Monte Carlo Integration

Digital Image Synthesis

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with slides by Pat Hanrahan and Torsten Moller

Introduction

- The integral equations generally don't have analytic solutions, so we must turn to numerical methods.
- Standard methods like Trapezoidal integration or Gaussian quadrature are not effective for high-dimensional and discontinuous integrals.

$$L_o(\mathrm{p},\omega_o) = L_e(\mathrm{p},\omega_o) + \int_{S^2} f(\mathrm{p},\omega_o,\omega_i) L_i(\mathrm{p},\omega_i) |\cos\theta_i| d\omega_i$$

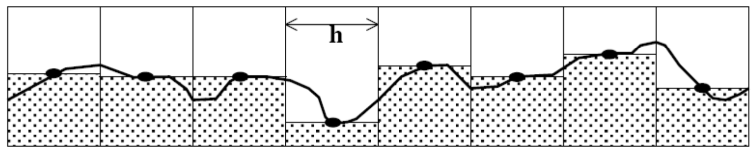
Numerical quadrature

- Suppose we want to calculate $I = \int_a^b f(x)dx$, but can't solve it analytically. The approximations through quadrature rules have the form

$$\hat{I} = \sum_{i=1}^{n} w_i f(x_i)$$

which is essentially the weighted sum of samples of the function at various points

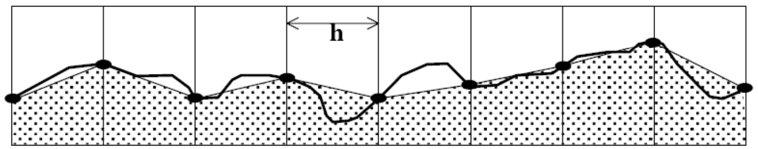
Midpoint rule



$$\hat{I} = h\sum_{i=1}^{n} f(a + (i - \frac{1}{2})h)$$
$$= h\left[f(a+\frac{h}{2}) + f(a+\frac{3h}{2}) + \cdots + f(b-\frac{h}{2})\right]$$

convergence $\hat{I} - I = -\frac{(b-a)^3}{24n^2} f''(\xi) = O(n^{-2})$

Trapezoid rule



$$\hat{I} = \sum_{i=1}^{n} \frac{h}{2}[f(a+(i-1)h) + f(a+ih)]$$
$$= h\left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \cdots + f(b-h) + \frac{1}{2}f(b)\right]$$

convergence $\hat{I} - I = \frac{(b-a)^3}{12n^2} f''(\xi^*) = O(n^{-2})$

Simpson's rule

- Similar to trapezoid but using a quadratic polynomial approximation

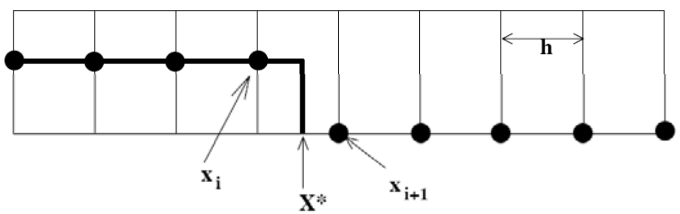
$$\hat{I} = h\left[\frac{1}{3}f(a) + \frac{4}{3}f(a+h) + \frac{2}{3}f(a+2h) + \frac{4}{3}f(a+3h) + \frac{2}{3}f(a+4h) + \cdots + \frac{4}{3}f(b-h) + \frac{1}{3}f(b)\right]$$

convergence $|\hat{I} - I| = \frac{(b-a)^5}{180(2n)^4} f^{(4)}(\xi) = O(n^{-4})$

assuming f has a continuous fourth derivative.

Curse of dimensionality and discontinuity

- For an sd function f, $\hat{I} = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_s=1}^{n} w_{i_1}w_{i_2}\cdots w_{i_s} f(x_{i_1}, x_{i_2}, \ldots, x_{i_s})$
- If the 1d rule has a convergence rate of $O(n^{-r})$, the sd rule would require a much larger number (n^s) of samples to work as well as the 1d one. Thus, the convergence rate is only $O(n^{-r/s})$.
- If f is discontinuous, convergence is $O(n^{-1/s})$ for sd.



Randomized algorithms

- *Las Vegas* v.s. *Monte Carlo*
- *Las Vegas:* always gives the right answer by using randomness.
- *Monte Carlo*: gives the right answer *on the average*. Results depend on random numbers used, but statistically likely to be close to the right answer.

Monte Carlo integration

- Monte Carlo integration: uses sampling to estimate the values of integrals. It only requires to be able to evaluate the integrand at arbitrary points, making it *easy to implement* and *applicable to many problems*.
- If n samples are used, its converges at the rate of $O(n^{-1/2})$. That is, to cut the error in half, it is necessary to evaluate four times as many samples.
- Images by Monte Carlo methods are often noisy. Most current methods try to reduce noise.

Monte Carlo methods

- Advantages
  - Easy to implement
  - Easy to think about (but be careful of statistical bias)
  - Robust when used with complex integrands and domains (shapes, lights, …)
  - Efficient for high dimensional integrals
- Disadvantages
  - Noisy
  - Slow (many samples needed for convergence)

Basic concepts

- X is a random variable
- Applying a function to a random variable gives another random variable, $Y=f(X)$.
- CDF (cumulative distribution function)

$$P(x) \equiv \Pr\{X \leq x\}$$

- PDF (probability density function): nonnegative, sum to 1

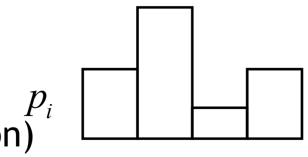
$$p(x) \equiv \frac{dP(x)}{dx}$$

- canonical uniform random variable ξ (provided by standard library and easy to transform to other distributions)

Discrete probability distributions

- Discrete events $\boldsymbol{X_i}$ with probability $\boldsymbol{p_i}$

$$p_i \geq 0 \qquad \sum_{i=1}^{n} p_i = 1$$



- Cumulative PDF (distribution)

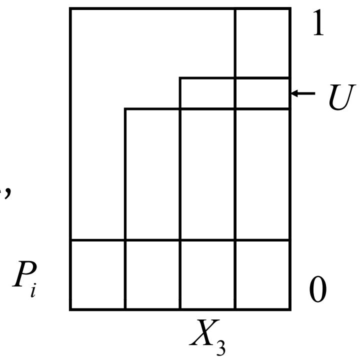
$$P_j = \sum_{i=1}^{j} p_i$$

- Construction of samples:

  To randomly select an event,

  Select $\boldsymbol{X_i}$ if $P_{i-1} < U \leq P_i$

  ↑ **Uniform random variable**



Continuous probability distributions

- PDF $p(x)$

$$p(x) \geq 0$$

- CDF $P(x)$

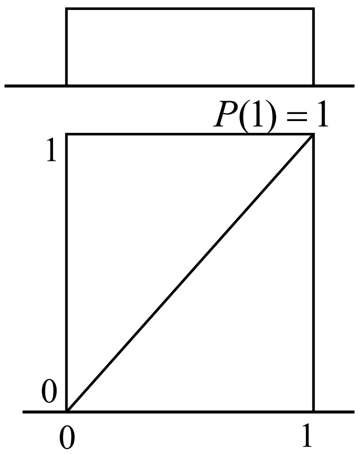
$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x)$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$

Uniform



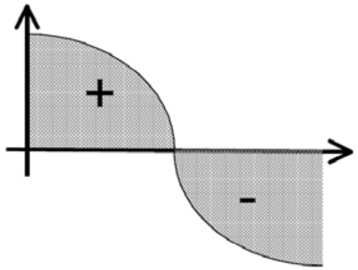
Expected values

- Average value of a function *f(x)* over some distribution of values *p(x)* over its domain *D*

$$E_p\left[f(x)\right] = \int_D f(x)p(x)dx$$

- Example: *cos* function over [0, π], *p* is uniform

$$E_p\left[\cos(x)\right] = \int_0^\pi \cos x \frac{1}{\pi} dx = 0$$



Variance

- Expected deviation from the expected value
- Fundamental concept of quantifying the error in Monte Carlo methods

$$V[f(x)] = E\left[\left(f(x) - E\left[f(x)\right]\right)^2\right]$$

Properties

$$E\left[af(x)\right] = aE\left[f(x)\right]$$

$$E\left[\sum_i f(X_i)\right] = \sum_i E\left[f(X_i)\right]$$

$$V\left[af(x)\right] = a^2 V\left[f(x)\right]$$

$$\longrightarrow V\left[f(x)\right] = E\left[\left(f(x)\right)^2\right] - E\left[f(x)\right]^2$$

Monte Carlo estimator

- Assume that we want to evaluate the integral of $f(x)$ over $[a,b]$ $\int_a^b f(x)dx$
- Given a uniform random variable X_i over $[a,b]$, Monte Carlo estimator

$$F_N = \frac{b-a}{N}\sum_{i=1}^{N} f(X_i)$$

says that the expected value $E[F_N]$ of the estimator F_N equals the integral

$$E[F_N] = E\left[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\right]$$
$$= \frac{b-a}{N}\sum_{i=1}^{N} E[f(X_i)]$$
$$= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)p(x)dx$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_a^b f(x)dx$$
$$= \int_a^b f(x)dx$$

General Monte Carlo estimator

- Given a random variable X drawn from an arbitrary PDF $p(x)$, then the estimator is

$$F_N = \frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$

$$E[F_N] = E\left[\frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right]$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_a^b \frac{f(x)}{p(x)} p(x)dx$$
$$= \int_a^b f(x)dx$$

- Although the converge rate of MC estimator is $O(N^{1/2})$, slower than other integral methods, its converge rate is independent of the dimension, making it the only practical method for high dimensional integral

Convergence of Monte Carlo

- Chebyshev's inequality: let X be a random variable with expected value μ and variance σ^2. For any real number k>0,

$$\Pr\{|X-\mu| \ge k\sigma\} \le \frac{1}{k^2}$$

- For example, for k=$\sqrt{2}$, it shows that at least half of the value lie in the interval $(\mu - \sqrt{2}\sigma, \mu + \sqrt{2}\sigma)$
- Let $Y_i = f(X_i)/p(X_i)$, the MC estimate F_N becomes

$$F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

Convergence of Monte Carlo

- According to Chebyshev's inequality,

$$\Pr\left\{|F_N - E[F_N]| \ge \left(\frac{V[F_N]}{\delta}\right)^{1/2}\right\} \le \delta$$

$$V[F_N] = V\left[\frac{1}{N}\sum_{i=1}^{N} Y_i\right] = \frac{1}{N^2} V\left[\sum_{i=1}^{N} Y_i\right] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N} V[Y]$$

- Plugging into Chebyshev's inequality,

$$\Pr\left\{|F_N - I| \ge \frac{1}{\sqrt{N}}\left(\frac{V[Y]}{\delta}\right)^{1/2}\right\} \le \delta$$

So, for a fixed threshold, the error decreases at the rate $N^{-1/2}$.

Properties of estimators

- An estimator F_N is called unbiased if for all N

$$E[F_N] = Q$$

That is, the expected value is independent of N.

- Otherwise, the bias of the estimator is defined as

$$\beta[F_N] = E[F_N] - Q$$

- If the bias goes to zero as N increases, the estimator is called consistent

$$\lim_{N\to\infty} \beta[F_N] = 0$$

$$\lim_{N\to\infty} E[F_N] = Q$$

Example of a biased consistent estimator

- Suppose we are doing antialiasing on a 1d pixel, to determine the pixel value, we need to evaluate $I = \int_0^1 w(x) f(x) dx$, where $w(x)$ is the filter function with $\int_0^1 w(x) dx = 1$
- A common way to evaluate this is

$$F_N = \frac{\sum_{i=1}^N w(X_i) f(X_i)}{\sum_{i=1}^N w(X_i)}$$

- When N=1, we have

$$E[F_1] = E\left[\frac{w(X_1) f(X_1)}{w(X_1)}\right] = E[f(X_1)] = \int_0^1 f(x) dx \neq I$$

Example of a biased consistent estimator

- When N=2, we have

$$E[F_2] = \int_0^1 \int_0^1 \frac{w(x_1) f(x_1) + w(x_2) f(x_2)}{w(x_1) + w(x_2)} dx_1 dx_2 \neq I$$

- However, when N is very large, the bias approaches to zero

$$F_N = \frac{\frac{1}{N}\sum_{i=1}^N w(X_i) f(X_i)}{\frac{1}{N}\sum_{i=1}^N w(X_i)}$$

$$\lim_{N\to\infty} E[F_N] = \frac{\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N w(X_i) f(X_i)}{\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N w(X_i)} = \frac{\int_0^1 w(x) f(x) dx}{\int_0^1 w(x) dx} = \int_0^1 w(x) f(x) dx = I$$

Choosing samples

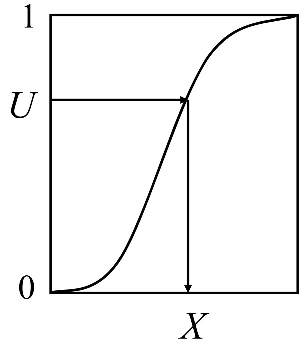
- $F_N = \frac{1}{N}\sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$
- Carefully choosing the PDF from which samples are drawn is an important technique to reduce variance. We want the f/p to have a low variance. Hence, it is necessary to be able to draw samples from the chosen PDF.
- How to sample an arbitrary distribution from a variable of uniform distribution?
  - Inversion
  - Rejection
  - Transform

Inversion method

- Cumulative probability distribution function

$$P(x) = \Pr(X < x)$$

- Construction of samples
  Solve for $\boldsymbol{X=P^{-1}(U)}$
- Must know:
  1. The integral of $\boldsymbol{p(x)}$
  2. The inverse function $\boldsymbol{P^{-1}(x)}$



Proof for the inversion method

- Let U be an uniform random variable and its CDF is $P_u(x)=x$. We will show that $Y=P^{-1}(U)$ has the CDF $P(x)$.

$$\Pr\{Y \le x\} = \Pr\{P^{-1}(U) \le x\} = \Pr\{U \le P(x)\} = P_u(P(x)) = P(x)$$

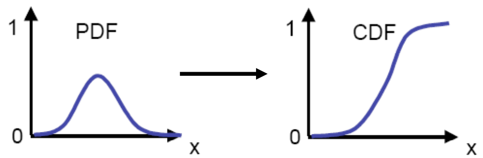
because P is monotonic,

$$P(x_1) \le P(x_2) \text{ if } x_1 \le x_2$$

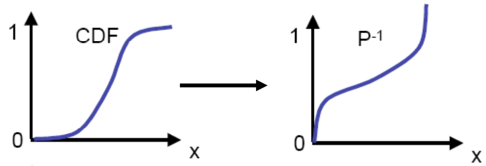
Thus, Y's CDF is exactly $P(x)$.

Inversion method

- Compute CDF P(x)



- Compute $P^{-1}(x)$



- Obtain ξ
- Compute $X_i=P^{-1}(\xi)$

Example: power function

It is used in sampling Blinn's microfacet model.

- Assume

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick (It only works for sampling power distribution)

$$Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

Example: exponential distribution

$p(x) = ce^{-ax}$ useful for rendering participating media.

$$\int_0^\infty ce^{-ax}dx = 1 \longrightarrow c = a$$

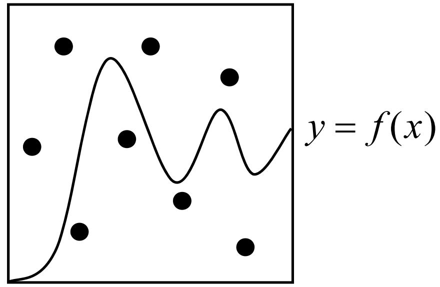
- Compute CDF P(x) $P(x) = \int_0^x ae^{-as}ds = 1 - e^{-ax}$
- Compute $P^{-1}(x)$ $P^{-1}(x) = -\frac{1}{a}\ln(1-x)$
- Obtain ξ
- Compute $X_i=P^{-1}(\xi)$ $X = -\frac{1}{a}\ln(1-\xi) = -\frac{1}{a}\ln\xi$

Rejection method

- Sometimes, we can't integrate into CDF or invert CDF

$$I = \int_0^1 f(x)\,dx$$

$$= \iint_{y<f(x)} dx\,dy$$



- Algorithm

  Pick U_1 and U_2

  Accept U_1 if $U_2 < f(U_1)$

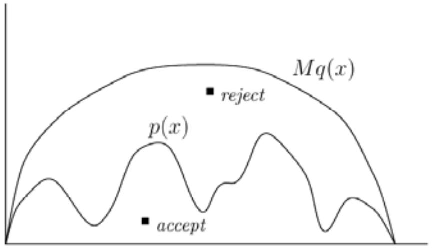
- Wasteful? **Efficiency = Area / Area of rectangle**

Rejection method

- Rejection method is a dart-throwing method without performing integration and inversion.

1. Find $q(x)$ so that $p(x)<Mq(x)$
2. Dart throwing
   a. Choose a pair (X, ξ), where X is sampled from $q(x)$
   b. If $(\xi<p(X)/Mq(X))$ return X

- Equivalently, we pick point $(X, \xi Mq(X))$. If it lies beneath $p(X)$ then we are fine.



Why it works

- For each iteration, we generate X_i from q. The sample is returned if $\xi<p(X)/Mq(X)$, which happens with probability $p(X)/Mq(X)$.
- So, the probability to return x is

$$q(x)\frac{p(x)}{Mq(x)} = \frac{p(x)}{M}$$

whose integral is 1/M

- Thus, when a sample is returned (with probability 1/M), X_i is distributed according to $p(x)$.

Example: sampling a unit disk

```
void RejectionSampleDisk(float *x, float *y) {
  float sx, sy;
  do {
    sx = 1.f -2.f * RandomFloat();
    sy = 1.f -2.f * RandomFloat();
  } while (sx*sx + sy*sy > 1.f)
  *x = sx;  *y = sy;
}
```

π/4～78.5% good samples, gets worse in higher dimensions, for example, for sphere, π/6～52.4%

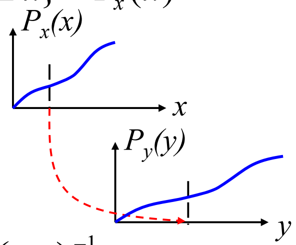
Transformation of variables

- Given a random variable X from distribution $p_x(x)$ to a random variable $Y=y(X)$, where y is one-to-one, i.e. monotonic. We want to derive the distribution of Y, $p_y(y)$.
- $P_y(y(x)) = \Pr\{Y \le y(x)\} = \Pr\{X \le x\} = P_x(x)$
- PDF:

$$\frac{dP_y(y(x))}{dx} = \frac{dP_x(x)}{dx}$$

$$p_y(y)\frac{dy}{dx} = \frac{dP_y(y)}{dy}\frac{dy}{dx} \qquad p_x(x)$$

$$p_y(y) = \left(\frac{dy}{dx}\right)^{-1} p_x(x)$$



Example

$$p_x(x) = 2x$$

$$Y = \sin X$$

$$p_y(y) = (\cos x)^{-1} p_x(x) = \frac{2x}{\cos x} = \frac{2\sin^{-1} y}{\sqrt{1-y^2}}$$

Transformation method

- A problem to apply the above method is that we usually have some PDF to sample from, not a given transformation.
- Given a source random variable X with $p_x(x)$ and a target distribution $p_y(y)$, try transform X into to another random variable Y so that Y has the distribution $p_y(y)$.
- We first have to find a transformation $y(x)$ so that $P_x(x)=P_y(y(x))$. Thus,

$$y(x) = P_y^{-1}(P_x(x))$$

Transformation method

- Let's prove that the above transform works.

  We first prove that the random variable $Z = P_x(x)$ has a uniform distribution. If so, then $P_y^{-1}(Z)$ should have distribution $P_x(x)$ from the inversion method.

$$\Pr\{Z \le x\} = \Pr\{P_x(X) \le x\} = \Pr\{X \le P_x^{-1}(x)\} = P_x(P_x^{-1}(x)) = x$$

  Thus, Z is uniform and the transformation works.
- It is an obvious generalization of the inversion method, in which X is uniform and $P_x(x)=x$.

Example

$$p_x(x) = x \xrightarrow{y} p_y(y) = e^y$$

$$P_x(x) = \frac{x^2}{2} \qquad P_y(y) = e^y$$

$$P_y^{-1}(y) = \ln y$$

$$y(x) = P_y^{-1}(P_x(x)) = \ln(\frac{x^2}{2}) = 2\ln x - \ln 2$$

Thus, if X has the distribution $p_x(x) = x$, then the random variable $Y = 2\ln X - \ln 2$ has the distribution $p_y(y) = e^y$

Multiple dimensions

- Easily generalized - using the Jacobian of Y=T(X) $p_y(T(x)) = |J_T(x)|^{-1} p_x(x)$
- Example - polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$J_T(x) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$p(x,y) = r^{-1} p(r,\theta)$$

We often need the other way around, $p(r,\theta) = r\,p(x,y)$

Spherical coordinates

- The spherical coordinate representation of directions is

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$|J_T| = r^2\sin\theta$$

$$p(r,\theta,\phi) = r^2\sin\theta\, p(x,y,z)$$

Spherical coordinates

- Now, look at relation between spherical directions and a solid angles

$$d\omega = \sin\theta d\theta d\phi$$

- Hence, the density in terms of θ, ϕ

$$p(\theta,\phi)d\theta d\phi = p(\omega)d\omega$$

$$\boxed{p(\theta,\phi) = \sin\theta p(\omega)}$$

Multidimensional sampling

- Separable case: independently sample X from p_x and Y from p_y. $p(x,y)=p_x(x)p_y(y)$
- Often, this is not possible. Compute the marginal density function $p(x)$ first.

$$p(x) = \int p(x,y)dy$$

- Then, compute the conditional density function

$$p(y \mid x) = \frac{p(x,y)}{p(x)}$$

- Use 1D sampling with $p(x)$ and $p(y|x)$.

Sampling a hemisphere

- Sample a hemisphere uniformly, *i.e.* $p(\omega) = c$

$$1 = \int_\Omega p(\omega) \qquad c = \frac{1}{2\pi} \longrightarrow p(\omega) = \frac{1}{2\pi}$$

$$\downarrow$$

- Sample θ first

$$p(\theta,\phi) = \frac{\sin\theta}{2\pi}$$

$$p(\theta) = \int_0^{2\pi} p(\theta,\phi)d\phi = \int_0^{2\pi} \frac{\sin\theta}{2\pi} d\phi = \sin\theta$$

- Now sampling ϕ

$$p(\phi \mid \theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{1}{2\pi}$$

Sampling a hemisphere

- Now, we use inversion technique in order to sample the PDF's

$$P(\theta) = \int_0^\theta \sin\theta' d\theta' = 1 - \cos\theta$$

$$P(\phi \mid \theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}$$

- Inverting these:

$$\theta = \cos^{-1}\xi_1$$

$$\phi = 2\pi\xi_2$$

Sampling a hemisphere

- Convert these to Cartesian coordinate

$$\theta = \cos^{-1} \xi_1 \qquad \phi = 2\pi\xi_2$$

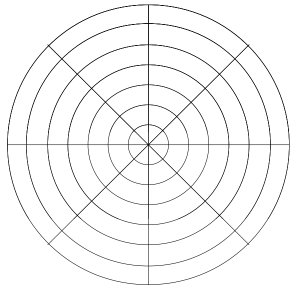
$$\longrightarrow$$

$$x = \sin\theta\cos\phi = \cos(2\pi\xi_1)\sqrt{1-\xi_1^2}$$
$$y = \sin\theta\sin\phi = \sin(2\pi\xi_1)\sqrt{1-\xi_1^2}$$
$$z = \cos\theta = \xi_1$$

- Similar derivation for a full sphere

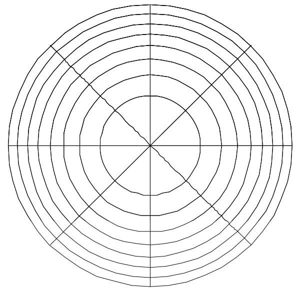
Sampling a disk

WRONG ≠ Equi-Areal



$$\theta = 2\pi U_1$$
$$r = U_2$$

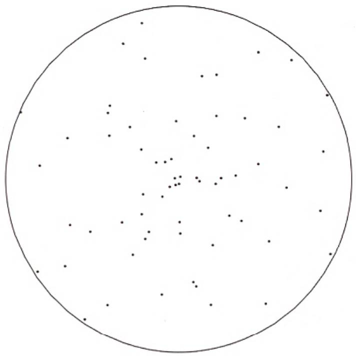
RIGHT = Equi-Areal



$$\theta = 2\pi U_1$$
$$r = \sqrt{U_2}$$

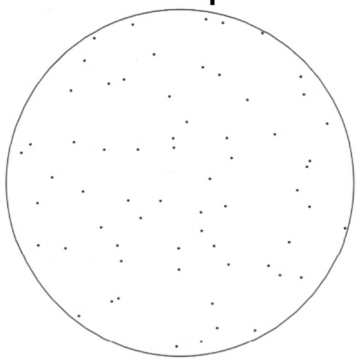
Sampling a disk

WRONG ≠ Equi-Areal



$$\theta = 2\pi U_1$$
$$r = U_2$$

RIGHT = Equi-Areal

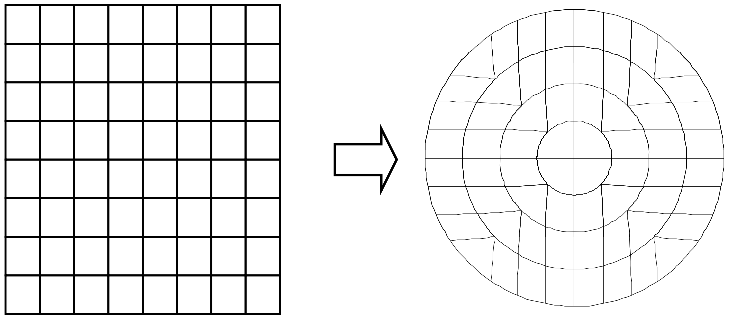


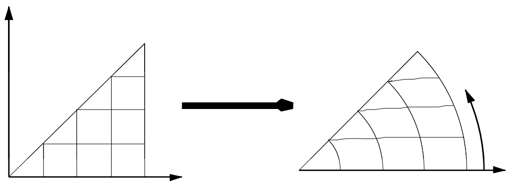
$$\theta = 2\pi U_1$$
$$r = \sqrt{U_2}$$

Sampling a disk

- Uniform $p(x,y) = \frac{1}{\pi}$ $\quad p(r,\theta) = rp(x,y) = \frac{r}{\pi}$
- Sample r first. $$p(r) = \int_0^{2\pi} p(r,\theta)d\theta = 2r$$
- Then, sample θ. $$p(\theta \mid r) = \frac{p(r,\theta)}{p(r)} = \frac{1}{2\pi}$$
- Invert the CDF. $$P(r) = r^2 \qquad P(\theta \mid r) = \frac{\theta}{2\pi}$$ $$r = \sqrt{\xi_1} \qquad \theta = 2\pi\xi_2$$

Shirley's mapping



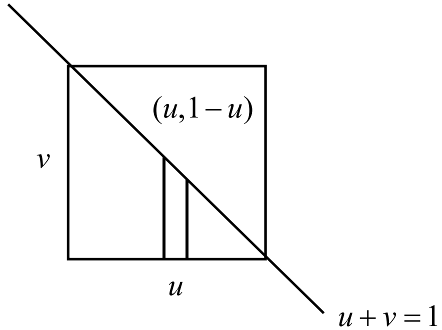


$$r = U_1$$

$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

Sampling a triangle

$$u \geq 0$$
$$v \geq 0$$
$$u + v \leq 1$$



$$A = \int_0^1 \int_0^{1-u} dv\,du = \int_0^1 (1-u)\,du = -\frac{(1-u)^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

Sampling a triangle

- Here u and v are not independent! $p(u,v) = 2$
- Conditional probability

$$p(u) \equiv \int p(u,v)dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(v)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$u_0 = 1 - \sqrt{U_1}$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\,du \quad = (1-u_0)^2$$

$$p(v \mid u) = \frac{1}{(1-u)}$$

$$v_0 = \sqrt{U_1}U_2$$

$$P(v_0 \mid u_0) = \int_0^{v_o} p(v \mid u_0)\,dv \quad = \int_0^{v_o} \frac{1}{(1-u_0)}dv \quad = \frac{v_0}{(1-u_0)}$$

Cosine weighted hemisphere

$$p(\omega) \propto \cos\theta$$

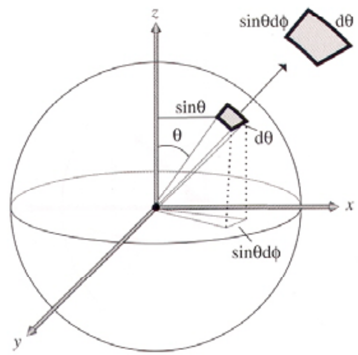
$$1 = \int_\Omega p(\omega)d\omega$$

$$1 = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} c\cos\theta\sin\theta d\theta d\phi$$

$$1 = c2\pi\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta$$

$$c = \frac{1}{\pi}$$

$$\boxed{p(\theta,\phi) = \frac{1}{\pi}\cos\theta\sin\theta}$$



$$d\omega = \sin\theta d\theta d\phi$$

Cosine weighted hemisphere

$$p(\theta,\phi)=\frac{1}{\pi}\cos\theta\sin\theta$$

$$p(\theta)=\int_0^{2\pi}\frac{1}{\pi}\cos\theta\sin\theta d\phi=2\cos\theta\sin\theta=\sin 2\theta$$

$$p(\phi\,|\,\theta)=\frac{p(\theta,\phi)}{p(\theta)}=\frac{1}{2\pi}$$

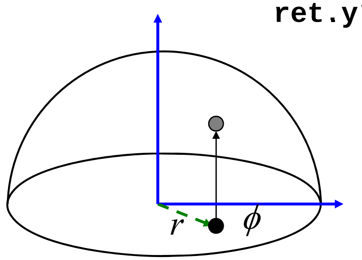
$$P(\theta)=-\frac{1}{2}\cos 2\theta+\frac{1}{2}=\xi_1 \qquad \theta=\frac{1}{2}\cos^{-1}(1-2\xi_1)$$

$$P(\phi\,|\,\theta)=\frac{\phi}{2\pi}=\xi_2 \qquad \phi=2\pi\xi_2$$

Cosine weighted hemisphere

- Malley's method: uniformly generates points on the unit disk and then generates directions by projecting them up to the hemisphere above it.

```
Vector CosineSampleHemisphere(float u1,float u2){
  Vector ret;
  ConcentricSampleDisk(u1, u2, &ret.x, &ret.y);
  ret.z = sqrtf(max(0.f,1.f - ret.x*ret.x -
                        ret.y*ret.y));
  return ret;
}
```



Cosine weighted hemisphere

- Why deos Malley's method works?
- Unit disk sampling $p(r,\phi)=\frac{r}{\pi}$
- Map to hemisphere $(r,\phi)\Rightarrow(\sin\theta,\phi)$

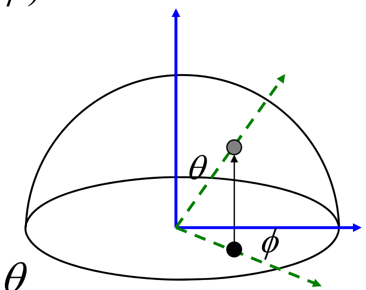
$$Y=(r,\phi)\xleftarrow{T}X=(\theta,\phi)$$

$$r=\sin\theta$$

$$\phi=\phi$$

$$p_y(T(x))=|J_T(x)|^{-1}p_x(x)$$

$$|J_T(x)|=\left|\begin{pmatrix}\cos\theta & 0\\ 0 & 1\end{pmatrix}\right|=\cos\theta$$



Cosine weighted hemisphere

$$Y=(r,\phi)\xleftarrow{T}X=(\theta,\phi)$$

$$r=\sin\theta$$

$$\phi=\phi$$

$$p_y(T(x))=|J_T(x)|^{-1}p_x(x)$$

$$|J_T(x)|=\left|\begin{pmatrix}\cos\theta & 0\\ 0 & 1\end{pmatrix}\right|=\cos\theta$$

$$p(\theta,\phi)=|J_T|\,p(r,\phi)=\frac{\cos\theta\sin\theta}{\pi}$$

Sampling Phong lobe

$$p(\omega) \propto \cos^n \theta$$

$$p(\omega) = c\cos^n \theta \rightarrow \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} c\cos^n \theta \sin\theta d\theta d\phi = 1$$

$$\rightarrow -2\pi c \int_{\cos\theta=1}^{0} \cos^n \theta d\cos\theta = 1 \rightarrow \frac{2\pi c}{n+1} = 1$$

$$\rightarrow c = \frac{n+1}{2\pi}$$

$$p(\theta,\phi) = \frac{n+1}{2\pi}\cos^n \theta \sin\theta$$

Sampling Phong lobe

$$p(\theta,\phi) = \frac{n+1}{2\pi}\cos^n \theta \sin\theta$$

$$p(\theta) = \int_{\phi=0}^{2\pi} \frac{n+1}{2\pi}\cos^n \theta \sin\theta d\phi = (n+1)\cos^n \theta \sin\theta$$

$$P(\theta') = \int_{\theta=0}^{\theta'} (n+1)\cos^n \theta \sin\theta d\theta$$

$$= -(n+1)\int_{\theta=0}^{\theta'} \cos^n \theta d\cos\theta = -(n+1)\left.\frac{\cos^{n+1}\theta}{n+1}\right|_{\cos\theta=1}^{\cos\theta'}$$

$$= 1 - \cos^{n+1}\theta'$$

$$\theta = \cos^{-1}\left(\sqrt[n+1]{\xi_1}\right)$$

Sampling Phong lobe

$$p(\theta,\phi) = \frac{n+1}{2\pi}\cos^n \theta \sin\theta$$

$$p(\phi \mid \theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{\frac{n+1}{2\pi}\cos^n \theta \sin\theta}{(n+1)\cos^n \theta \sin\theta} = \frac{1}{2\pi}$$

$$P(\phi' \mid \theta) = \int_{\phi=0}^{\phi'} \frac{1}{2\pi} d\phi = \frac{\phi'}{2\pi}$$

$$\phi = 2\pi\xi_2$$

Sampling Phong lobe

When n=1, it is actually equivalent to cosine-weighted hemisphere

$$\mathrm{n}=1, (\theta,\phi) = (\cos^{-1}\sqrt{\xi_1}, 2\pi\xi_2) \qquad (\theta,\phi) = \left(\frac{1}{2}\cos^{-1}(1-2\xi_1), 2\pi\xi_2\right)$$

$$P(\theta) = 1 - \cos^{n+1}\theta = 1 - \cos^2\theta \qquad P(\theta) = -\frac{1}{2}\cos 2\theta + \frac{1}{2}$$

$$-\frac{1}{2}\cos 2\theta + \frac{1}{2} = -\frac{1}{2}(1 - 2\sin^2\theta) + \frac{1}{2} = \sin^2\theta = 1 - \cos^2\theta$$