Geometry and Transformations

Digital Image Synthesis

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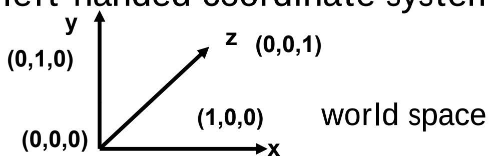
with slides by Pat Hanrahan

Geometric classes

- Representation and operations for the basic mathematical constructs like points, vectors and rays.
- Actual scene geometry such as triangles and spheres are defined in the “Shapes” chapter.
- core/geometry.* and core/transform.*
- Purposes of learning this chapter
  - Get used to the style of learning by tracing source code
  - Get familiar to the basic geometry utilities because you will use them intensively later on

Coordinate system

- Points, vectors and normals are represented with three floating-point coordinate values: x, y, z defined under a coordinate system.
- A coordinate system is defined by an origin p_o and a frame (linearly independent vectors v_i).
- A vector $v= s_1 v_1 + \ldots + s_n v_n$ represents a direction, while a point $p= p_o + s_1 v_1 + \ldots + s_n v_n$ represents a position. They are not freely interchangeable.
- pbrt uses left-handed coordinate system.



Vectors

```
class Vector {
  public:
     <Vector Public Methods>
     float x, y, z;
}
```

no need to use selector (getX) and mutator (setX) because the design gains nothing and adds bulk to its usage

Provided operations: **`Vector u, v; float a;`**

```
v+u, v-u, v+=u, v-=u
-v
(v==u)
a*v, v*=a, v/a, v/=a
a=v[i], v[i]=a
```

Dot and cross product

`Dot(v, u)` $v \cdot u = \|v\|\|u\|\cos\theta$

`AbsDot(v, u)`

`Cross(v, u)`

$$\|v \times u\| = \|v\|\|u\|\sin\theta$$

Vectors v, u, v×u form a frame

$$(v \times u)_x = v_y u_z - v_z u_y$$
$$(v \times u)_y = v_z u_x - v_x u_z$$
$$(v \times u)_z = v_x u_y - v_y u_z$$

Normalization

`a=LengthSquared(v)`

`a=Length(v)`

`u=Normalize(v)` *return a vector, does not normalize in place*

Coordinate system from a vector

Construct a local coordinate system from a vector.

```
inline void CoordinateSystem(const Vector &v1,
                       Vector *v2, Vector *v3)
{
  if (fabsf(v1.x) > fabsf(v1.y)) {
    float invLen = 1.f/sqrtf(v1.x*v1.x + v1.z*v1.z);
    *v2 = Vector(-v1.z * invLen, 0.f, v1.x * invLen);
  }
  else {
    float invLen = 1.f/sqrtf(v1.y*v1.y + v1.z*v1.z);
    *v2 = Vector(0.f, v1.z * invLen, -v1.y * invLen);
  }
  *v3 = Cross(v1, *v2);
}
```

Points

Points are different from vectors; given a coordinate system (p_0, v_1, v_2, v_3), a point p and a vector v with the same (x,y,z) essentially means

$p=(x,y,z,1)[v_1\ v_2\ v_3\ p_0]^T$

$v=(x,y,z,0)[v_1\ v_2\ v_3\ p_0]^T$

`explicit Vector(const Point &p);`

You have to convert a point to a vector explicitly (i.e. you know what you are doing).

☒ `Vector v=p;`

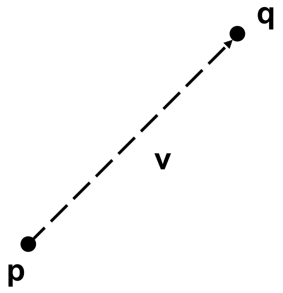
☑ `Vector v=Vector(p);`

Operations for points

```
Vector v; Point p, q, r; float a;

q=p+v;
q=p-v;
v=q-p;

r=p+q;
a*p; p/a;
```

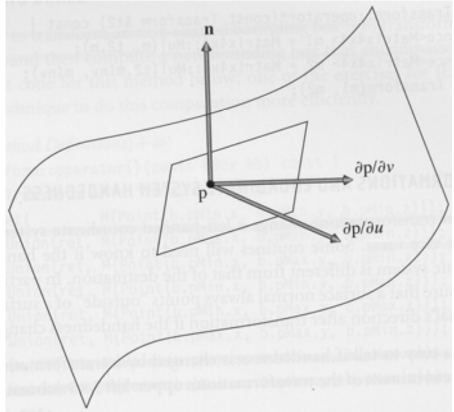


(This is only for the operation αp+βq.)

```
Distance(p,q);
DistanceSquared(p,q);
```

Normals

- A *surface normal* (or just *normal*) is a vector that is perpendicular to a surface at a particular position.



Normals

- Different than vectors in some situations, particularly when applying transformations.
- Implementation similar to **`Vector`**, but a normal cannot be added to a point and one cannot take the cross product of two normals.
- **`Normal`** is not necessarily normalized.
- Only explicit conversion between **`Vector`** and **`Normal`**.

Rays

```
class Ray {
public:
    Point o;
    Vector d;
    mutable float mint, maxt;
    float time;
    int depth;
};
```

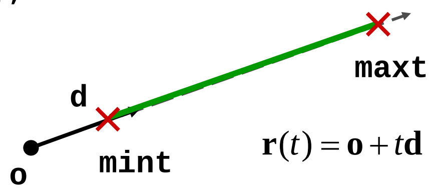
(They may be changed even if **`Ray`** is **`const`**. This ensures that o and d are not modified, but mint and maxt can be.)

Initialized as **`RAY_EPSILON`** to avoid self intersection.

`time`: (for motion blur)

`depth`: (how many times the ray has bounced)

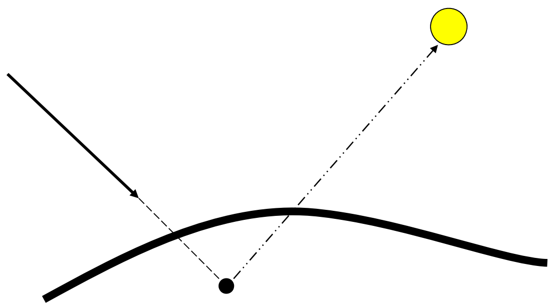
```
Ray r(o, d);
Point p=r(t);
```



$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$ $\qquad 0 \le t \le \infty$

Rays

```
Ray(): mint(RAY_EPSILON), maxt(INFINITY),
  time(0.f) {}
```

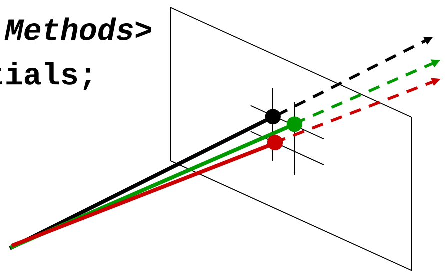


The reason why we need epsilon. Unfortunately, there is not a universal epsilon that works for all scenes.

Ray differentials

- Subclass of **Ray** with two auxiliary rays. Used to estimate the projected area for a small part of a scene and for antialiasing in **Texture**.

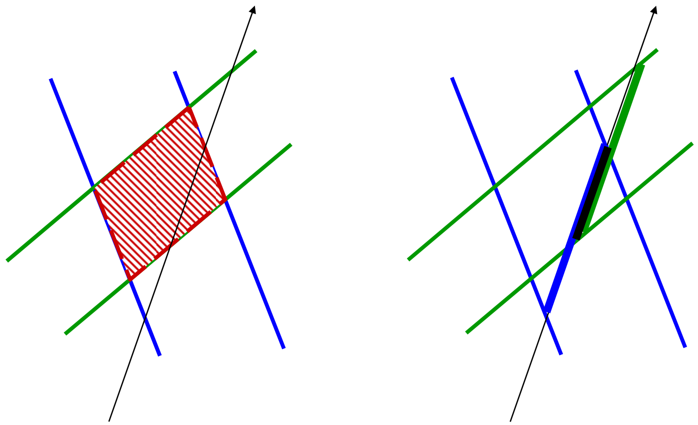
```
class RayDifferential : public Ray {
public:
  <RayDifferential Methods>
  bool hasDifferentials;
  Ray rx, ry;
};
```



Bounding boxes

- To avoid intersection test inside a volume if the ray doesn't hit the *bounding volume*.
- Benefits depends on the expense of testing volume *v.s.* objects inside and the tightness of the bounding volume.
- Popular bounding volume, sphere, axis-aligned bounding box (AABB), oriented bounding box (OBB) and slab.

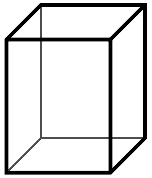
Bounding volume (slab)



Bounding boxes

```
class BBox {
public:
  <BBox Public Methods>
  Point pMin, pMax;
}
```

two options of storing



```
Point p,q; BBox b; float delta; bool s;
b = BBox(p,q)   // no order for p, q
b = Union(b,p);  b = Union(b,b2); b = b.Expand(delta)
s = b.Overlaps(b2);  s = b.Inside(p)
Volume(b)
Vector v=Offset(Point &p)
```
relative position of p inside the box (pMax=(1,1,1) and pMin=(0,0,0))

`p=Lerp(tx, ty, tz)` *the point with relative position (tx, ty, tz)*

`b.MaximumExtent()` *which axis is the longest; for building kd-tree*

`b.BoundingSphere(c, r)` *for generating samples*

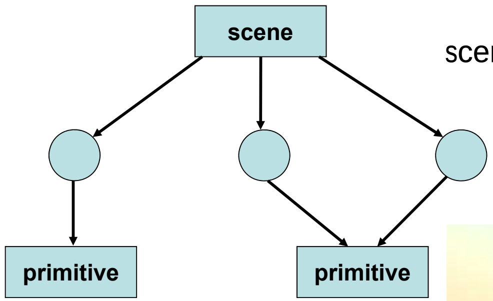
Transformations

- p’=T(p); v’=T(v)
- Only supports transforms with the following properties:
  - Linear: $T(a\mathbf{v}+b\mathbf{u})=aT(\mathbf{v})+bT(\mathbf{u})$
  - Continuous: T maps the neighbors of p to ones of p’
  - Ont-to-one and invertible: T maps p to single p’ and T^{-1} exists
- Represented with a 4x4 matrix; homogeneous coordinates are used implicitly
- Can be applied to points, vectors and normals
- Simplify implementations (e.g. cameras and shapes)

Transformations

- More convenient, instancing

scene graph



Transformations

```
class Transform {
...
private:
  Reference<Matrix4x4> m, mInv;
}
```
save space, but can’t be modified after construction
Usually not a problem because transforms are pre-specified in the scene file and won’t be changed during rendering.

```
Transform() {m = mInv = new Matrix4x4; }
Transform(float mat[4][4]);
Transform(const Reference<Matrix4x4> &mat);
Transform(const Reference<Matrix4x4> &mat,
          const Reference<Matrix4x4> &minv);
```
A better way to initialize

Transformations

- `Translate(Vector(dx,dy,dz))`
- `Scale(sx,sy,sz)`
- `RotateX(a)`

$$T(dx,dy,dz)=\begin{pmatrix}1&0&0&dx\\0&1&0&dy\\0&0&1&dz\\0&0&0&1\end{pmatrix}\qquad R_x(\theta)=\begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}$$

$$S(sx,sy,sz)=\begin{pmatrix}sx&0&0&0\\0&sy&0&0\\0&0&sz&0\\0&0&0&1\end{pmatrix}$$

$$R_x(\theta)^{-1}=R_x(\theta)^{\mathrm{T}}$$

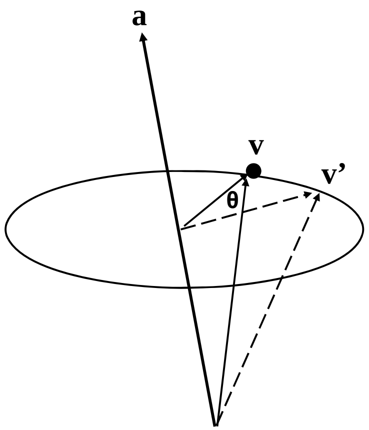
because R is orthogonal

Example for creating common transforms

```
Transform Translate(const Vector &delta) {
  Matrix4x4 *m, *minv;
  m = new Matrix4x4(1, 0, 0, delta.x,
                    0, 1, 0, delta.y,
                    0, 0, 1, delta.z,
                    0, 0, 0,       1);
  minv = new Matrix4x4(1, 0, 0, -delta.x,
                       0, 1, 0, -delta.y,
                       0, 0, 1, -delta.z,
                       0, 0, 0,        1);
  return Transform(m, minv);
}
```

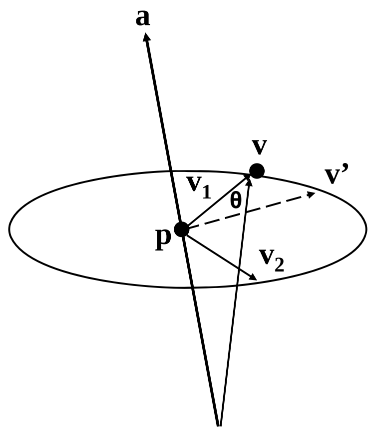
Rotation around an arbitrary axis

- Rotate(theta, axis) *axis is normalized*



Rotation around an arbitrary axis

- Rotate(theta, axis) *axis is normalized*



$$\mathbf{p}=\mathbf{a}(\mathbf{v}\cdot\mathbf{a})$$

$$\mathbf{v}_1=\mathbf{v}-\mathbf{p}$$

$$\mathbf{v}_2=\mathbf{a}\times\mathbf{v}_1 \qquad |\mathbf{v}_2|=|\mathbf{v}_1|$$

$$\mathbf{v}'=p+\mathbf{v}_1\cos\theta+\mathbf{v}_2\sin\theta$$

Rotation around an arbitrary axis

- Rotate(theta, axis) *axis is normalized*

$$\mathbf{p} = \mathbf{a}(\mathbf{v}\cdot\mathbf{a})$$

M v

$$\mathbf{v}_1 = \mathbf{v} - \mathbf{p}$$

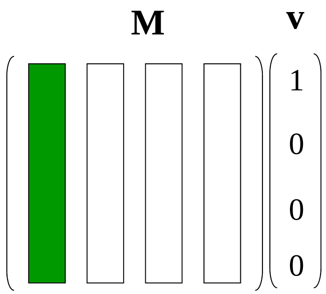
$$\mathbf{v}_2 = \mathbf{a} \times \mathbf{v}_1 \qquad |\mathbf{v}_2| = |\mathbf{v}_1|$$

$$= \quad \mathbf{v}' = p + \mathbf{v}_1 \cos\theta + \mathbf{v}_2 \sin\theta$$

Rotation around an arbitrary axis

```
m[0][0]=a.x*a.x  + (1.f-a.x*a.x)*c;
m[1][0]=a.x*a.y*(1.f-c) + a.z*s;
m[2][0]=a.x*a.z*(1.f-c) - a.y*s;
```

$$\mathbf{p} = \mathbf{a}(\mathbf{v}\cdot\mathbf{a})$$



$$\mathbf{v}_1 = \mathbf{v} - \mathbf{p}$$

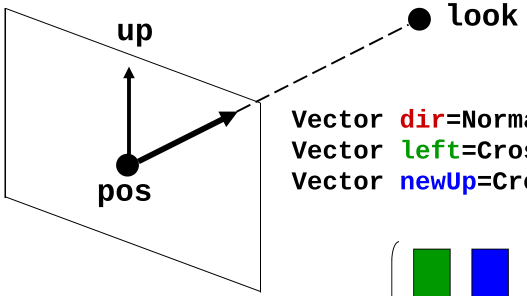
$$\mathbf{v}_2 = \mathbf{a} \times \mathbf{v}_1 \qquad |\mathbf{v}_2| = |\mathbf{v}_1|$$

$$= \quad \mathbf{v}' = p + \mathbf{v}_1 \cos\theta + \mathbf{v}_2 \sin\theta$$

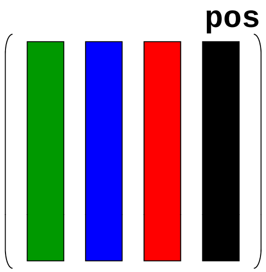
Look-at

- `LookAt(Point &pos, Point look, Vector &up)`

up is not necessarily perpendicular to dir




```
Vector dir=Normalize(look-pos);
Vector left=Cross(Normalize(up),dir);
Vector newUp=Cross(dir, left);
```

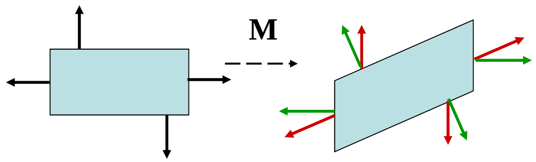


Applying transformations

- **Point**: `q=T(p), T(p,&q)`
  use homogeneous coordinates implicitly

Point: (p, 1)
Vector: (v, 0)

- **Vector**: `u=T(v), T(u, &v)`
- **Normal**: treated differently than vectors because of anisotropic transformations



$$\mathbf{n}\cdot\mathbf{t} = \mathbf{n}^T\mathbf{t} = 0$$

$$(\mathbf{n}')^T\mathbf{t}' = 0$$

$$(\mathbf{S}\mathbf{n})^T\mathbf{M}\mathbf{t} = 0$$

$$\mathbf{n}^T\mathbf{S}^T\mathbf{M}\mathbf{t} = 0$$

$$\mathbf{S}^T\mathbf{M} = \mathbf{I}$$

$$\boxed{\mathbf{S} = \mathbf{M}^{-T}}$$

- **Transform** should keep its inverse
- For orthonormal matrix, **S=M**

Applying transformations

- **BBox**: transforms its eight corners and expand to include all eight points.

```
BBox Transform::operator()(const BBox &b) const {
  const Transform &M = *this;
  BBox ret(        M(Point(b.pMin.x, b.pMin.y, b.pMin.z)));
  ret = Union(ret,M(Point(b.pMax.x, b.pMin.y, b.pMin.z)));
  ret = Union(ret,M(Point(b.pMin.x, b.pMax.y, b.pMin.z)));
  ret = Union(ret,M(Point(b.pMin.x, b.pMin.y, b.pMax.z)));
  ret = Union(ret,M(Point(b.pMin.x, b.pMax.y, b.pMax.z)));
  ret = Union(ret,M(Point(b.pMax.x, b.pMax.y, b.pMin.z)));
  ret = Union(ret,M(Point(b.pMax.x, b.pMin.y, b.pMax.z)));
  ret = Union(ret,M(Point(b.pMax.x, b.pMax.y, b.pMax.z)));
  return ret;
}
```

Differential geometry

- **DifferentialGeometry**: a self-contained representation for a particular point on a surface so that all the other operations in pbrt can be executed without referring to the original shape. It contains
- Position
- Parameterization (u,v)
- Parametric derivatives (dp/du, dp/dv)
- Surface normal (derived from (dp/du)x(dp/dv))
- Derivatives of normals
- Pointer to shape