Camera calibration

Digital Visual Effects, Spring 2007

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2007/4/17

with slides by Richard Szeliski, Steve Seitz, and Marc Pollefyes

Announcements

- Project #2 is due next Tuesday before the class

Outline

- Camera projection models
- Camera calibration (tools)
- Nonlinear least square methods
- Bundle adjustment

Camera projection models

Pinhole camera

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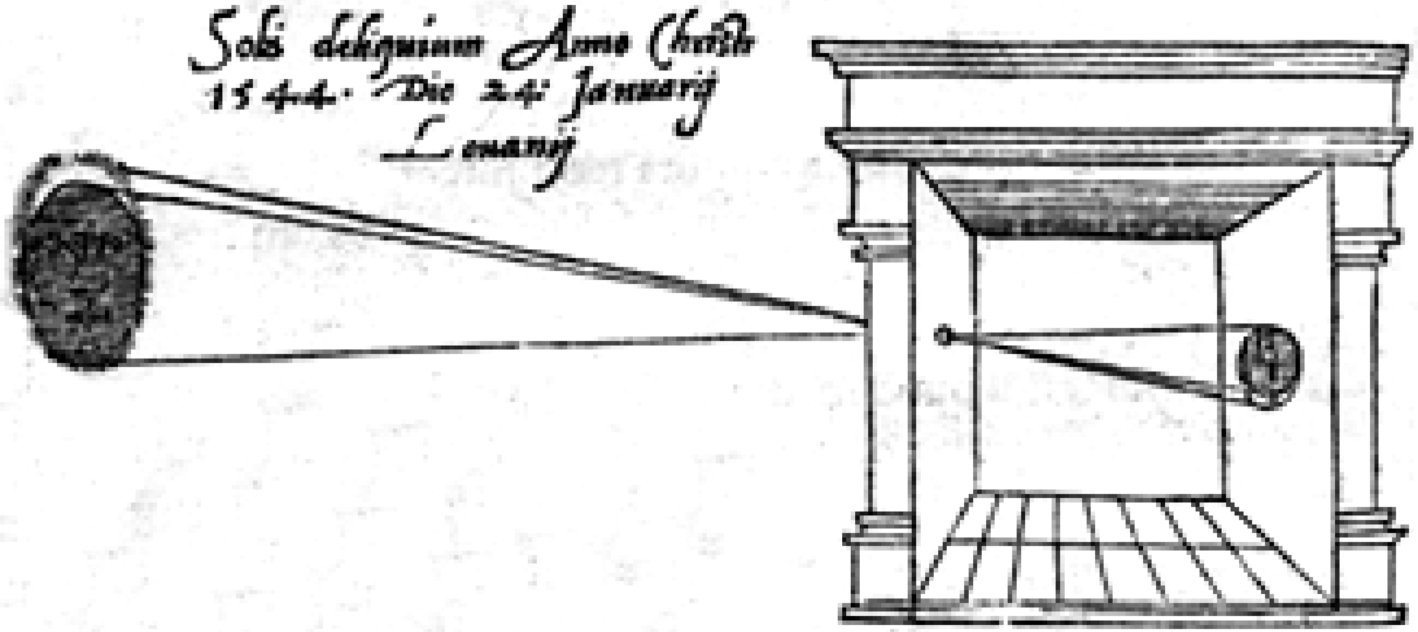

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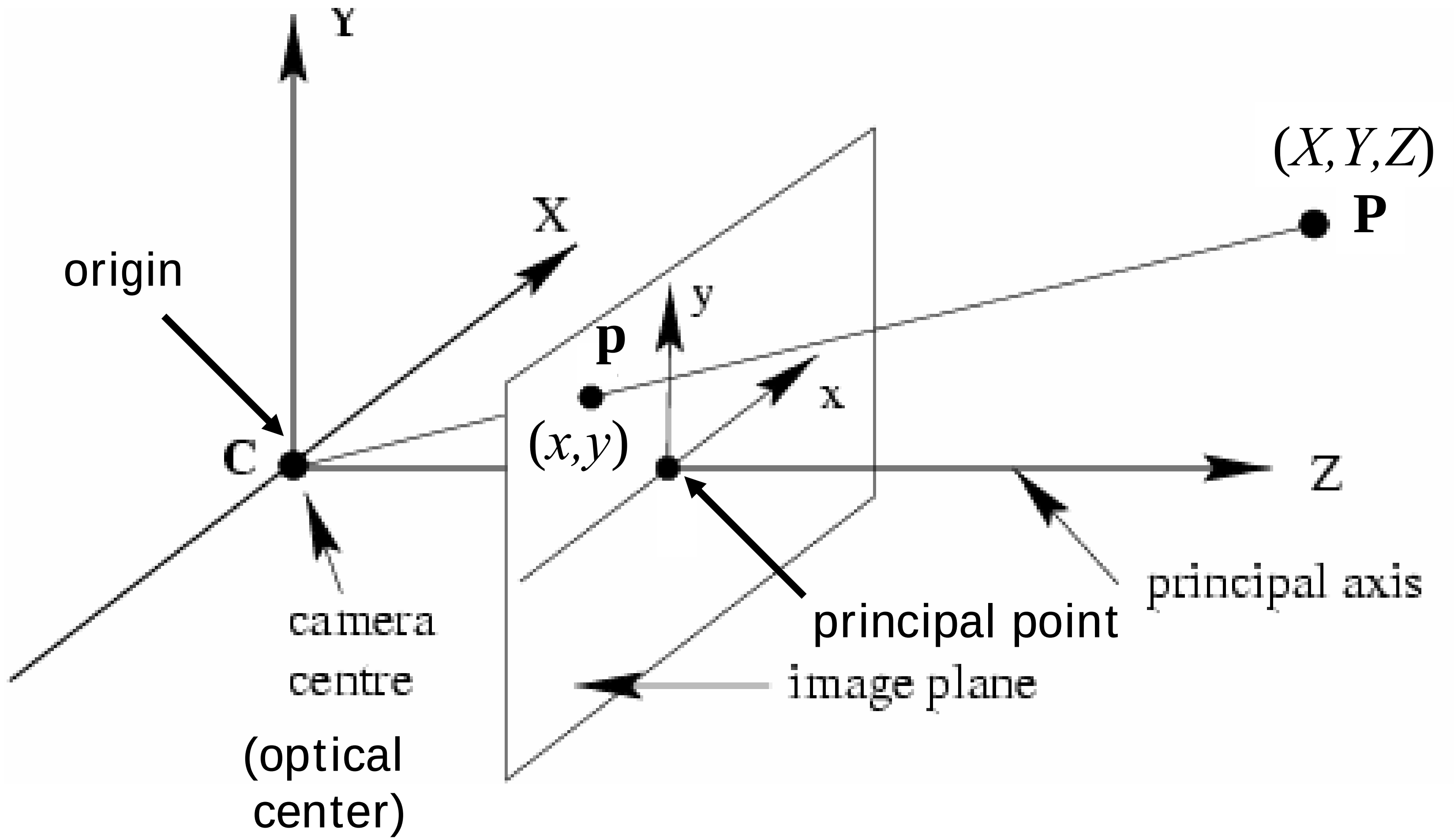
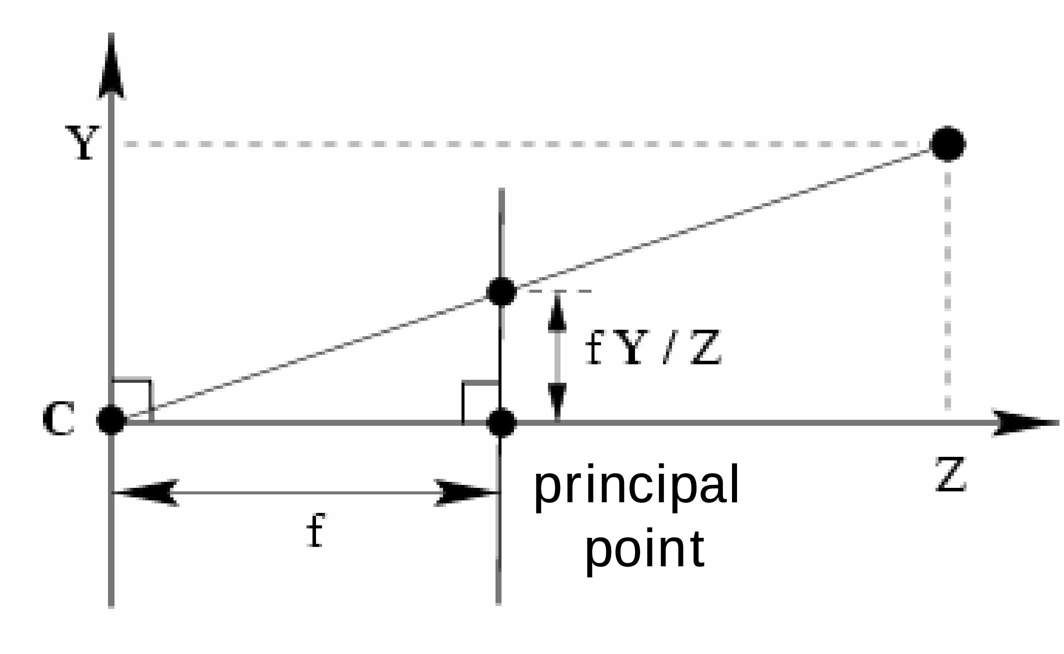
Pinhole camera model



Pinhole camera model

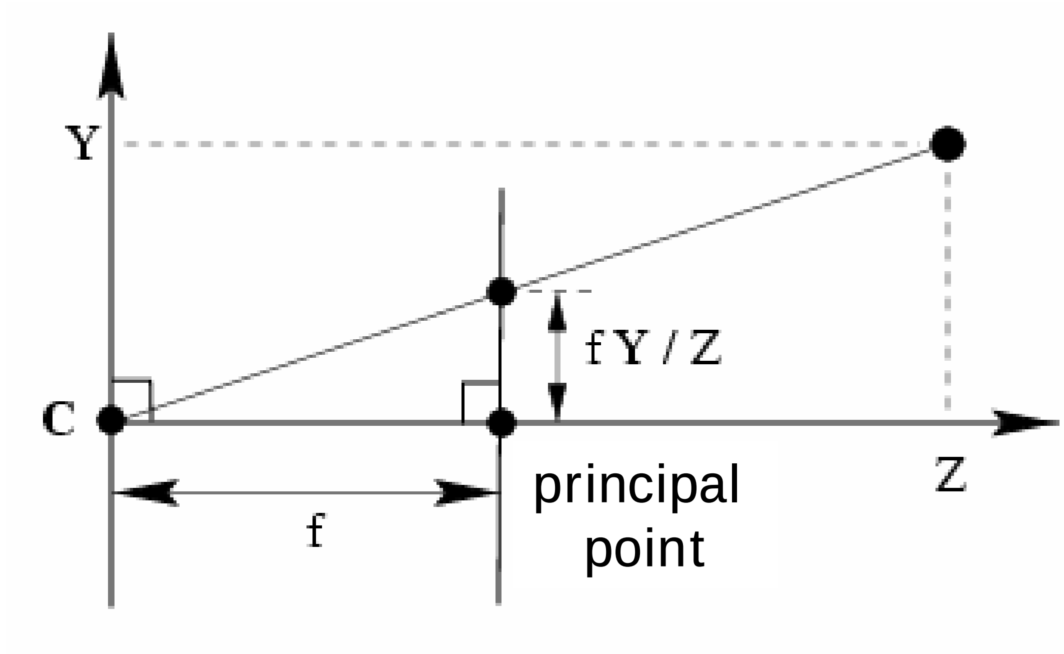


$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

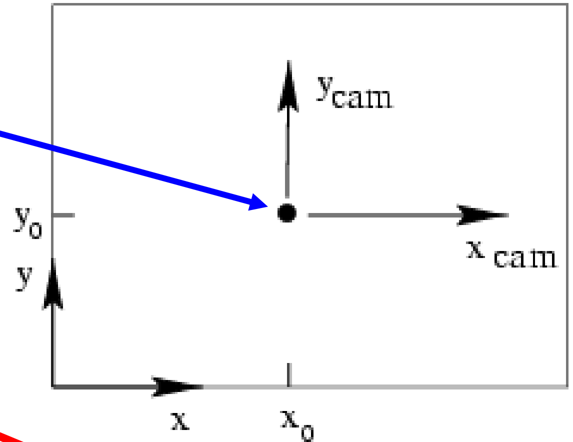
Pinhole camera model



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset

principal point



intrinsic matrix
only related to camera projection

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Intrinsic matrix

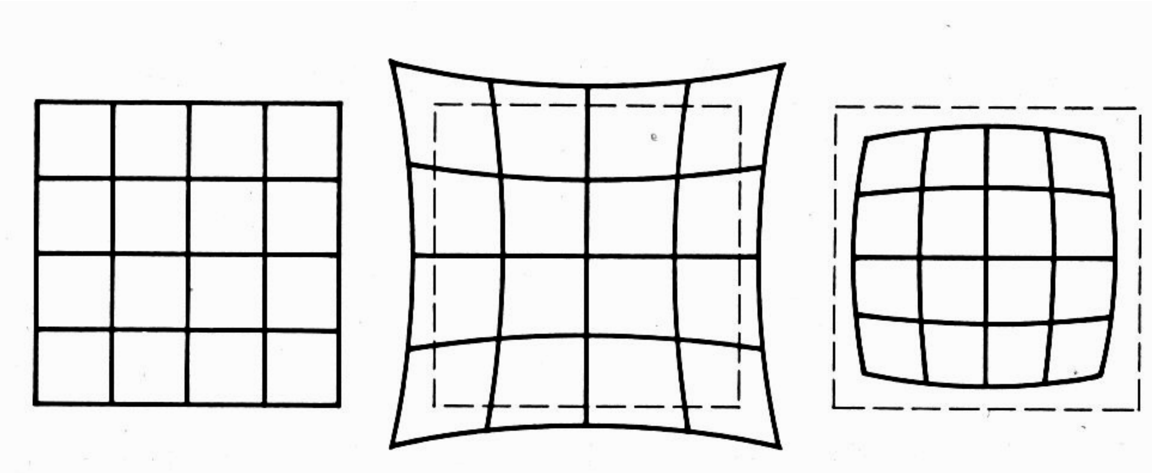
Is this form of K good enough?

$$\mathbf{K} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

- non-square pixels (digital video)
- skew
- radial distortion

$$\mathbf{K} = \begin{bmatrix} fa & s & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

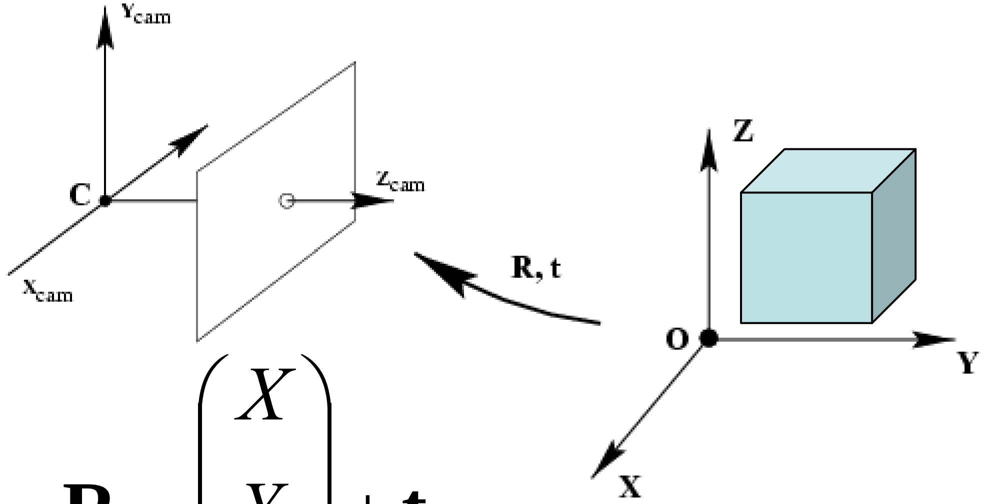
Distortion



No distortion   Pin cushion   Barrel

- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

Camera rotation and translation



$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}_{3\times 3} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mathbf{t}$$

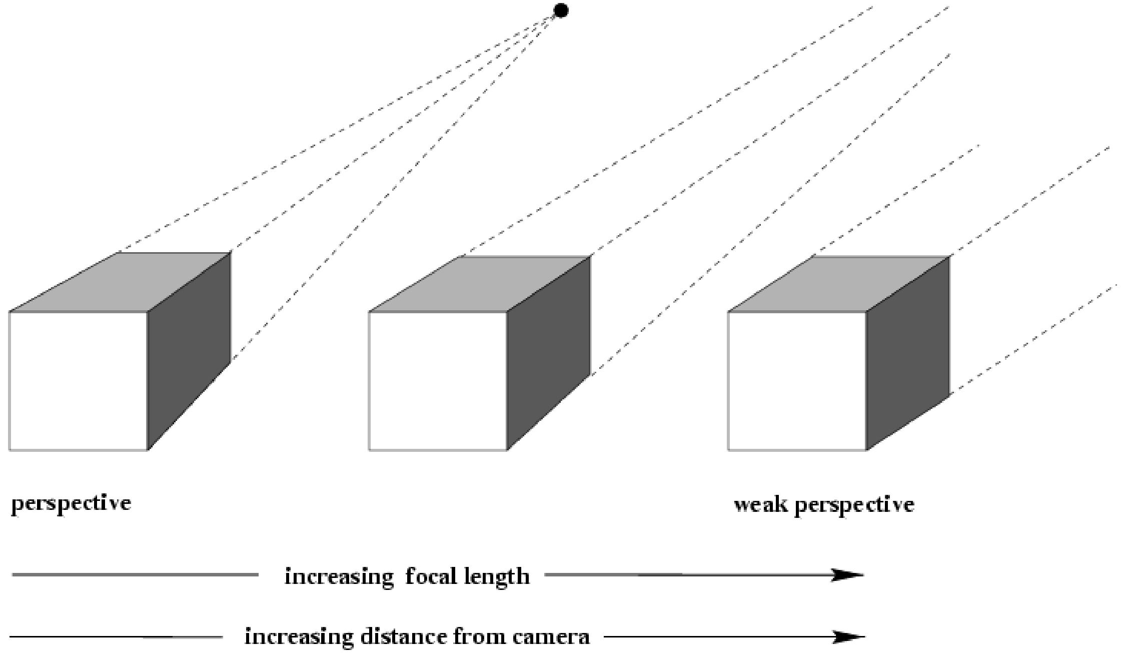
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \middle| \mathbf{t}\right] \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} \sim \mathbf{K} \underbrace{\left[\mathbf{R} \middle| \mathbf{t}\right]}_{\text{extrinsic matrix}} \mathbf{X}$$

Two kinds of parameters

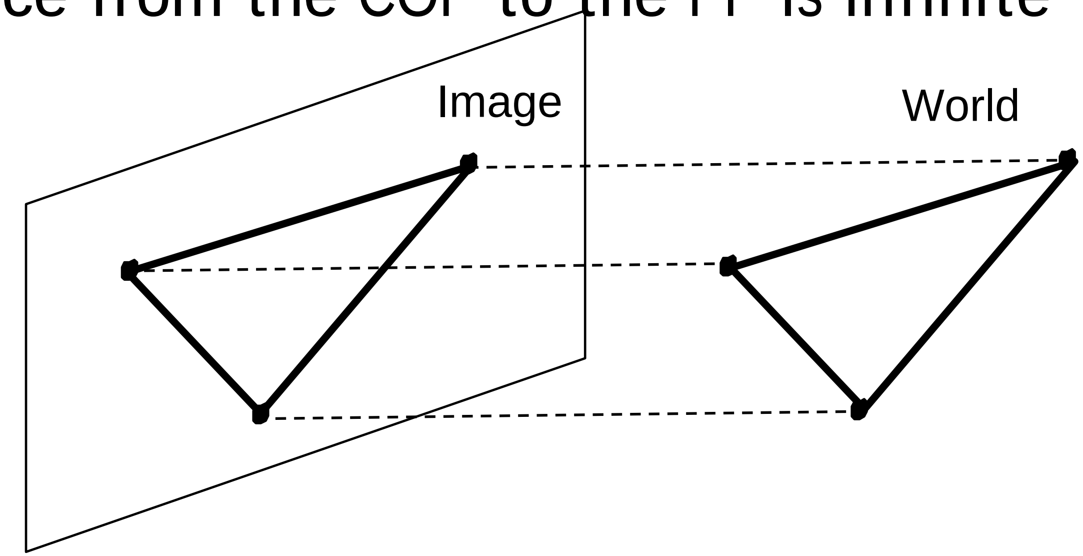
- *internal* or *intrinsic* parameters such as focal length, optical center, aspect ratio: *what kind of camera*?
- *external* or *extrinsic* (pose) parameters including rotation and translation: *where is the camera*?

Other projection models



Orthographic projection

- Special case of perspective projection
  - Distance from the COP to the PP is infinite



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

  - Also called “parallel projection”: $(x, y, z) \rightarrow (x, y)$

Other types of projections

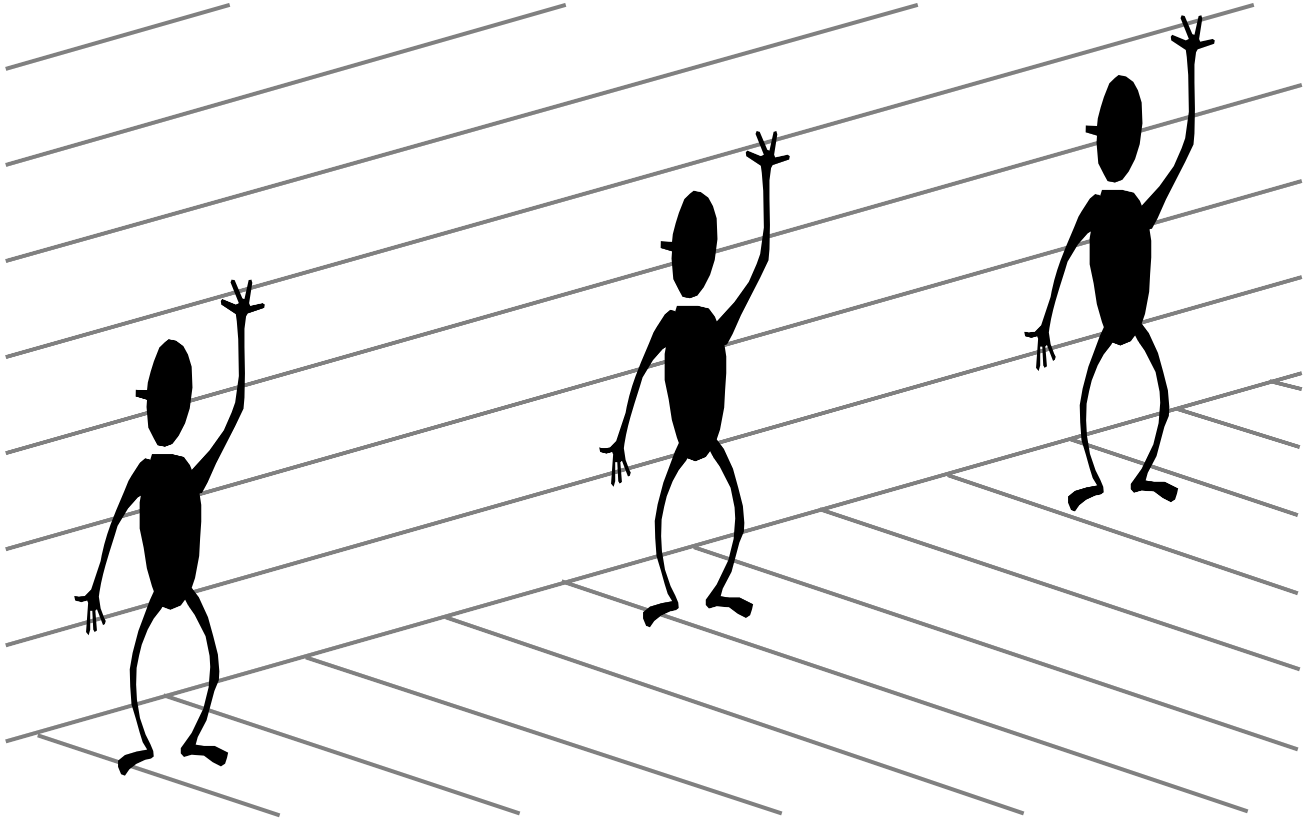
- Scaled orthographic
  - Also called "weak perspective"

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/d \end{bmatrix} \Rightarrow (dx, dy)$$

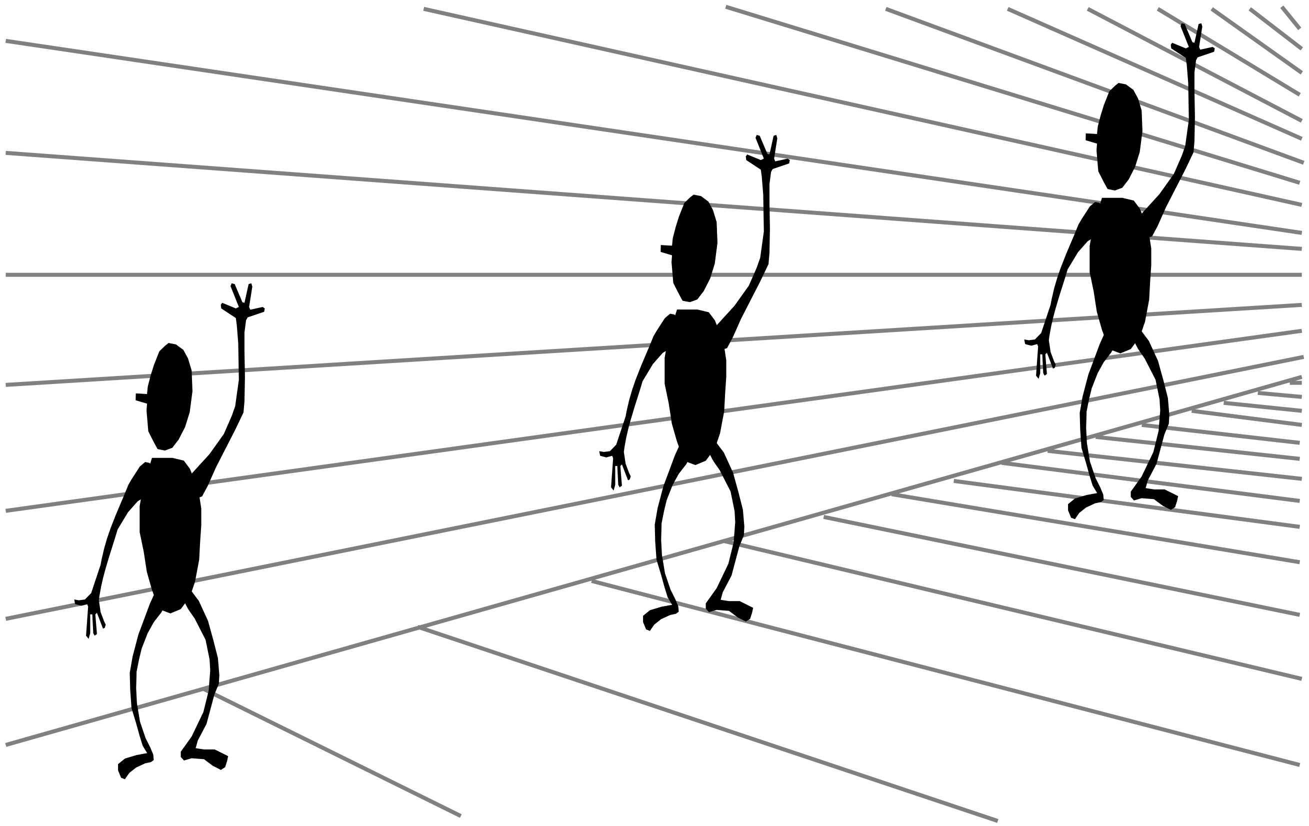
- Affine projection
  - Also called "paraperspective"

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

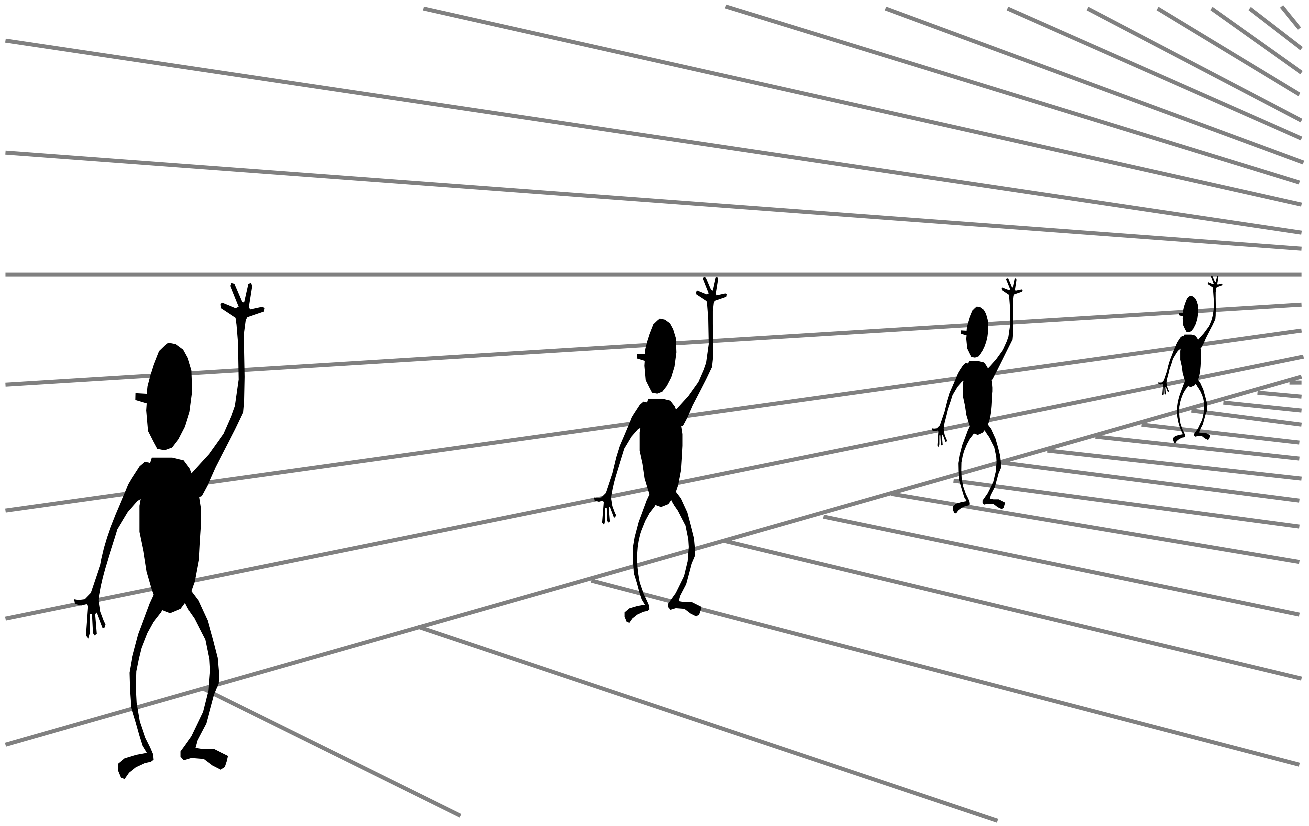
Fun with perspective



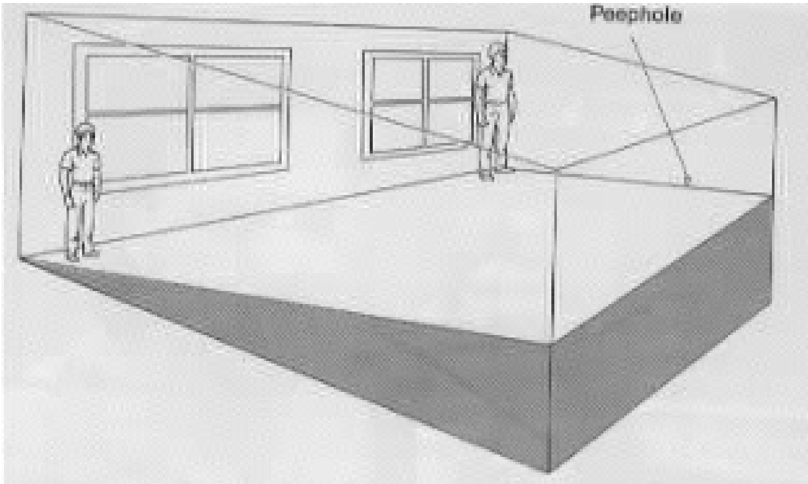
Perspective cues

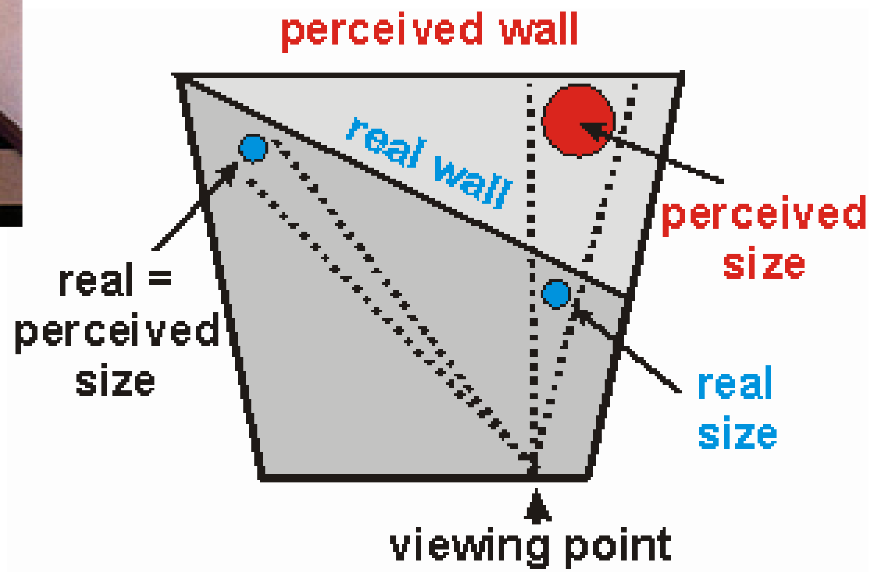


Perspective cues



Fun with perspective





[Ames room]()

Forced perspective in LOTR

Camera calibration

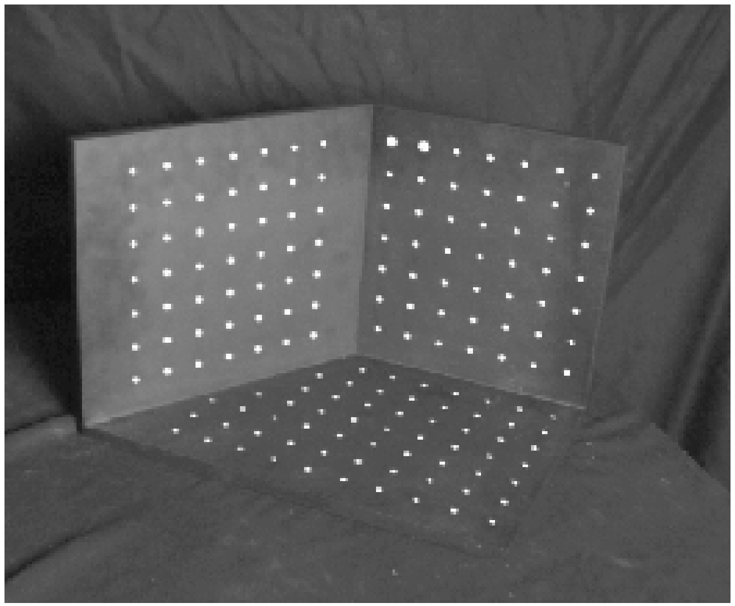
Camera calibration

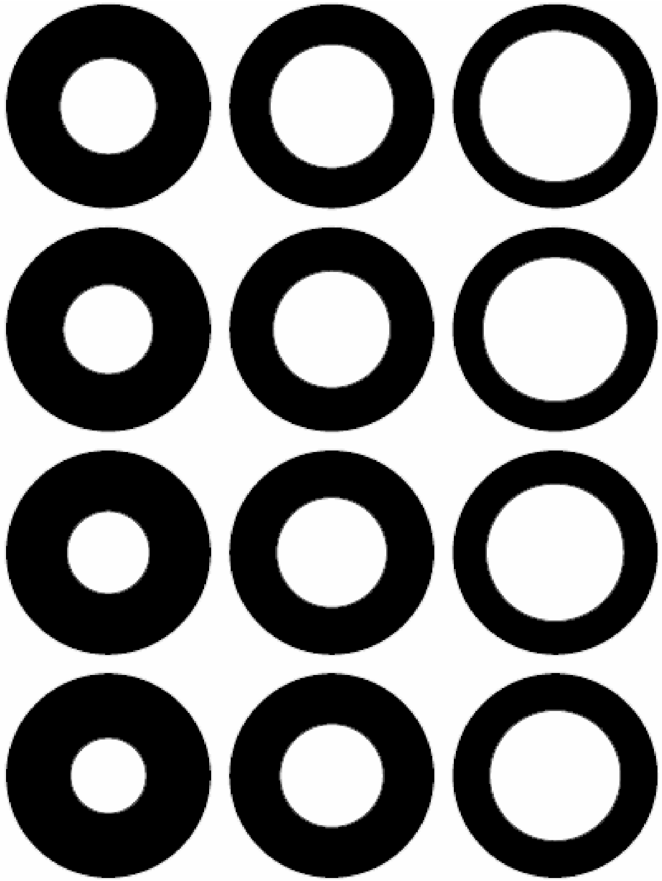
- Estimate both intrinsic and extrinsic parameters
- Mainly, two categories:

1. Photometric calibration: uses reference objects with known geometry
2. Self calibration: only assumes static scene, e.g. structure from motion

Camera calibration approaches

1. linear regression (least squares)
2. nonlinear optimization
3. multiple planar patterns





Chromaglyphs (HP research)

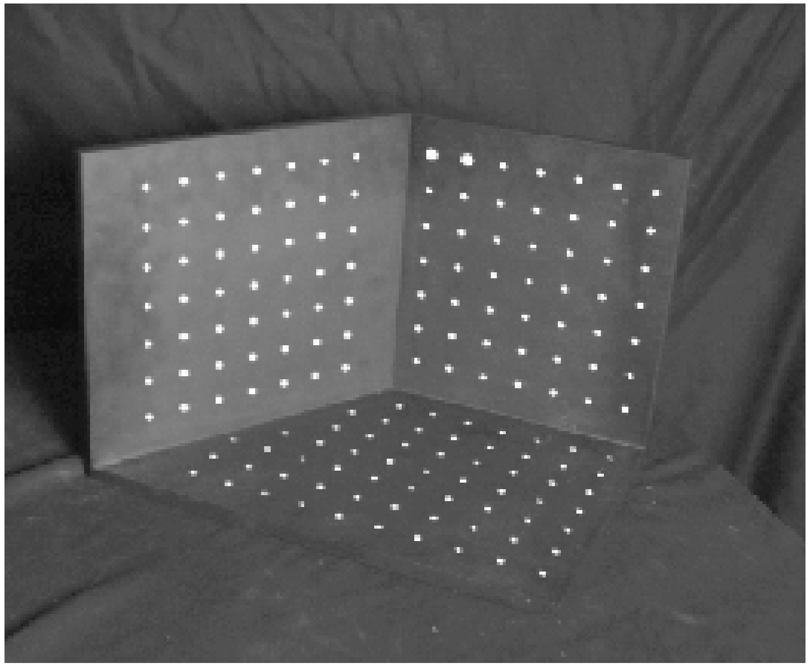
Linear regression

$$\mathbf{x} \sim \mathbf{K}\left[\mathbf{R}\middle|\mathbf{t}\right]\mathbf{X} = \mathbf{M}\mathbf{X}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear regression

- Directly estimate 11 unknowns in the **M** matrix using known 3D points (X_i,Y_i,Z_i) and measured feature positions (u_i,v_i)



Linear regression

$$u_i = \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$v_i = \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

$$u_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}$$

$$v_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1) = m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}$$

Linear regression

$$\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i & -u_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i & -v_i \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear regression

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Solve for Projection Matrix M using least-square techniques

Normal equation

Given an overdetermined system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

the normal equation is that which minimizes the sum of the square differences between left and right sides

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Why?

Normal equation

$$E(\mathbf{x}) = \left(\mathbf{A}\mathbf{x} - \mathbf{b}\right)^2$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

nxm, n equations, m variables

Normal equation

$$\mathbf{Ax}-\mathbf{b}=\begin{bmatrix}\sum_{j=1}^{m}a_{1j}x_j\\ \vdots\\ \sum_{j=1}^{m}a_{ij}x_j\\ \vdots\\ \sum_{j=1}^{m}a_{nj}x_j\end{bmatrix}-\begin{bmatrix}b_1\\ \vdots\\ b_i\\ \vdots\\ b_n\end{bmatrix}=\begin{bmatrix}\left(\sum_{j=1}^{m}a_{1j}x_j\right)-b_1\\ \vdots\\ \left(\sum_{j=1}^{m}a_{ij}x_j\right)-b_i\\ \vdots\\ \left(\sum_{j=1}^{m}a_{nj}x_j\right)-b_n\end{bmatrix}$$

$$E(\mathbf{x})=(\mathbf{Ax}-\mathbf{b})^2=\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{m}a_{ij}x_j\right)-b_i\right]^2$$

Normal equation

$$E(\mathbf{x}) = (\mathbf{Ax} - \mathbf{b})^2 = \sum_{i=1}^{n} \left[\left(\sum_{j=1}^{m} a_{ij} x_j \right) - b_i \right]^2$$

$$0 = \frac{\partial E}{\partial x_1} = \sum_{i=1}^{n} 2 \left[\left(\sum_{j=1}^{m} a_{ij} x_j \right) - b_i \right] a_{i1}$$

$$= 2 \sum_{i=1}^{n} a_{i1} \sum_{j=1}^{m} a_{ij} x_j - 2 \sum_{i=1}^{n} a_{i1} b_i$$

$$0 = \frac{\partial E}{\partial \mathbf{x}} = 2(\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - \mathbf{A}^{\mathrm{T}}\mathbf{b}) \rightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Normal equation

$$(\mathbf{Ax}-\mathbf{b})^2$$

$$=(\mathbf{Ax}-\mathbf{b})^T(\mathbf{Ax}-\mathbf{b})$$

$$=\left((\mathbf{Ax})^T-\mathbf{b}^T\right)(\mathbf{Ax}-\mathbf{b})$$

$$=\left(\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}-\mathbf{b}^{\mathrm{T}}\right)(\mathbf{Ax}-\mathbf{b})$$

$$=\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax}-\mathbf{b}^{\mathrm{T}}\mathbf{Ax}-\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b}+\mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$=\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax}-\left(\mathbf{A}^{\mathrm{T}}\mathbf{b}\right)^{\mathrm{T}}\mathbf{x}-\left(\mathbf{A}^{\mathrm{T}}\mathbf{b}\right)^{\mathrm{T}}\mathbf{x}+\mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}=2\mathbf{A}^{\mathrm{T}}\mathbf{Ax}-2\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Linear regression

- Advantages:
  - All specifics of the camera summarized in one matrix
  - Can predict where any world point will map to in the image
- Disadvantages:
  - Doesn't tell us about particular parameters
  - Mixes up internal and external parameters
    - pose specific: move the camera and everything breaks

Nonlinear optimization

- A probabilistic view of least square
- Feature measurement equations

$$
\begin{aligned}
u_i &= f(\boxed{\mathbf{M}}, \mathbf{x}_i) + n_i = \widehat{u}_i + n_i, \quad n_i \sim N(0, \sigma) \\
v_i &= g(\boxed{\mathbf{M}}, \mathbf{x}_i) + m_i = \widehat{v}_i + m_i, \quad m_i \sim N(0, \sigma)
\end{aligned}
$$

- Likelihood of *M* given $\{(u_i, v_i)\}$

$$
\begin{aligned}
L &= \prod_i p(u_i|\widehat{u}_i)p(v_i|\widehat{v}_i) \\
&= \prod_i e^{-(u_i-\widehat{u}_i)^2/\sigma^2} e^{-(v_i-\widehat{v}_i)^2/\sigma^2}
\end{aligned}
$$

Optimal estimation

- Log likelihood of *M* given $\{(u_i, v_i)\}$

$$C = -\log L = \sum_i (u_i - \hat{u}_i)^2/\sigma_i^2 + (v_i - \hat{v}_i)^2/\sigma_i^2$$

- It is a least square problem (but not necessarily linear least square)
- How do we minimize *C*?

Optimal estimation

- Non-linear regression (least squares), because the relations between $\hat{u}_i$ and u_i are non-linear functions M

unknown parameters

We could have terms like $f \cos\theta$ in this

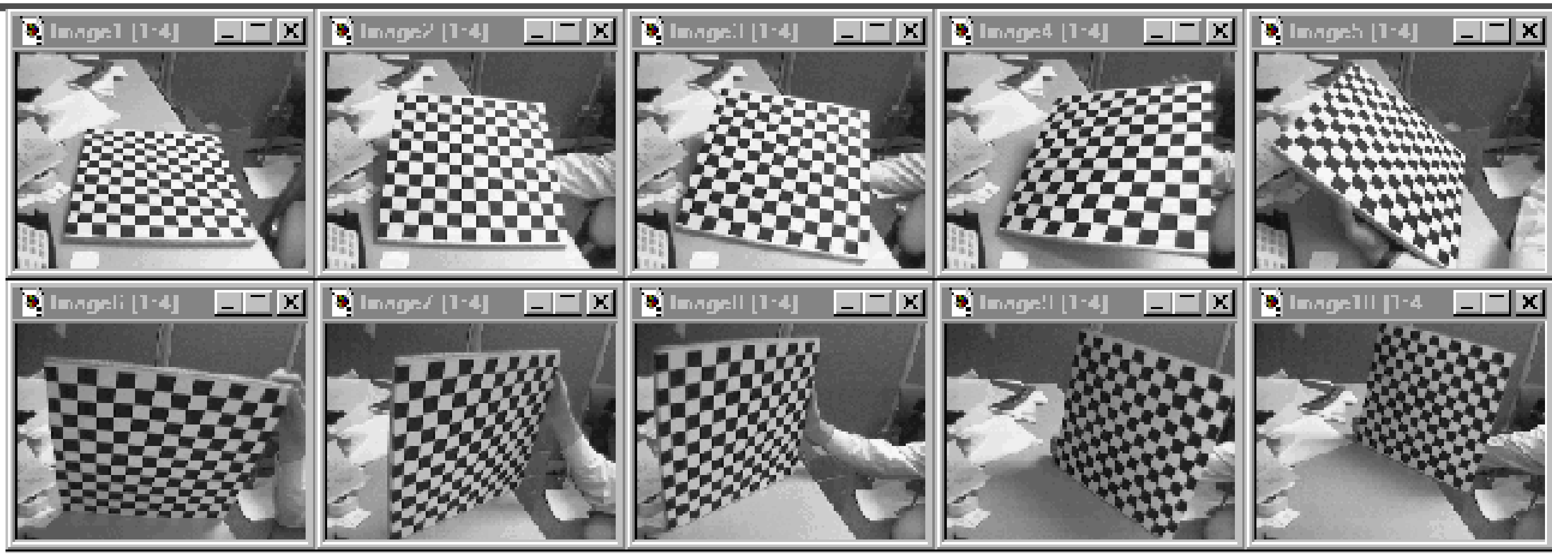
$$\mathbf{u} - \hat{\mathbf{u}} \sim \mathbf{u} - \mathbf{K}\left[\mathbf{R} \middle| \mathbf{t}\right]\mathbf{X}$$

known constant

- We can use Levenberg-Marquardt method to minimize it

A popular calibration tool

Multi-plane calibration

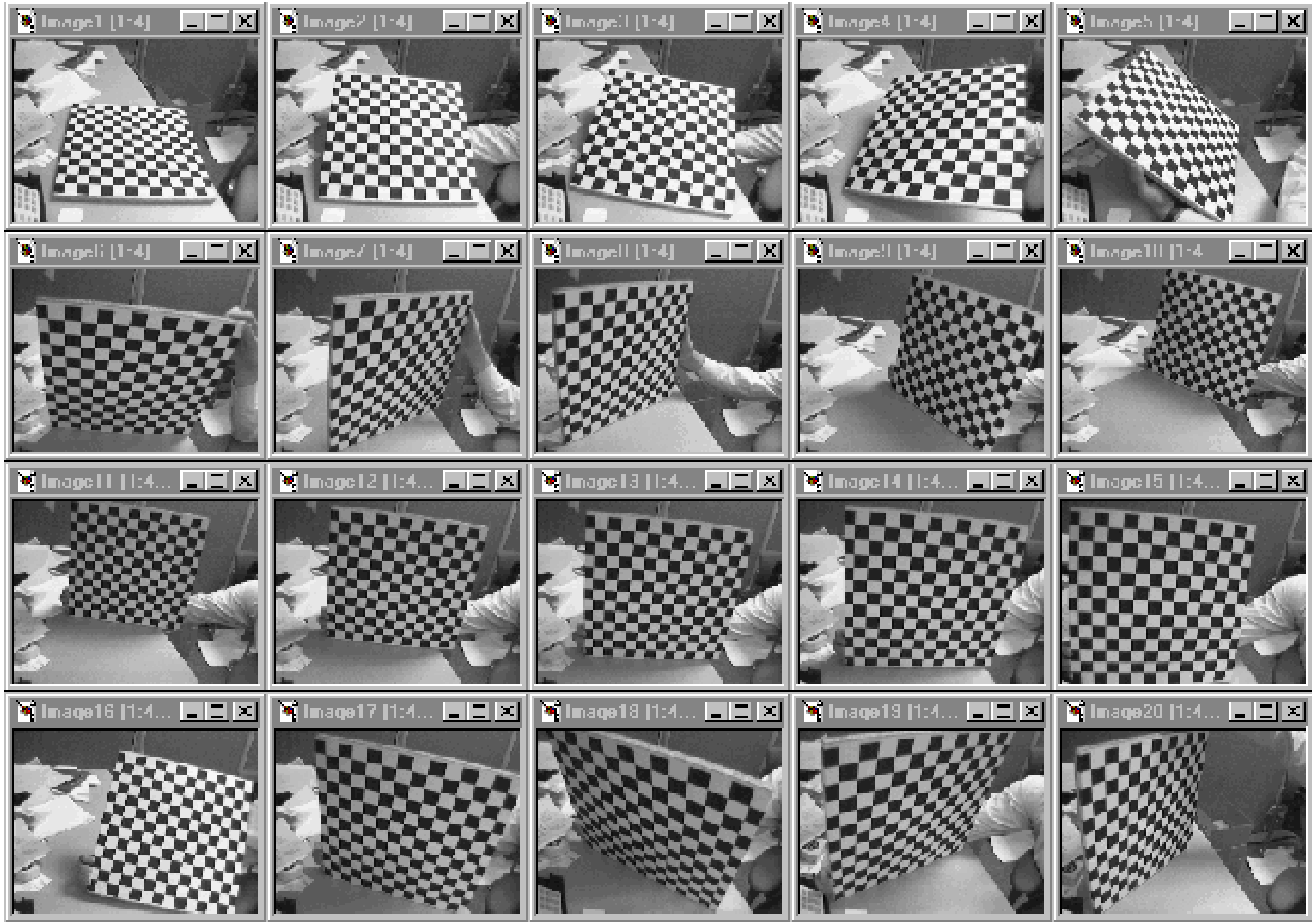


Images courtesy Jean-Yves Bouguet, Intel Corp.

Advantage

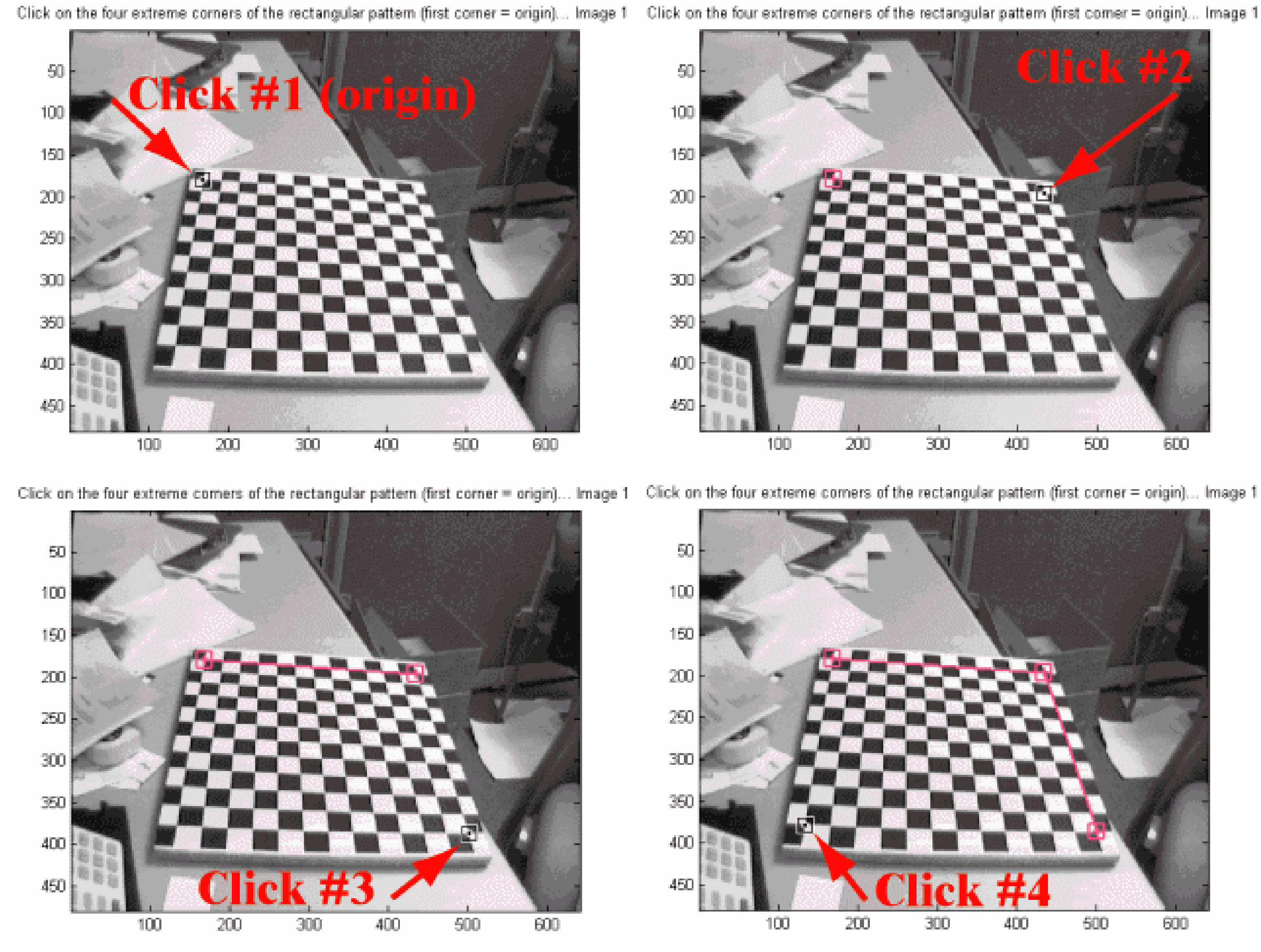
- Only requires a plane
- Don't have to know positions/orientations
- Good code available online!
  - Intel's OpenCV library: http://www.intel.com/research/mrl/research/opencv/
  - Matlab version by Jean-Yves Bouget: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Zhengyou Zhang's web site: http://research.microsoft.com/~zhang/Calib/

Step 1: data acquisition

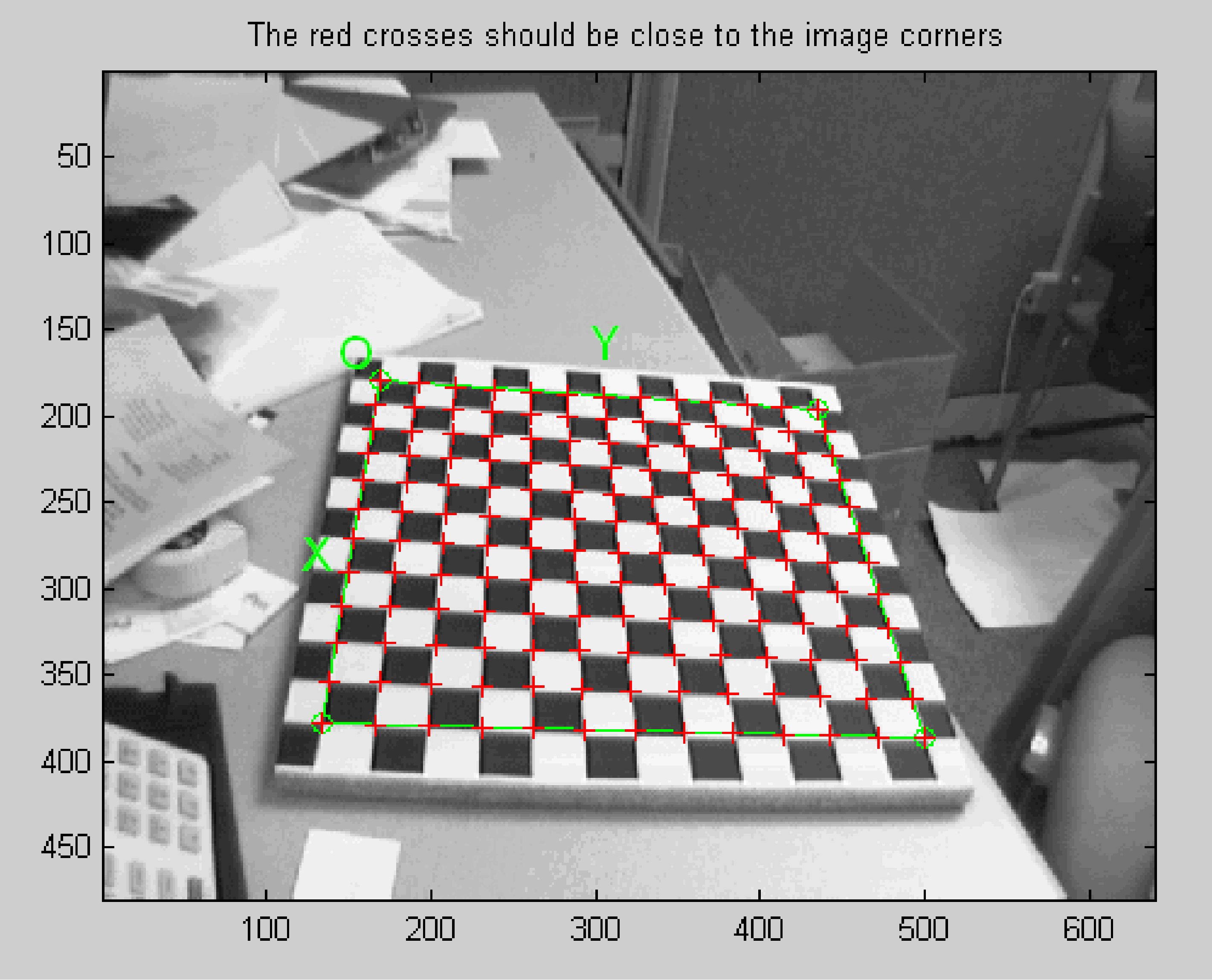


Step 2: specify corner order

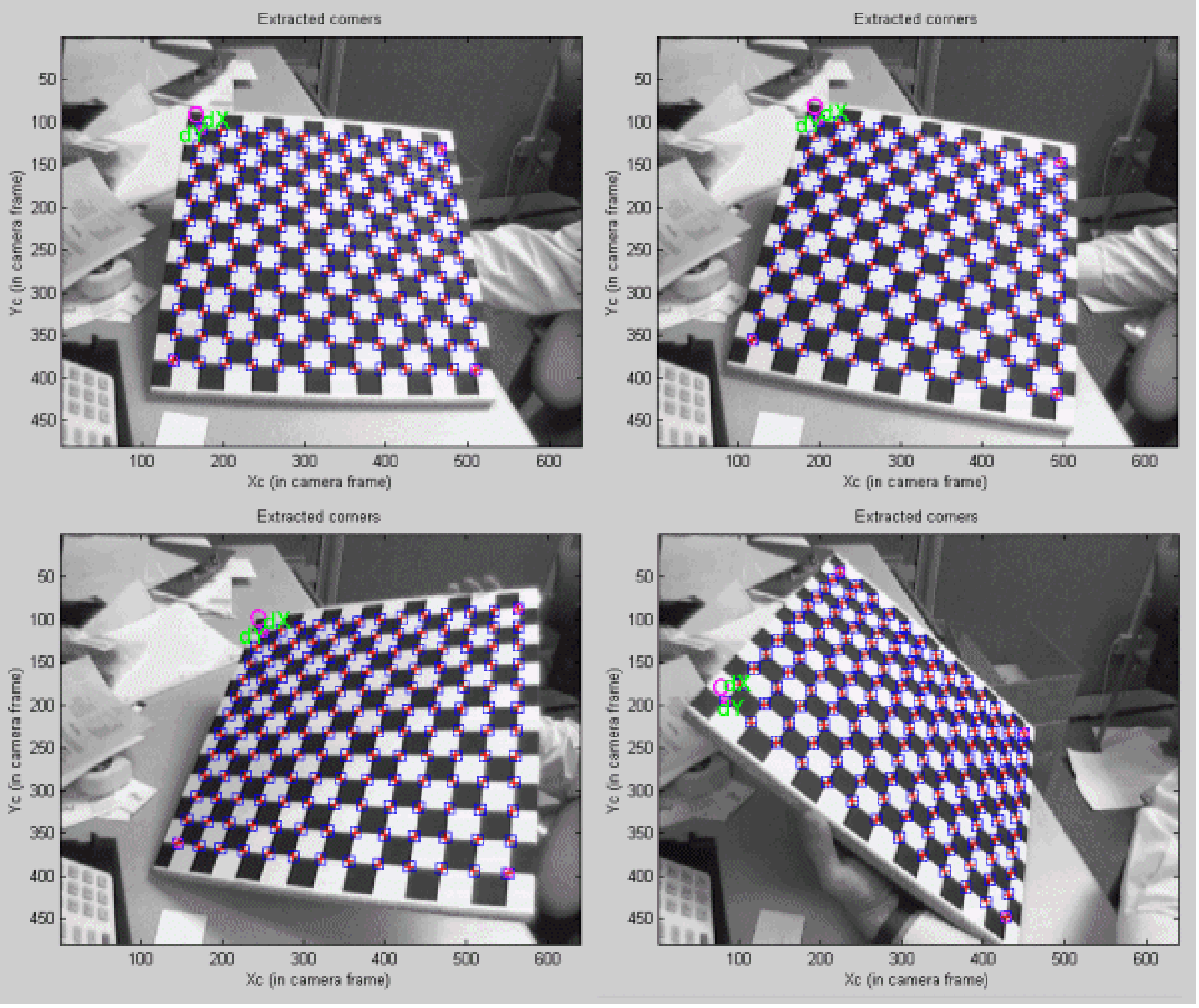
DigiVFX



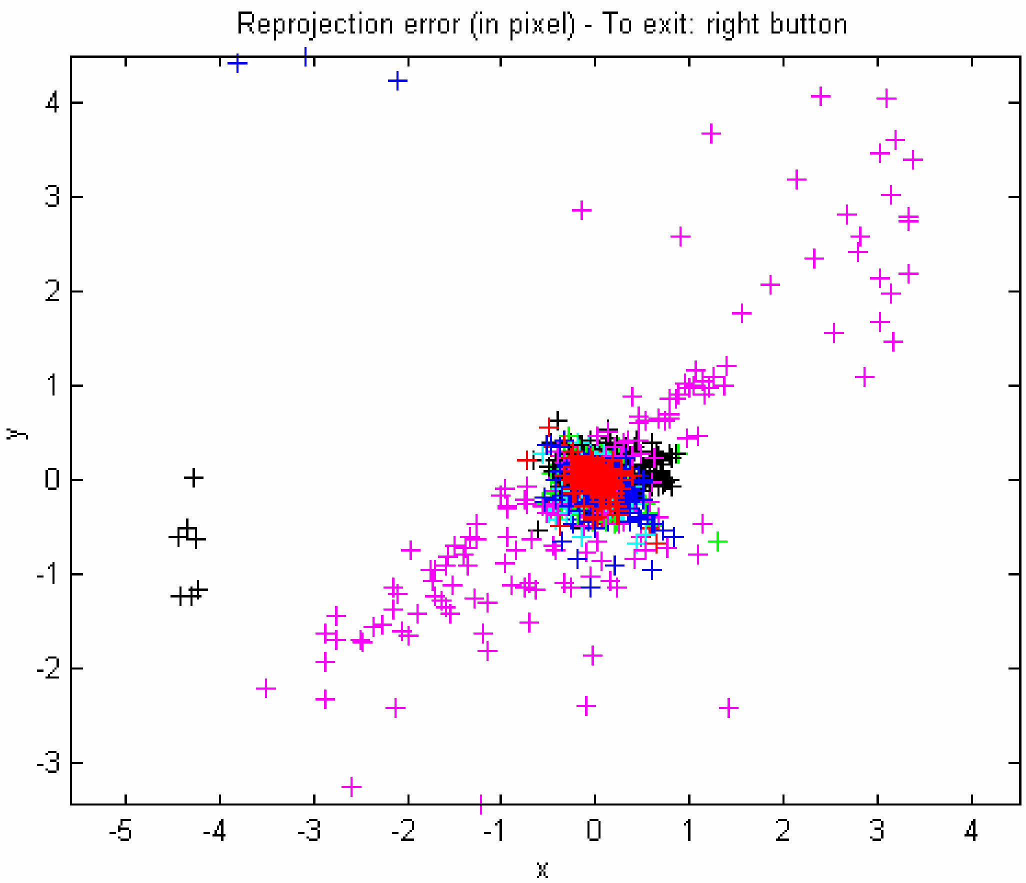
Step 3: corner extraction



Step 3: corner extraction



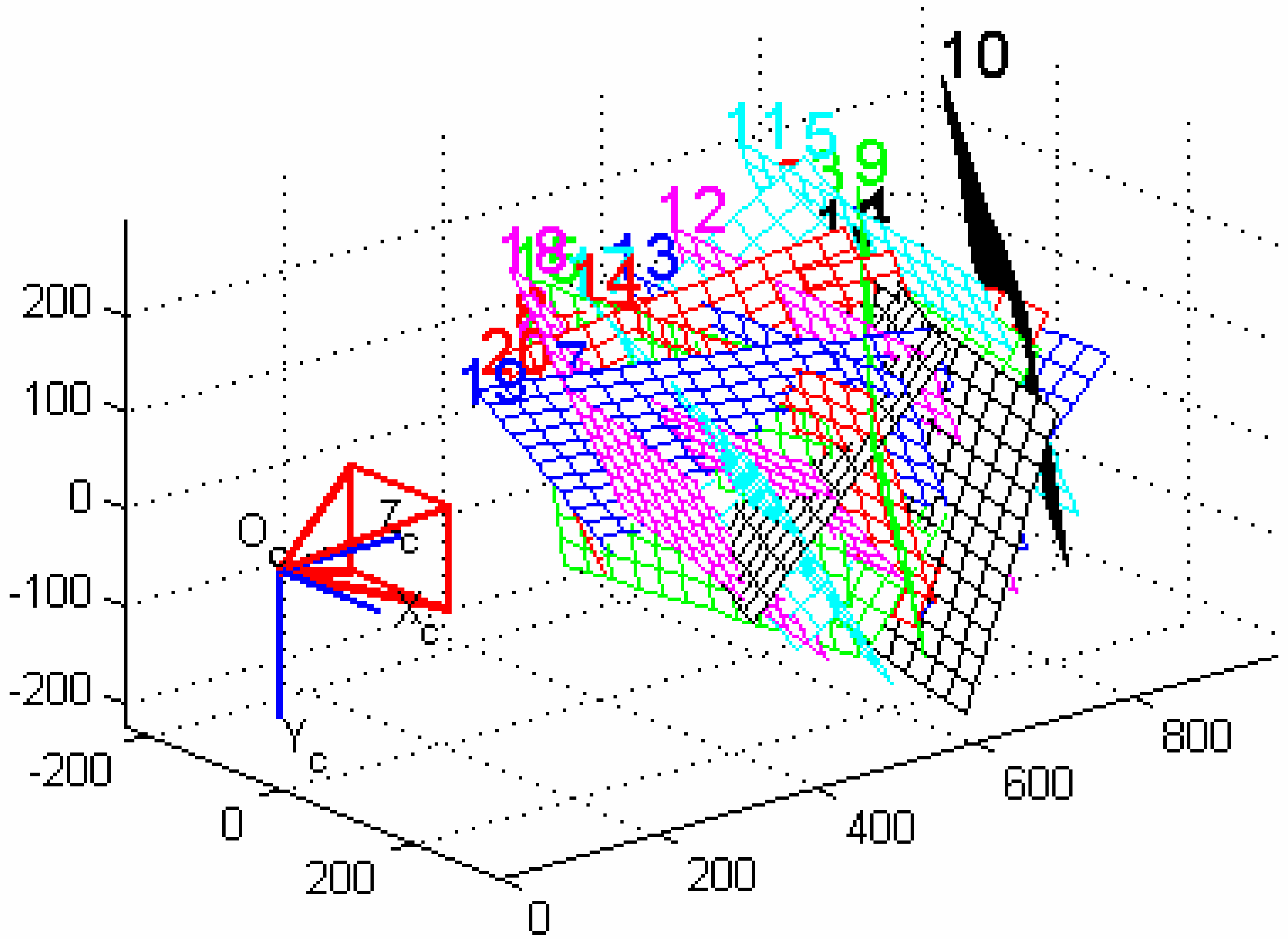
Step 4: minimize projection error



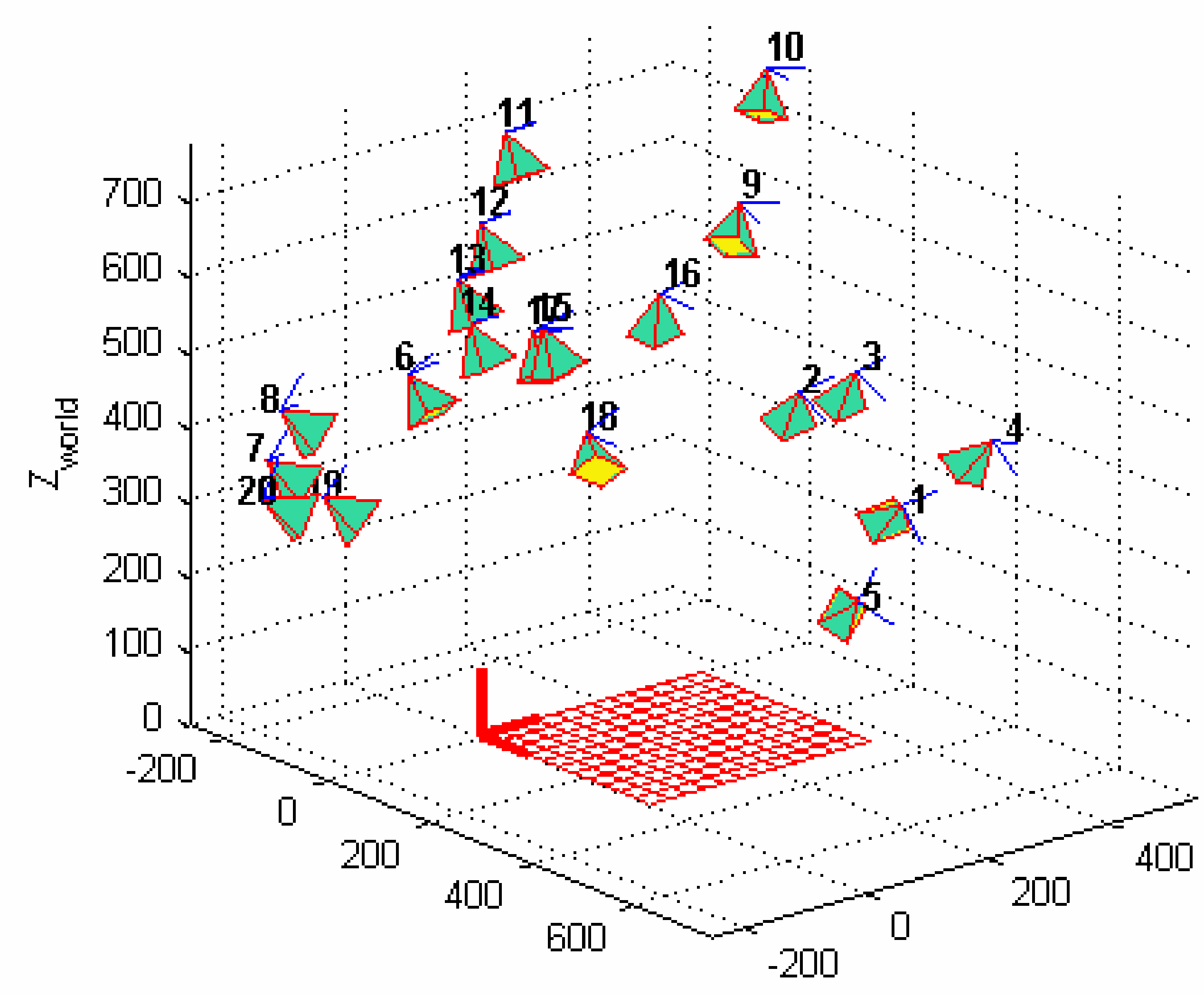

```
Calibration res

Focal Length:          fc = [ 657.46290   657.94673 ] ± [ 0.31819   0.34046 ]
Principal point:       cc = [ 303.13665   242.56935 ] ± [ 0.64682   0.59218 ]
Skew:             alpha_c = [ 0.00000 ] ± [ 0.00000  ]   => angle of pixel axes =
Distortion:            kc = [ -0.25403   0.12143   -0.00021   0.00002   0.00000 ]
Pixel error:          err = [ 0.11689   0.11500 ]
```

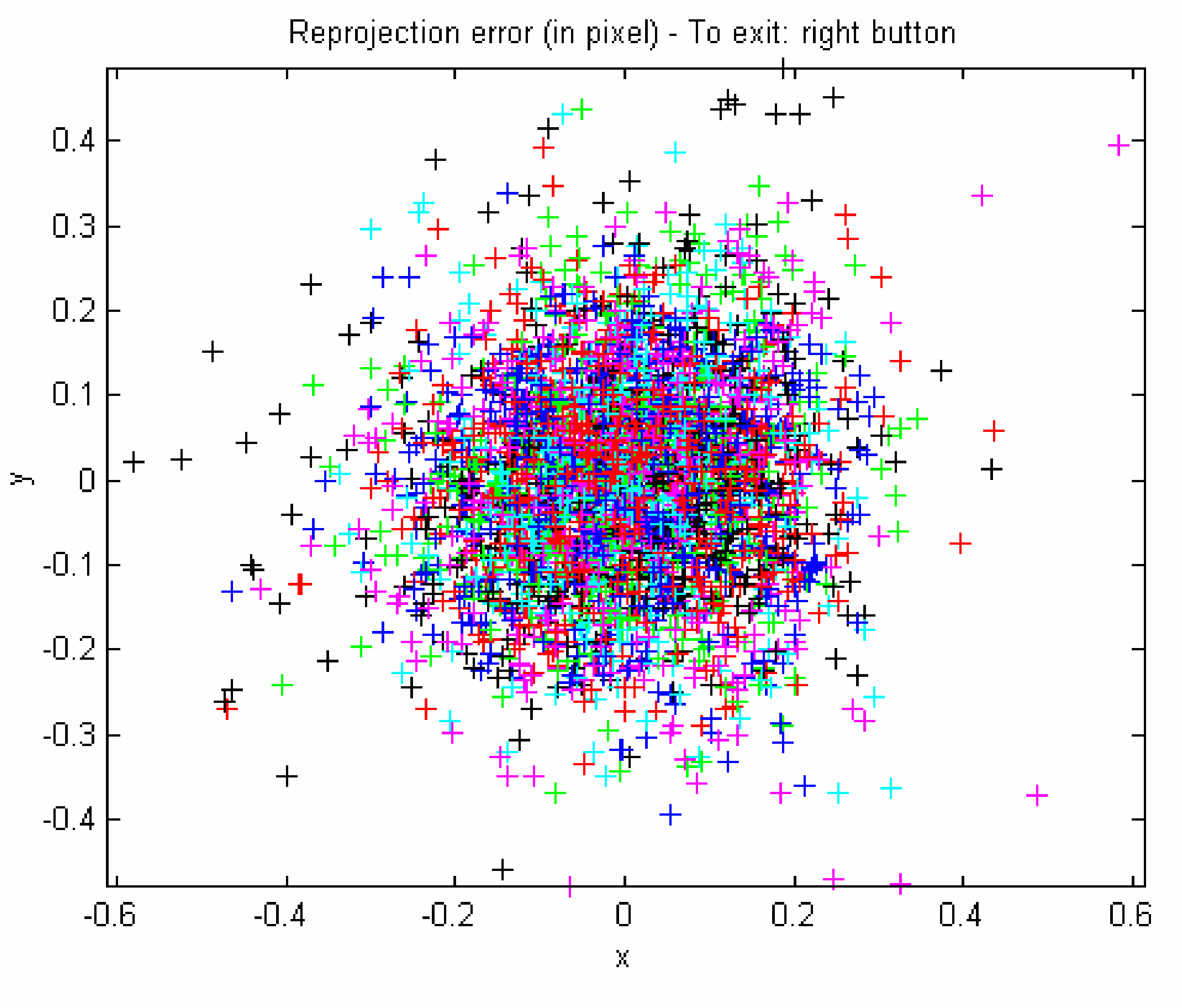
Step 4: camera calibration



Step 4: camera calibration



Step 5: refinement



Nonlinear least square methods

Least square fitting

Least Squares Problem

Find $\mathbf{x}^*$, a local minimizer for

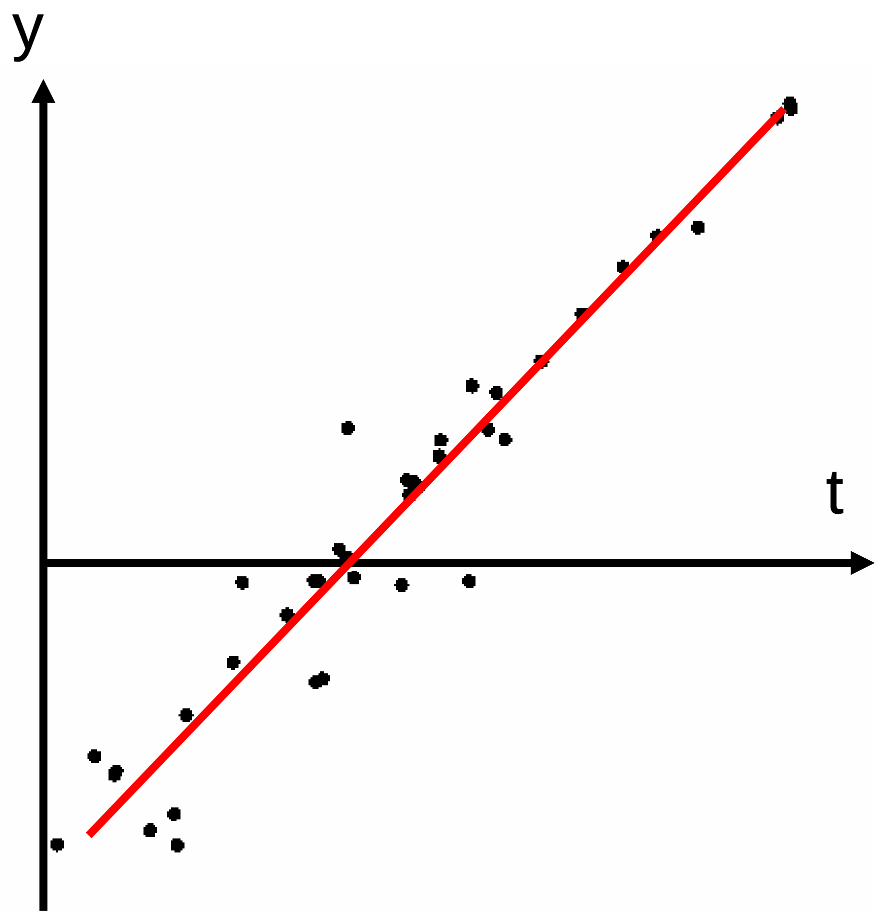
$$F(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}(f_i(\mathbf{x}))^2 ,$$

where $f_i : \mathbb{R}^n \mapsto \mathbb{R},\ i=1,\ldots,m$ are given functions, and $m \geq n$.

number of data points

number of parameters

Linear least square fitting



model parameters

$$y(t) = M(x,t) = x_0 + x_1 t$$

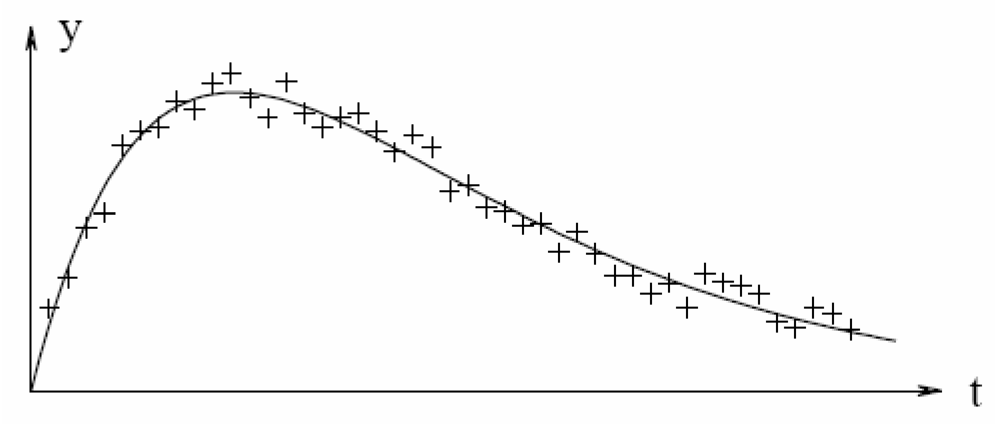
$$f_i(x) = y_i - \boxed{M(x,t_i)}$$

prediction

residual

$M(x,t) = x_0 + x_1 t + x_2 t^3$ is linear, too.

Nonlinear least square fitting



model $M(\mathbf{x}, t) = x_3 e^{x_1 t} + x_4 e^{x_2 t}$

parameters $\mathbf{x} = [x_1, x_2, x_3, x_4]^\top$

residuals
$$\begin{aligned} f_i(\mathbf{x}) &= y_i - M(\mathbf{x}, t_i) \\ &= y_i - x_3 e^{x_1 t_i} - x_4 e^{x_2 t_i} \end{aligned}$$

Function minimization

Least square is related to function minimization.

Global Minimizer

Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find

$$\mathbf{x}^+ = \operatorname{argmin}_{\mathbf{x}} \{F(\mathbf{x})\} .$$

It is very hard to solve in general. Here, we only consider a simpler problem of finding local minimum.

Local Minimizer

Given $F : \mathbb{R}^n \mapsto \mathbb{R}$. Find $\mathbf{x}^*$ so that

$$F(\mathbf{x}^*) \le F(\mathbf{x}) \quad \text{for} \quad \|\mathbf{x} - \mathbf{x}^*\| < \delta .$$

Function minimization

We assume that the cost function F is differentiable and so smooth that the following *Taylor expansion* is valid,[2)]

$$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathbf{h}^\top \mathbf{g} + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}\,\mathbf{h} + O(\|\mathbf{h}\|^3),$$

where g is the *gradient*,

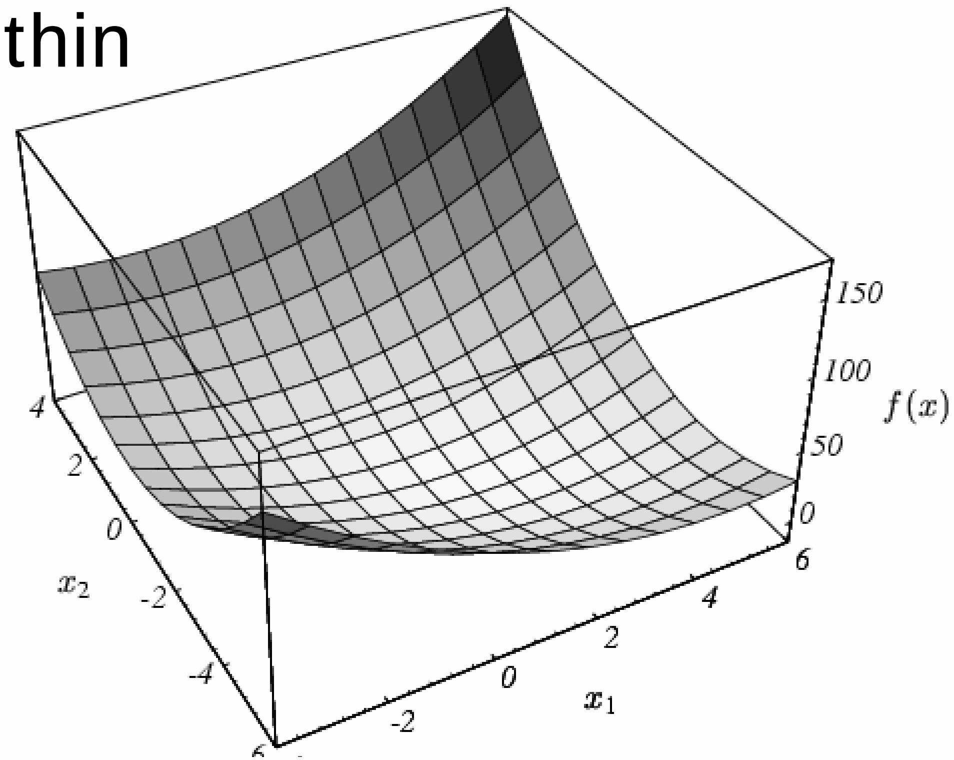
$$\mathbf{g} \equiv \mathbf{F}'(\mathbf{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial F}{\partial x_n}(\mathbf{x}) \end{bmatrix},$$

and $\mathbf{H}$ is the *Hessian*,

$$\mathbf{H} \equiv \mathbf{F}''(\mathbf{x}) = \left[\frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{x})\right].$$

Quadratic functions

Approximate the function with a quadratic function within a small neighborhood



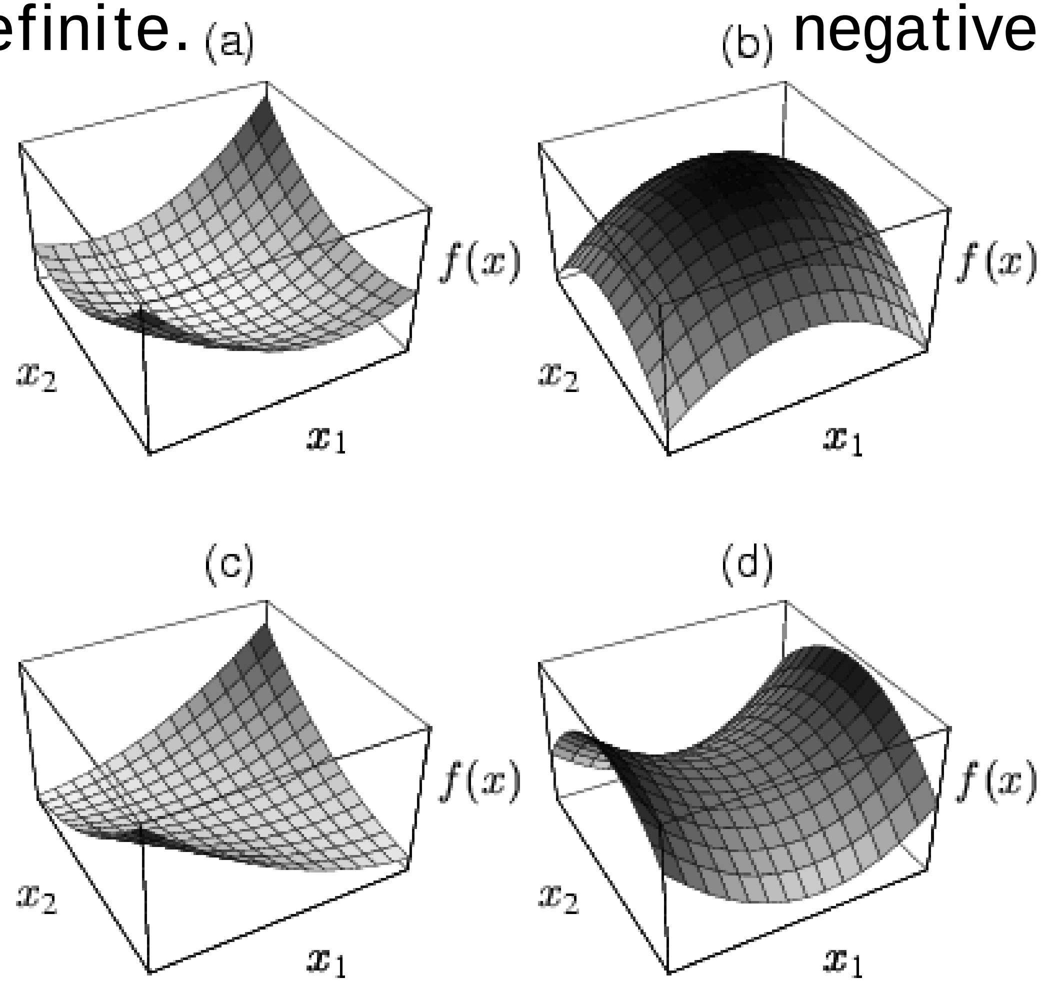
$$f(x) = \frac{1}{2}x^T A x - b^T x + c$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \qquad c = 0.$$

Quadratic functions

$\mathbf{A}$ is positive definite.
All eigenvalues
are positive.
Fall all x,
$x^T Ax>0$.

negative definite



A is singular

A is indefinite

Function minimization

Theorem 1.5. Necessary condition for a local minimizer.
If $\mathbf{x}^*$ is a local minimizer, then

$$\mathbf{g}^* \equiv \mathbf{F}'(\mathbf{x}^*) = \mathbf{0}\,.$$

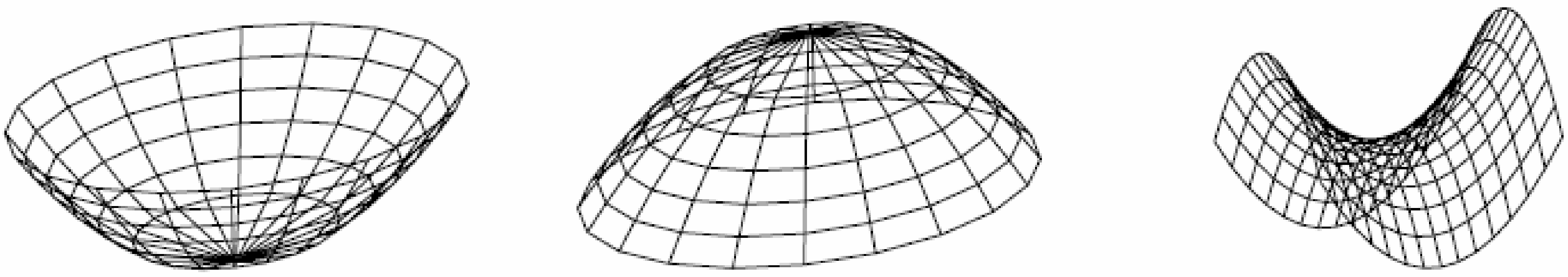
Definition 1.6. Stationary point. If

$$\mathbf{g}_s \equiv \mathbf{F}'(\mathbf{x}_s) = \mathbf{0}\,,$$

then $\mathbf{x}_s$ is said to be a *stationary point* for F.

$$F(\mathbf{x}_s+\mathbf{h}) = F(\mathbf{x}_s) + \tfrac{1}{2}\mathbf{h}^\top \mathbf{H}_s\,\mathbf{h} + O(\|\mathbf{h}\|^3)$$

$\mathbf{H}_s$ is *positive definite*



a) minimum *b) maximum* *c) saddle point*

Descent methods

$$\mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_2,\ \ldots\ ,\ \mathbf{x}_k\ \rightarrow\ \mathbf{x}^* \quad \text{for} \quad k \rightarrow \infty$$

1. Find a descent direction $\mathbf{h}_\mathrm{d}$
2. find a step length giving a good decrease in the F-value.

Algorithm Descent method

begin
  $k := 0;\ \ \mathbf{x} := \mathbf{x}_0;\ \ \mathit{found} := \mathbf{false}$ $\qquad\{\text{Starting point}\}$
  while $(\mathbf{not}\ \mathit{found})$ **and** $(k < k_{\max})$
    $\mathbf{h}_\mathrm{d} := \text{search\_direction}(\mathbf{x})$ $\qquad\{\text{From } \mathbf{x} \text{ and downhill}\}$
    if (no such $\mathbf{h}$ exists)
      $\mathit{found} := \mathbf{true}$ $\qquad\{\mathbf{x} \text{ is stationary}\}$
    else
      $\alpha := \text{step\_length}(\mathbf{x}, \mathbf{h}_\mathrm{d})$ $\qquad\{\text{from } \mathbf{x} \text{ in direction } \mathbf{h}_\mathrm{d}\}$
      $\mathbf{x} := \mathbf{x} + \alpha\mathbf{h}_\mathrm{d};\quad k := k+1$ $\qquad\{\text{next iterate}\}$
end

Descent direction

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

Definition Descent direction.

$\mathbf{h}$ is a descent direction for F at $\mathbf{x}$ if $\quad \mathbf{h}^\top\mathbf{F}'(\mathbf{x}) < 0$.

Steepest descent method

$$F(\mathbf{x}+\alpha\mathbf{h}) = F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) + O(\alpha^2)$$
$$\simeq F(\mathbf{x}) + \alpha\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) \quad \text{for } \alpha \text{ sufficiently small.}$$

$$\frac{F(\mathbf{x}) - F(\mathbf{x}+\alpha\mathbf{h})}{\alpha\|\mathbf{h}\|} = -\frac{1}{\|\mathbf{h}\|}\mathbf{h}^\top\mathbf{F}'(\mathbf{x}) = -\|\mathbf{F}'(\mathbf{x})\|\cos\theta$$

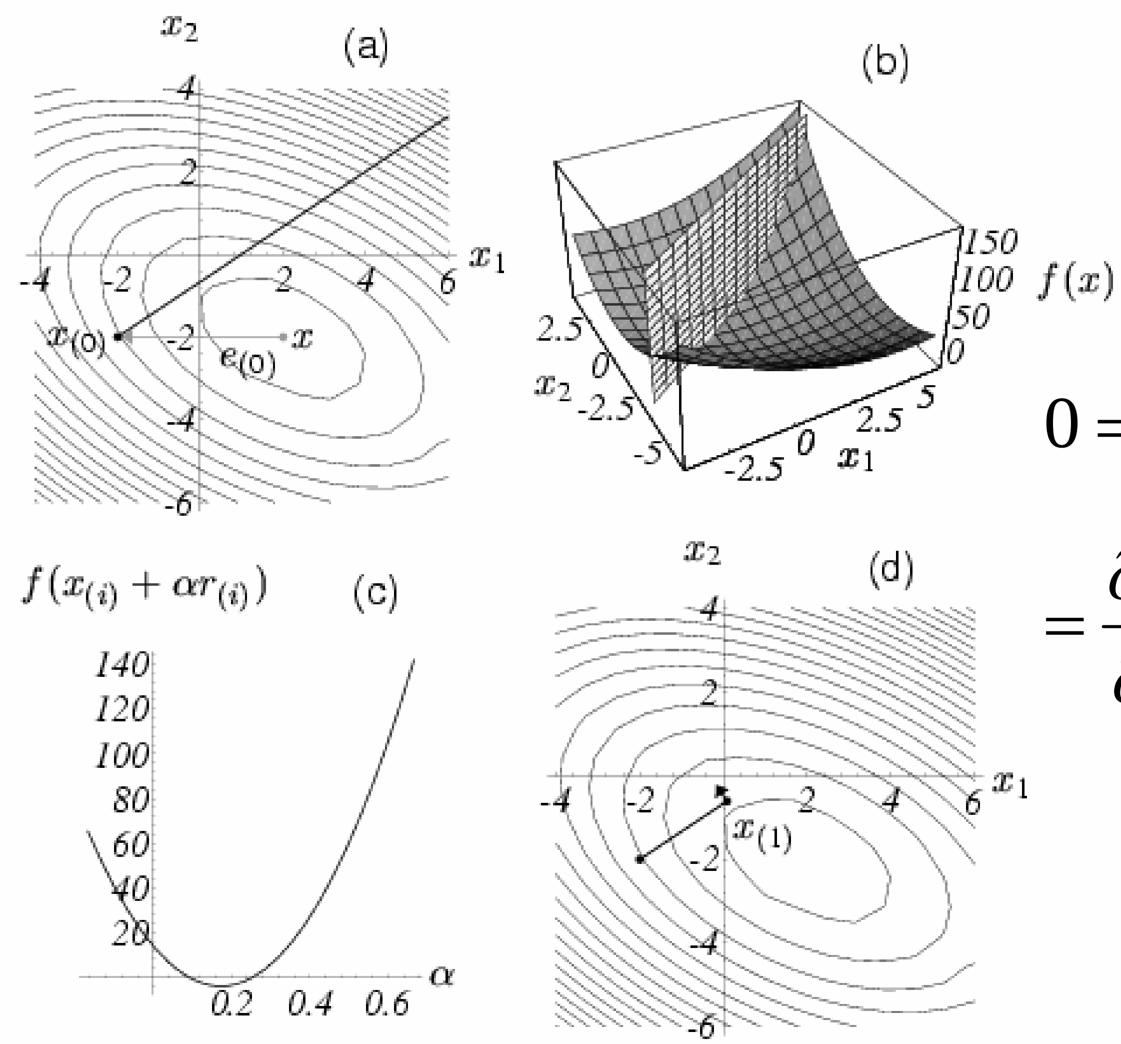
the decrease of ***F(x)*** per unit along h direction

$$\text{greatest gain rate if } \theta = \pi \;\rightarrow\; \mathbf{h}_{\mathrm{sd}} = -\mathbf{F}'(\mathbf{x})$$

h_{sd} is a descent direction because $h^T_{sd} F'(x) = - F'(x)^2 < 0$

Line search

$$\varphi(\alpha) = F(\mathbf{x}+\alpha\mathbf{h})\,, \quad \mathbf{x} \text{ and } \mathbf{h} \text{ fixed},\ \alpha \geq 0\,.$$



Find α so that

$$\varphi(\alpha) = \mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})$$

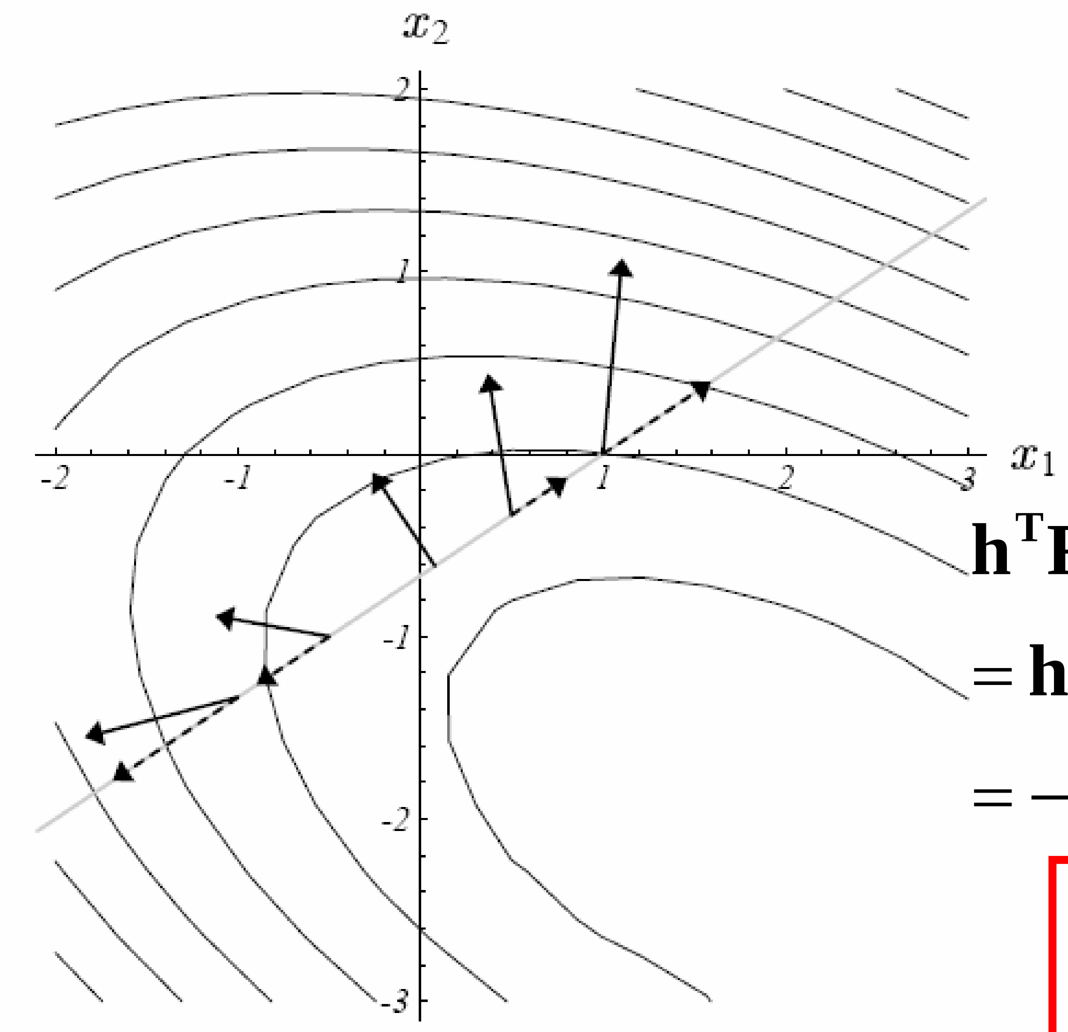
is minumum

$$0 = \frac{\partial\varphi(\alpha)}{\partial\alpha} = \frac{\partial\mathbf{F}(\mathbf{x}_0 + \alpha\mathbf{h})}{\partial\alpha}$$

$$= \frac{\partial\mathbf{F}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\alpha} = \mathbf{h}^T\mathbf{F}'(\mathbf{x}_0 + \alpha\mathbf{h})$$

$$\mathbf{h} = -\mathbf{F}'(\mathbf{x}_0)$$

Line search



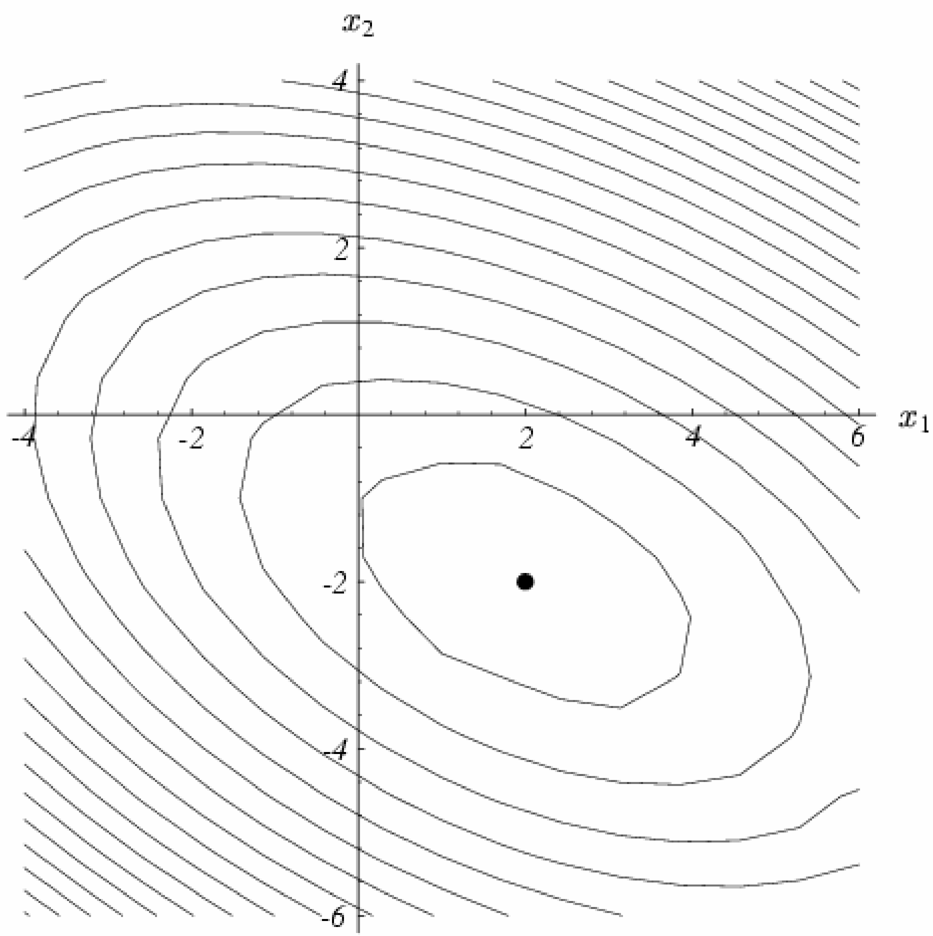
$$\mathbf{h}^{\mathrm{T}}\mathbf{F}'(\mathbf{x}_0+\alpha\mathbf{h})=0$$

$$\mathbf{h}=-\mathbf{F}'(\mathbf{x}_0)$$

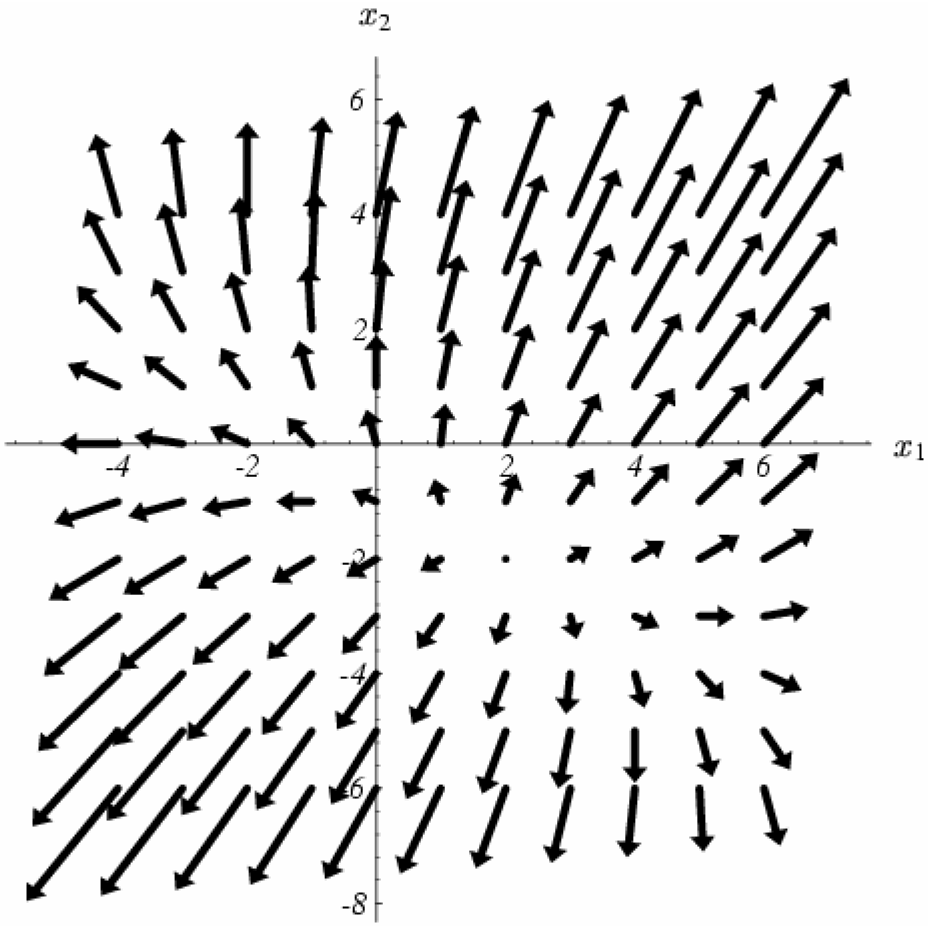
$$\mathbf{h}^{\mathrm{T}}\mathbf{F}'(\mathbf{x}_0+\alpha\mathbf{h})$$
$$=\mathbf{h}^{\mathrm{T}}\left(\mathbf{F}'(\mathbf{x}_0)+\alpha\mathbf{F}''(\mathbf{x}_0)^{\mathrm{T}}\mathbf{h}\right)$$
$$=-\mathbf{h}^{\mathrm{T}}\mathbf{h}+\alpha\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h}=0$$

$$\boxed{\alpha=\frac{\mathbf{h}^{\mathrm{T}}\mathbf{h}}{\mathbf{h}^{\mathrm{T}}\mathbf{H}\mathbf{h}}}$$

Steepest descent method

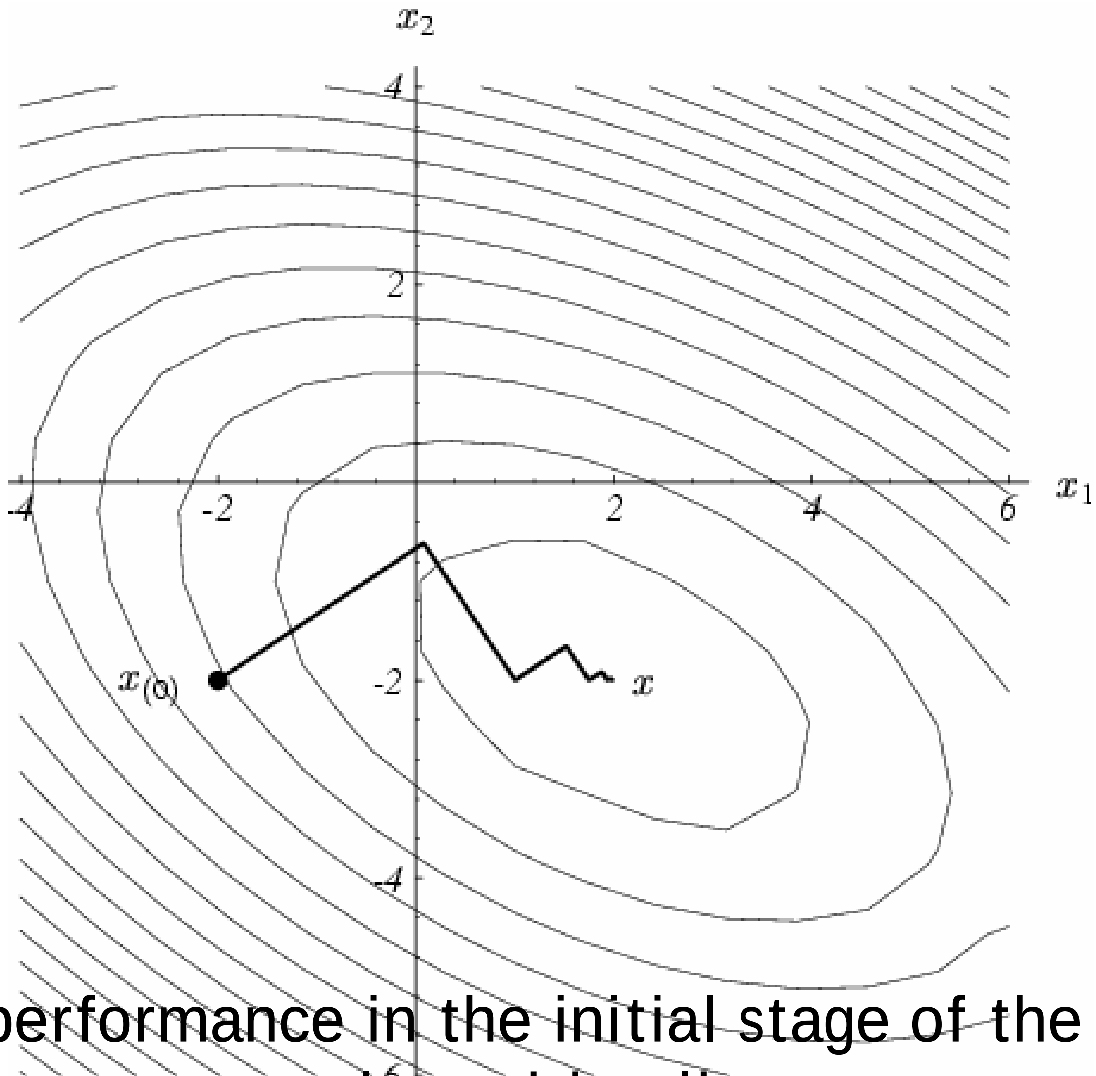


isocontour



gradient

Steepest descent method



It has good performance in the initial stage of the iterative process. Converge very slow with a linear rate.

Newton's method

$\mathbf{x}^*$ is a stationary point $\rightarrow$ it satisfies $\mathbf{F}'(\mathbf{x}^*) = \mathbf{0}$.

$$\begin{aligned} \mathbf{F}'(\mathbf{x}+\mathbf{h}) &= \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \\ &\simeq \mathbf{F}'(\mathbf{x}) + \mathbf{F}''(\mathbf{x})\mathbf{h} \quad \text{for } \|\mathbf{h}\| \text{ sufficiently small} \end{aligned}$$

$$\rightarrow \quad \mathbf{H}\,\mathbf{h}_n = -\mathbf{F}'(\mathbf{x}) \quad \text{with } \mathbf{H} = \mathbf{F}''(\mathbf{x})$$

$$\mathbf{x} := \mathbf{x} + \mathbf{h}_n$$

Suppose that $\mathbf{H}$ is positive definite

$\rightarrow \mathbf{u}^\top \mathbf{H}\,\mathbf{u} > 0$ for all nonzero $\mathbf{u}$.

$\rightarrow 0 < \mathbf{h}_n^\top \mathbf{H}\,\mathbf{h}_n = -\mathbf{h}_n^\top \mathbf{F}'(\mathbf{x})$ $\;\mathbf{h}_n$ is a descent direction

It has good performance in the final stage of the iterative process, where x is close to x*.

Hybrid method

$$
\begin{aligned}
&\textbf{if}\ \ \mathbf{F}''(\mathbf{x}) \text{ is positive definite} \\
&\quad \mathbf{h} := \mathbf{h}_{\mathrm{n}} \\
&\textbf{else} \\
&\quad \mathbf{h} := \mathbf{h}_{\mathrm{sd}} \\
&\mathbf{x} := \mathbf{x} + \alpha\mathbf{h}
\end{aligned}
$$

This needs to calculate second-order derivative which might not be available.