \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X} = \mathbf{M}\mathbf{X}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

## Linear regression

- Directly estimate 11 unknowns in the **M** matrix using known 3D points ($X_i$,$Y_i$,$Z_i$) and measured feature positions ($u_i$,$v_i$)



## Linear regression

$$u_i = \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$v_i = \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$u_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}$$

$$v_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}$$

## Linear regression

$$\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i & -u_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i & -v_i \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

## Linear regression

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Solve for Projection Matrix M using least-square techniques

## Normal equation

Given an overdetermined system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

the normal equation is that which minimizes the sum of the square differences between left and right sides

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

## Linear regression

- Advantages:
  - All specifics of the camera summarized in one matrix
  - Can predict where any world point will map to in the image
- Disadvantages:
  - Doesn't tell us about particular parameters
  - Mixes up internal and external parameters
    - pose specific: move the camera and everything breaks
  - More unknowns than true degrees of freedom

## Nonlinear optimization

- A probabilistic view of least square
- Feature measurement equations

$$
\begin{aligned}
u_i &= f(\boxed{\mathbf{M}}, \mathbf{x}_i) + n_i = \hat{u}_i + n_i, \quad n_i \sim N(0, \sigma) \\
v_i &= g(\boxed{\mathbf{M}}, \mathbf{x}_i) + m_i = \hat{v}_i + m_i, \ m_i \sim N(0, \sigma)
\end{aligned}
$$

- Probability of $\mathbf{M}$ given $\{(u_i, v_i)\}$

$$
\begin{aligned}
P &= \prod_i p(u_i|\hat{u}_i) p(v_i|\hat{v}_i) \\
&= \prod_i e^{-(u_i-\hat{u}_i)^2/\sigma^2} e^{-(v_i-\hat{v}_i)^2/\sigma^2}
\end{aligned}
$$

## Optimal estimation

- Likelihood of $\mathbf{M}$ given $\{(u_i, v_i)\}$

$$
L = -\log P = \sum_i (u_i - \hat{u}_i)^2/\sigma_i^2 + (v_i - \hat{v}_i)^2/\sigma_i^2
$$

- It is a least square problem (but not necessarily linear least square)
- How do we minimize $L$?

## Optimal estimation

- Non-linear regression (least squares), because the relations between $\hat{u}_i$ and $u_i$ are non-linear functions $\mathbf{M}$

unknown parameters

We could have terms like $f \cos\theta$ in this

$$
\mathbf{u} - \hat{\mathbf{u}} \sim \mathbf{u} - \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}
$$

known constant

- We can use Levenberg-Marquardt method to minimize it

# Nonlinear least square methods

## Least square fitting

**Least Squares Problem**

Find $\mathbf{x}^*$, a local minimizer for

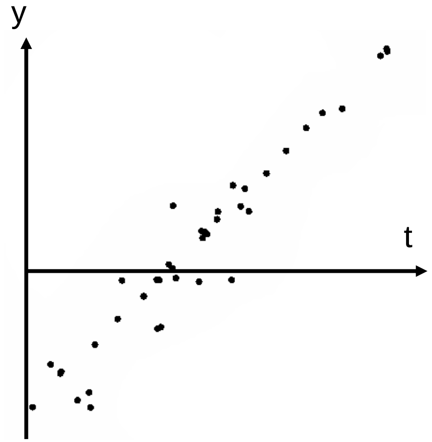
$$F(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}(f_i(\mathbf{x}))^2 ,$$

where $f_i : \mathbb{R}^n \mapsto \mathbb{R},\ i=1,\ldots,m$ are given functions, and $m \ge n$.

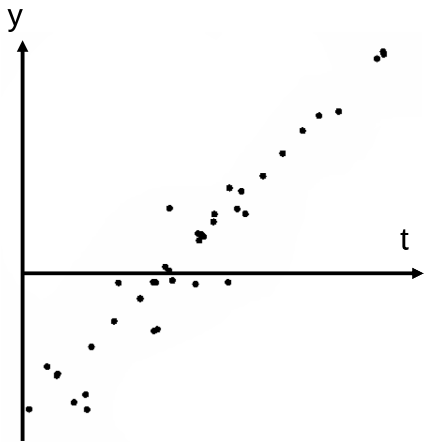
number of data points

number of parameters

## Linear least square fitting



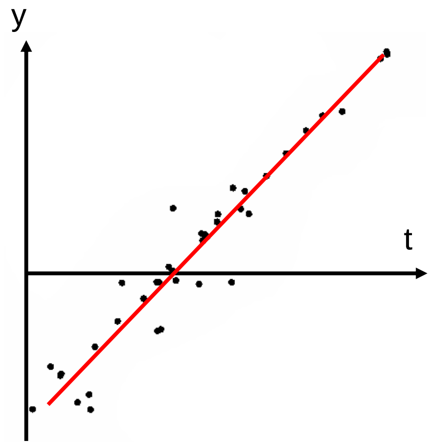

## Linear least square fitting




model parameters

$$y(t) = M(t;\mathbf{x})$$

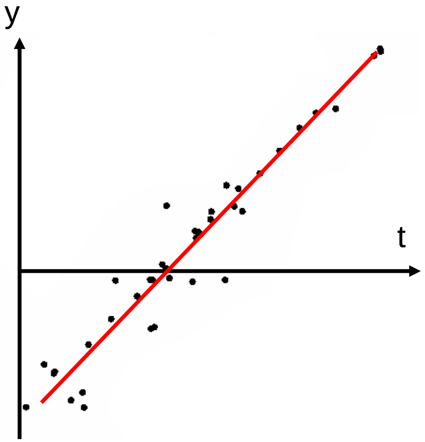
## Linear least square fitting




model parameters

$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

## Linear least square fitting




model parameters

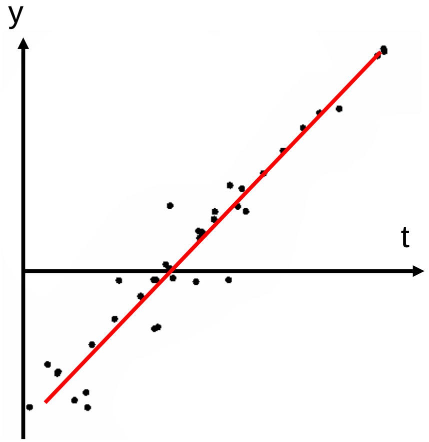
$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

$$f_i(x) = y_i - \boxed{M(t_i;\mathbf{x})}$$

prediction

residual

## Linear least square fitting




model parameters

$$y(t) = M(t;\mathbf{x}) = x_0 + x_1 t$$

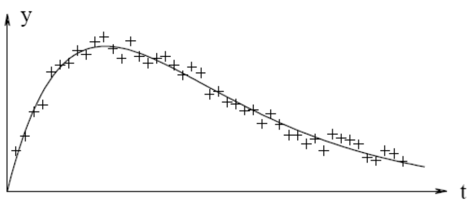
$$f_i(x) = y_i - \boxed{M(t_i;\mathbf{x})}$$

prediction

residual

$M(t;\mathbf{x}) = x_0 + x_1 t + x_2 t^3$ is linear, too.

## Nonlinear least square fitting




model $M(t;\mathbf{x}) = x_3 e^{x_1 t} + x_4 e^{x_2 t}$

parameters $\mathbf{x} = [x_1, x_2, x_4, x_4]^T$

residuals $f_i(\mathbf{x}) = y_i - M(t_i;\mathbf{x})$

$$= y_i - \left(x_3 e^{x_1 t} + x_4 e^{x_2 t}\right)$$

## Function minimization

Least square is related to function minimization.

**Global Minimizer**
Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find

$$\mathbf{x}^+ = \operatorname{argmin}_{\mathbf{x}} \{F(\mathbf{x})\} .$$

It is very hard to solve in general. Here, we only consider a simpler problem of finding local minimum.

**Local Minimizer**
Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find $\mathbf{x}^*$ so that

$$F(\mathbf{x}^*) \le F(\mathbf{x}) \quad \text{for} \quad \|\mathbf{x} - \mathbf{x}^*\| < \delta .$$

## Function minimization

We assume that the cost function $F$ is differentiable and so smooth that the following *Taylor expansion* is valid,[2)]

$$F(\mathbf{x}+\mathbf{h}) \;=\; F(\mathbf{x}) + \mathbf{h}^\top \mathbf{g} + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}\,\mathbf{h} + O(\|\mathbf{h}\|^3)\,,$$

where $\mathbf{g}$ is the *gradient*,

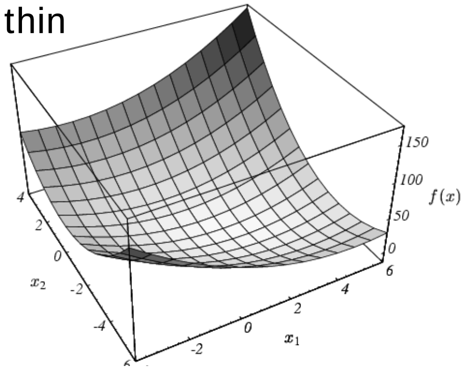
$$\mathbf{g} \;\equiv\; \mathbf{F}'(\mathbf{x}) \;=\; \begin{bmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial F}{\partial x_n}(\mathbf{x}) \end{bmatrix},$$

and $\mathbf{H}$ is the *Hessian*,

$$\mathbf{H} \;\equiv\; \mathbf{F}''(\mathbf{x}) \;=\; \left[\frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{x})\right].$$

## Quadratic functions

Approximate the function with
a quadratic function within
a small neighborhood



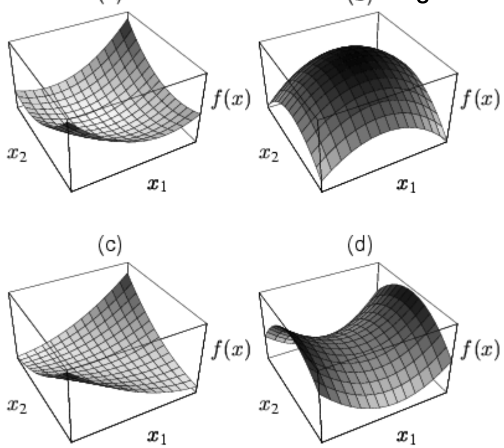

$$f(x) = \frac{1}{2}x^T A x - b^T x + c$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \qquad c = 0.$$

## Quadratic functions

**A** is positive definite.
All eigenvalues
are positive.
For all x,
$x^T A x > 0$.

negative definite




A is singular

A is indefinite

## Function minimization

**Theorem 1.5. Necessary condition for a local minimizer.**
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \;\equiv\; \mathbf{F}'(\mathbf{x}^*) \;=\; \mathbf{0}\,.$$

Why?

By definition, if $\mathbf{x}^*$ is a local minimizer,

$\|\mathbf{h}\|$ is small enough $\longrightarrow$ $\mathbf{F}(\mathbf{x}^* + \mathbf{h}) > \mathbf{F}(\mathbf{x}^*)$

$$\mathbf{F}(\mathbf{x}^* + \mathbf{h}) = \mathbf{F}(\mathbf{x}^*) + \mathbf{h}^T\mathbf{F}'(\mathbf{x}^*) + O(\|\mathbf{h}\|^2)$$

## Function minimization

**Theorem 1.5. Necessary condition for a local minimizer.**
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}\,.$$

**Definition 1.6. Stationary point.** If

$$\mathbf{g}_s \equiv \mathbf{F}'(\mathbf{x}_s) = \mathbf{0}\,,$$

then $\mathbf{x}_s$ is said to be a *stationary point* for $F$.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$

$\mathbf{H}_s$ is *positive definite*

*a) minimum* *b) maximum* *c) saddle point*

## Function minimization

**Theorem 1.8. Sufficient condition for a local minimizer.**
Assume that $\mathbf{x}_s$ is a stationary point and that $\mathbf{F}''(\mathbf{x}_s)$ is positive definite. Then $\mathbf{x}_s$ is a local minimizer.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$
$$\text{with } \mathbf{H}_s = \mathbf{F}''(\mathbf{x}_s)$$

If we request that $\mathbf{H}_s$ is *positive definite*, then its eigenvalues are greater than some number $\delta > 0$

$$\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} > \delta\,\|\mathbf{h}\|^2$$

## Descent methods

$$\mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_2,\ \ldots\ ,\ \mathbf{x}_k \to \mathbf{x}^* \quad \text{for} \quad k \to \infty$$

1. Find a descent direction $\mathbf{h}_d$
2. find a step length giving a good decrease in the $F$-value.

**Algorithm Descent method**

**begin**
  $k := 0$; $\mathbf{x} := \mathbf{x}_0$; *found* := **false** {Starting point}
  **while** (**not** *found*) **and** ($k < k_{max}$)
    $\mathbf{h}_d$ := search_direction($\mathbf{x}$) {From $\mathbf{x}$ and downhill}
    **if** (no such $\mathbf{h}$ exists)
      *found* := **true** {$\mathbf{x}$ is stationary}
    **else**
      $\alpha$ := step_length($\mathbf{x}, \mathbf{h}_d$) {from $\mathbf{x}$ in direction $\mathbf{h}_d$}
      $\mathbf{x} := \mathbf{x} + \alpha\mathbf{h}_d$; $k := k+1$ {next iterate}
**end**

## Descent direction

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

**Definition Descent direction.**

$\mathbf{h}$ is a descent direction for $F$ at $\mathbf{x}$ if $\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) < 0\,.$

## Steepest descent method

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^{\top}\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$

$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^{\top}\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

$$\frac{F(\mathbf{x}) - F(\mathbf{x}+\alpha\mathbf{h})}{\alpha\|\mathbf{h}\|} = -\frac{1}{\|\mathbf{h}\|}\mathbf{h}^{\top}\mathbf{F}'(\mathbf{x}) = -\|\mathbf{F}'(\mathbf{x})\|\cos\theta$$

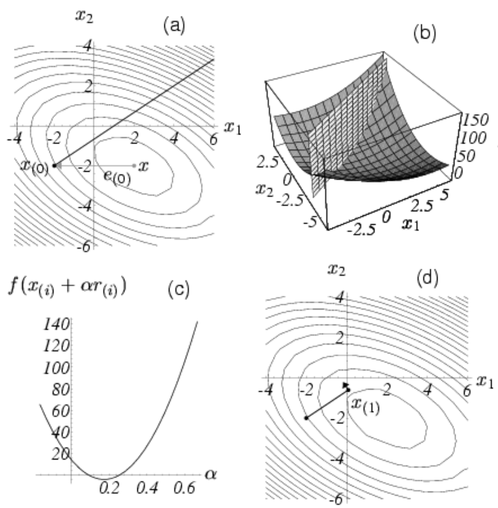
the decrease of $F(x)$ per unit along h direction

greatest gain rate if $\theta = \pi$ → $\mathbf{h}_{\mathrm{sd}} = -\mathbf{F}'(\mathbf{x})$

$h_{sd}$ is a descent direction because $h^T_{sd} F'(x) = -F'(x)^2 < 0$

## Line search

$\varphi(\alpha) = F(\mathbf{x}+\alpha\mathbf{h})$ , $\mathbf{x}$ and $\mathbf{h}$ fixed, $\alpha \geq 0$ .




Find $\alpha$ so that

$$\varphi(\alpha) = \mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})$$

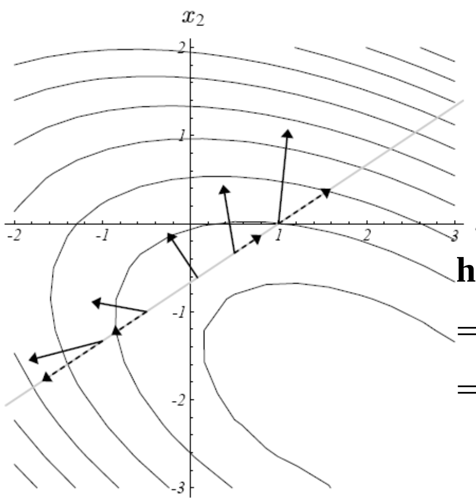
is minumum

$$0 = \frac{\partial\varphi(\alpha)}{\partial\alpha} = \frac{\partial\mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})}{\partial\alpha}$$

$$= \frac{\partial\mathbf{F}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\alpha} = \mathbf{h}^{\mathrm{T}}\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

## Line search



$$\mathbf{h}^{\mathrm{T}}\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h}) = 0$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

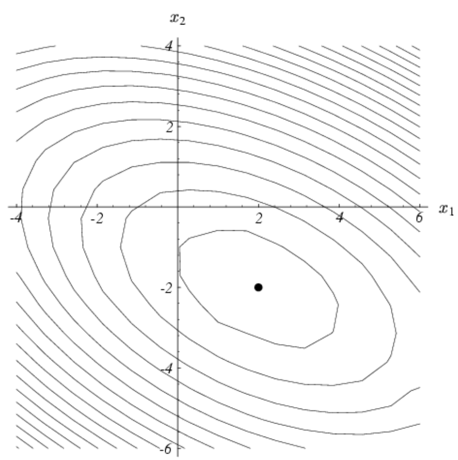
$$\mathbf{h}^{\mathrm{T}}\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$

$$= \mathbf{h}^{\mathrm{T}}\left(\mathbf{F}'(\mathbf{x}_0) + \alpha\mathbf{F}''(\mathbf{x}_0)^{\mathrm{T}}\mathbf{h}\right)$$

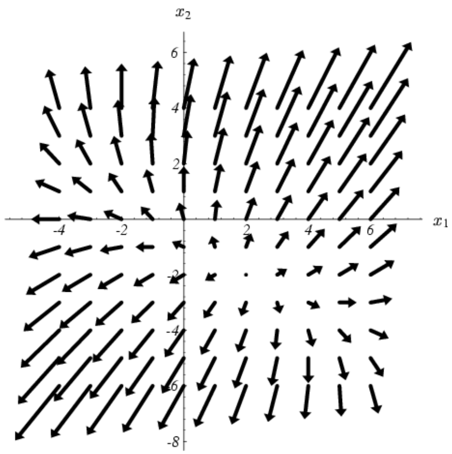
$$= -\mathbf{h}^{\mathrm{T}}\mathbf{h} + \alpha\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h} = 0$$

$$\boxed{\alpha = \frac{\mathbf{h}^{\mathrm{T}}\mathbf{h}}{\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h}}}$$

## Steepest descent method

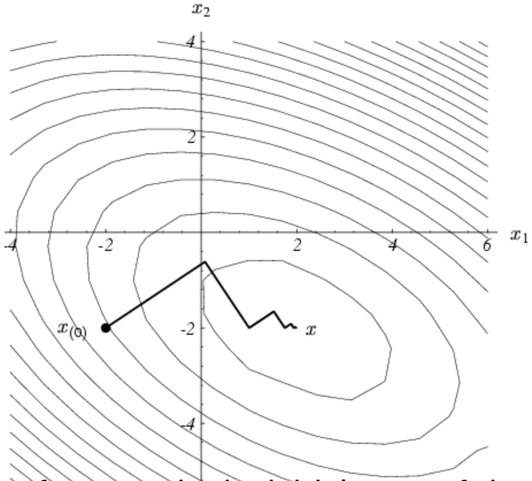


isocontour



gradient

## Steepest descent method




It has good performance in the initial stage of the iterative process. Converge very slow with a linear rate.

## Newton's method

$\mathbf{x}^*$ is a stationary point $\rightarrow$ it satisfies $\mathbf{F}'(\mathbf{x}^*) = \mathbf{0}$.

$$\begin{aligned} \mathbf{F}'(\mathbf{x}+\mathbf{h}) &= \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &\simeq \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} \quad \text{for } \|\mathbf{h}\| \text{ sufficiently small} \end{aligned}$$

$$\rightarrow \mathbf{H}\,\mathbf{h}_\mathrm{n} = -\mathbf{F}'(\mathbf{x}) \quad \text{with } \mathbf{H} = \mathbf{F}''(\mathbf{x})$$

$$\mathbf{x} := \mathbf{x} + \mathbf{h}_\mathrm{n}$$

Suppose that $\mathbf{H}$ is positive definite

$\rightarrow \mathbf{u}^\top \mathbf{H}\,\mathbf{u} > 0$ for all nonzero $\mathbf{u}$.

$\rightarrow 0 < \mathbf{h}_\mathrm{n}^\top \mathbf{H}\,\mathbf{h}_\mathrm{n} = -\mathbf{h}_\mathrm{n}^\top \mathbf{F}'(\mathbf{x})$ $\quad \mathbf{h}_\mathrm{n}$ is a descent direction

It has good performance in the final stage of the iterative process, where x is close to x*.

## Hybrid method

$$\begin{aligned} &\textbf{if } \mathbf{F}''(\mathbf{x}) \text{ is positive definite} \\ &\quad \mathbf{h} := \mathbf{h}_\mathrm{n} \\ &\textbf{else} \\ &\quad \mathbf{h} := \mathbf{h}_\mathrm{sd} \\ &\mathbf{x} := \mathbf{x} + \alpha\mathbf{h} \end{aligned}$$

This needs to calculate second-order derivative which might not be available.

## Levenberg-Marquardt method

- LM can be thought of as a combination of steepest descent and the Newton method. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method: slow, but guaranteed to converge. When the current solution is close to the correct solution, it becomes a Newton's method.

## Nonlinear least square

Given a set of measurements $\mathbf{x}$, try to find the best parameter vector $\mathbf{p}$ so that the squared distance $\varepsilon^T\varepsilon$ is minimal. Here, $\varepsilon = \mathbf{x} - \hat{\mathbf{x}}$, with $\hat{\mathbf{x}} = f(\mathbf{p})$.

## Levenberg-Marquardt method

For a small $||\delta_\mathbf{p}||$, $f(\mathbf{p}+\delta_\mathbf{p}) \approx f(\mathbf{p}) + \mathbf{J}\delta_\mathbf{p}$

$\mathbf{J}$ is the Jacobian matrix $\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}$

it is required to find the $\delta_\mathbf{p}$ that minimizes the quantity

$$||\mathbf{x} - f(\mathbf{p}+\delta_\mathbf{p})|| \approx ||\mathbf{x} - f(\mathbf{p}) - \mathbf{J}\delta_\mathbf{p}|| = ||\epsilon - \mathbf{J}\delta_\mathbf{p}||$$

$$\mathbf{J}^T\mathbf{J}\delta_\mathbf{p} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}\delta_\mathbf{p} = \mathbf{J}^T\epsilon$$

$$\mathbf{N}_{ii} = \underset{\textit{damping term}}{\underset{\uparrow}{\mu}} + \left[\mathbf{J}^T\mathbf{J}\right]_{ii}$$

## Levenberg-Marquardt method

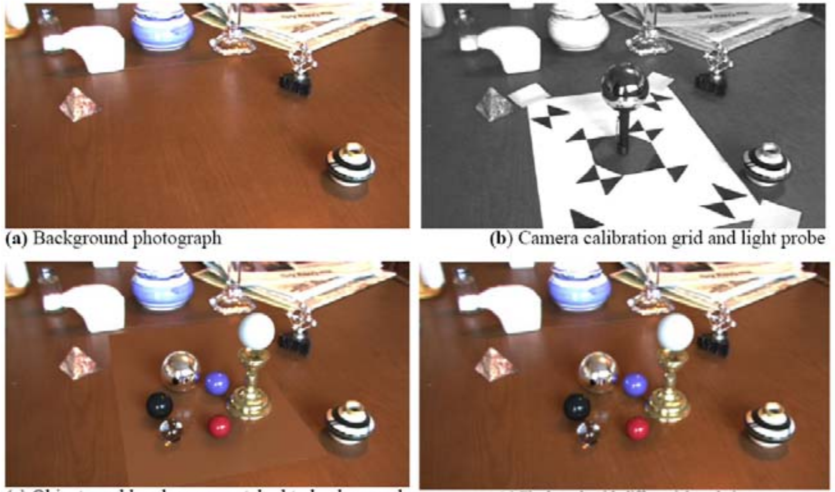
- $\mu$=0 → Newton's method
- $\mu \to \infty$ → steepest descent method

- Strategy for choosing $\mu$
  - Start with some small $\mu$
  - If F is not reduced, keep trying larger $\mu$ until it does
  - If F is reduced, accept it and reduce $\mu$ for the next iteration

## How is calibration used?

- Good for recovering intrinsic parameters; It is thus useful for many vision applications
- Since it requires a calibration pattern, it is often necessary to remove or replace the pattern from the footage or utilize it in some ways…

## Example of calibration



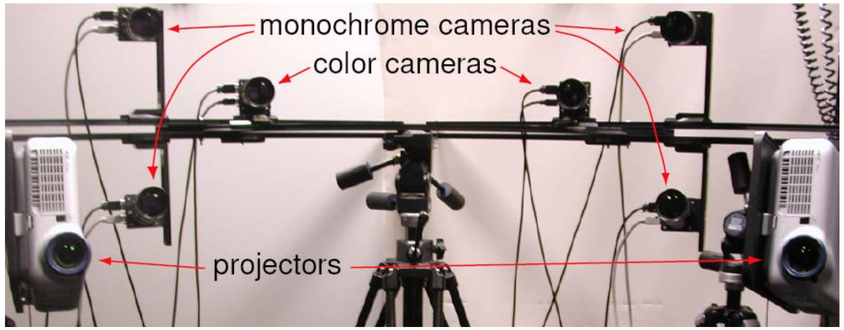
(a) Background photograph

(b) Camera calibration grid and light probe

(c) Objects and local scene matched to background

(g) Final result with differential rendering

## Example of calibration




## Example of calibration

- Videos from GaTech