Sampling and Reconstruction

Digital Image Synthesis

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with slides by Pat Hanrahan, Torsten Moller and Brian Curless

Sampling theory

- Sampling theory: the theory of taking discrete sample values *(grid of color pixels)* from functions defined over continuous domains *(incident radiance defined over the film plane)* and then using those samples to reconstruct new functions that are similar to the original (reconstruction).

`Sampler`: selects sample points on the image plane

`Filter`: blends multiple samples together

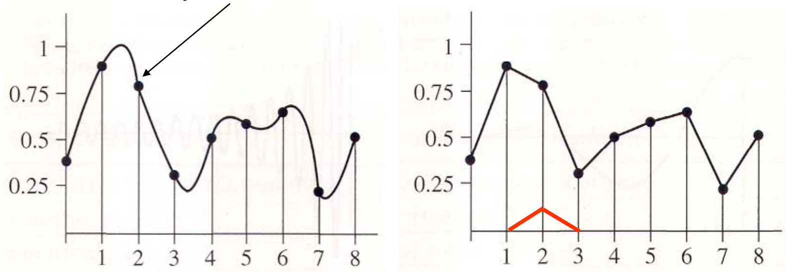
Aliasing

- Reconstruction generates an approximation to the original function. Error is called aliasing.

sampling

reconstruction

sample value



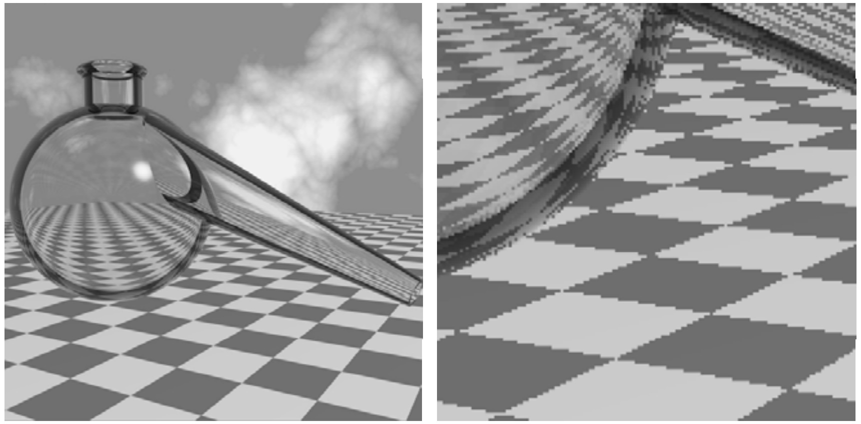
sample position

Sampling in computer graphics

- Artifacts due to sampling - Aliasing
  - Jaggies
  - Moire
  - Flickering small objects
  - Sparkling highlights
  - Temporal strobing (such as [Wagon-wheel effect])
- Preventing these artifacts - Antialiasing

Jaggies

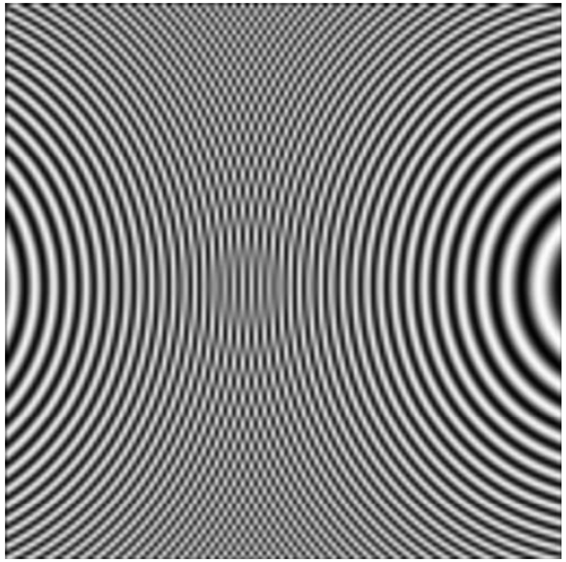
Retort sequence by Don Mitchell



Staircase pattern or jaggies

Moire pattern

- Sampling the equation
  $\sin(x^2 + y^2)$



Fourier analysis

- Can be used to evaluate the quality between the reconstruction and the original.
- The concept was introduced to Graphics by Robert Cook in 1986. (extended by Don Mitchell)

Rob Cook

V.P. of Pixar
1981 M.S. Cornell
1987 SIGGRAPH Achievement award
1999 Fellow of ACM
2001 Academic Award with Ed Catmull and Loren Carpenter (for Renderman)

Fourier transforms

- Most functions can be decomposed into a weighted sum of shifted sinusoids.
- Each function has two representations
  - Spatial domain - normal representation
  - Frequency domain - spectral representation
- The *Fourier transform* converts between the spatial and frequency domain

Spatial Domain $f(x)$

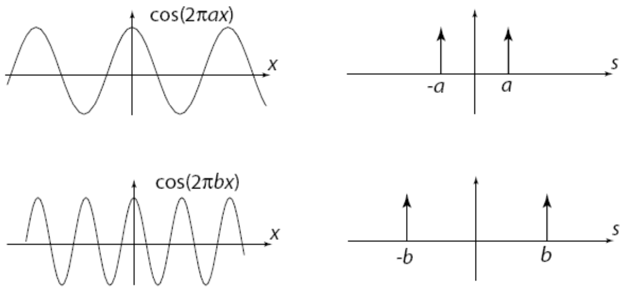
$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \Rightarrow$$

$$\Leftarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega \Leftarrow$$

Frequency Domain $F(\omega)$

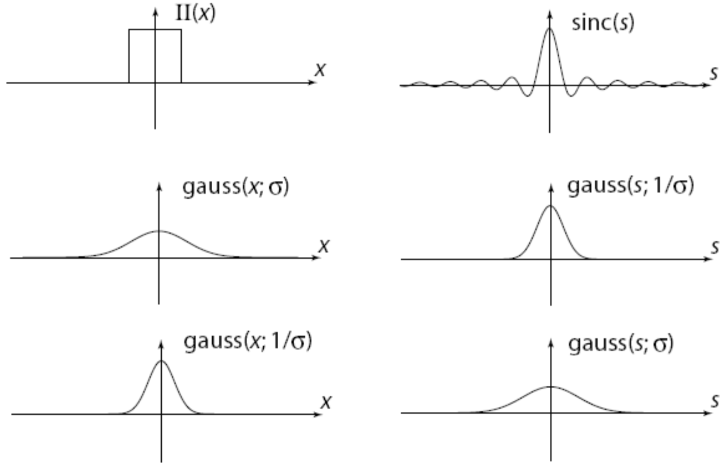
Fourier analysis

spatial domain *frequency domain*



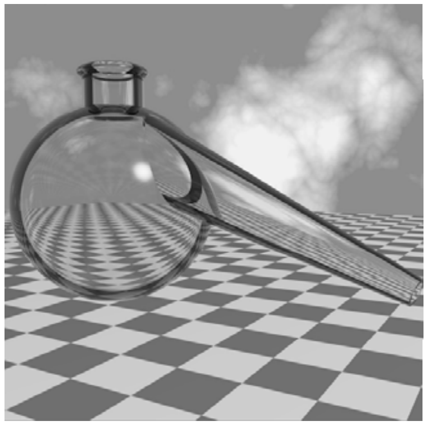
Fourier analysis

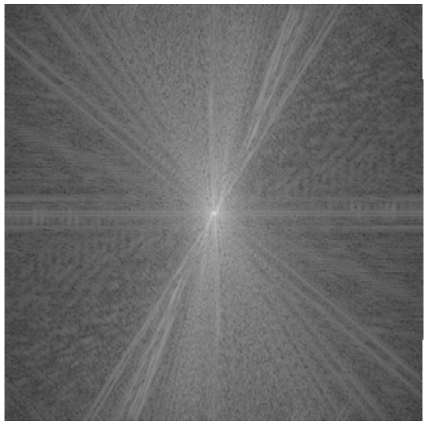
spatial domain *frequency domain*



Fourier analysis

spatial domain *frequency domain*





Convolution

- *Definition*

$$h(x) = f \otimes g = \int f(x')g(x-x')\,dx'$$

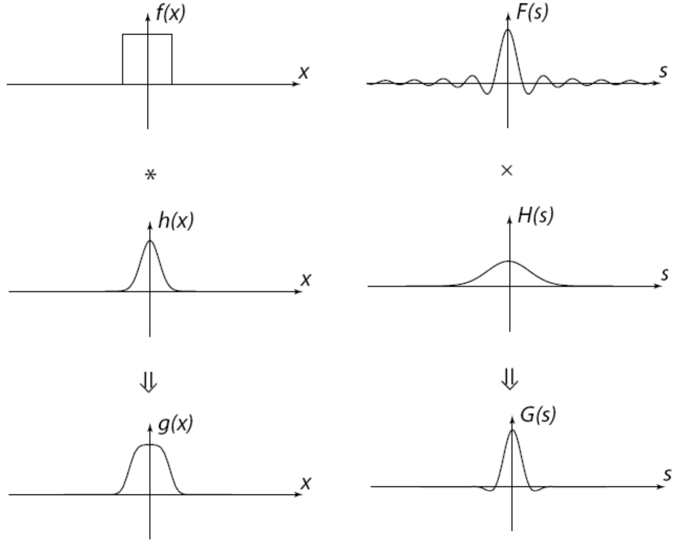
- *Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.

$$f \otimes g \leftrightarrow F \times G$$

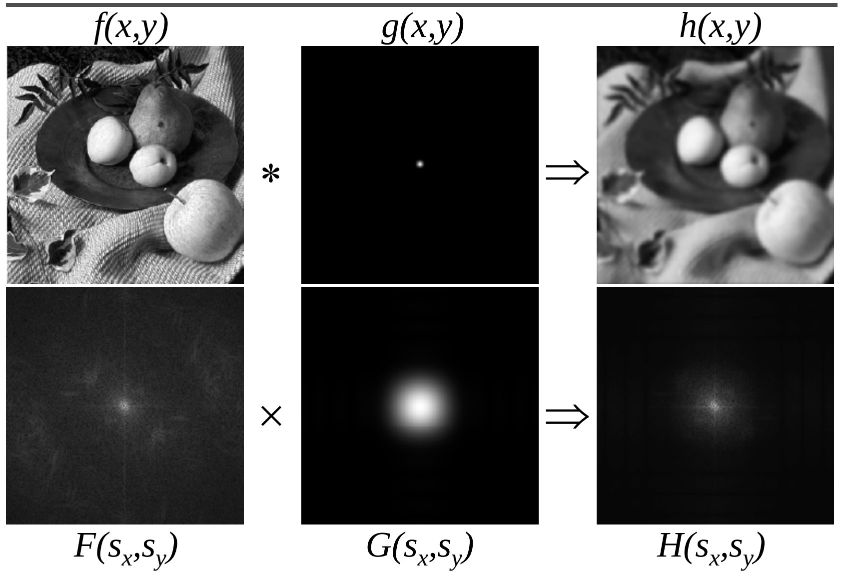
- *Symmetric Theorem*: Multiplication in the space domain is equivalent to convolution in the frequency domain.

$$f \times g \leftrightarrow F \otimes G$$

1D convolution theorem example

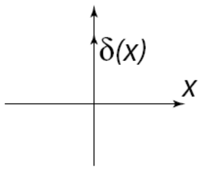


2D convolution theorem example



The delta function

- Dirac delta function, zero width, infinite height and unit area

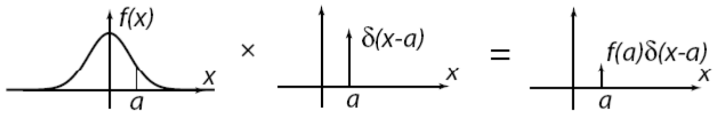


Sifting and shifting

Sifting:

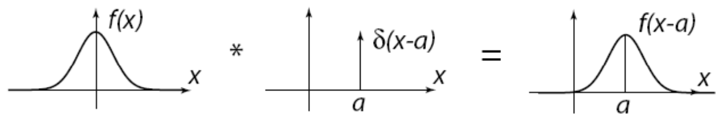
$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = \int_{a-\varepsilon}^{a-\varepsilon} f(x)\delta(x-a)dx = f(a)\int_{a-\varepsilon}^{a-\varepsilon} \delta(x-a)dx = f(a)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$



Shifting:

$$f(x) * \delta(x-a) = f(x-a)$$



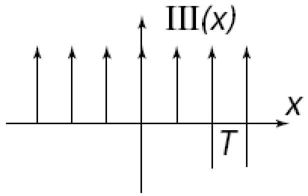
Shah/impulse train function

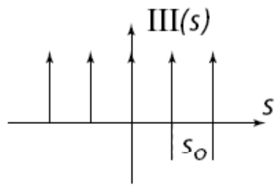
spatial domain

$$\mathrm{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT)$$

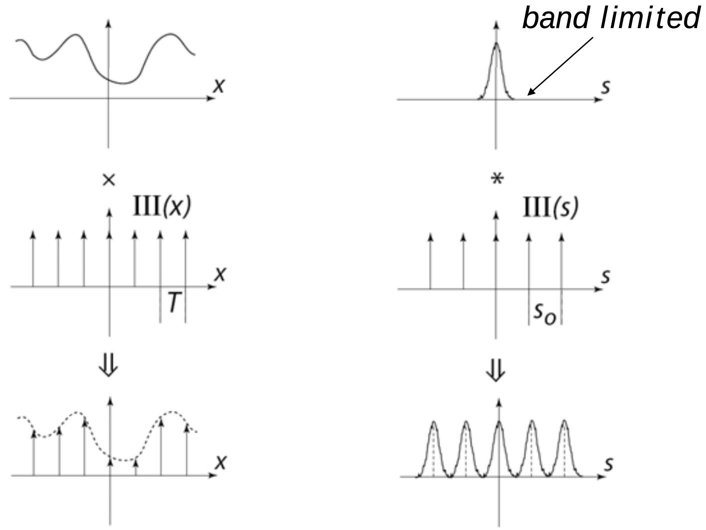
frequency domain

$$\mathrm{III}(s) = \sum_{n=-\infty}^{\infty} \delta(s - ns_o), \quad s_o = 1/T$$

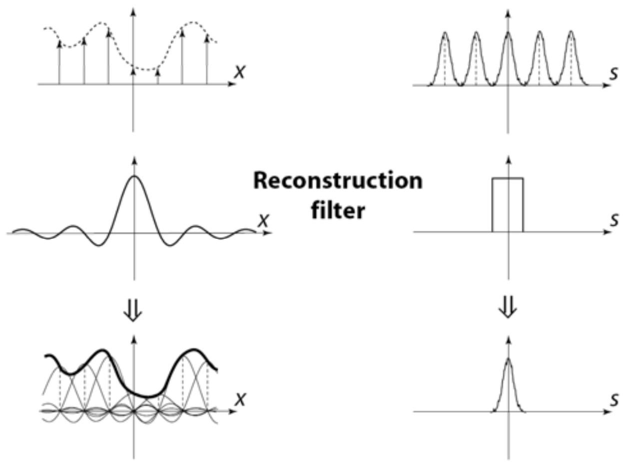




Sampling



Reconstruction



The reconstructed function is obtained by interpolating among the samples in some manner

In math forms

$$\tilde{F} = (F(s) * \mathrm{III}(s)) \times \Pi(s)$$

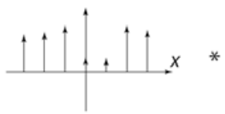
$$\tilde{f} = (f(x) \times \mathrm{III}(x)) * \mathrm{sinc}(x)$$

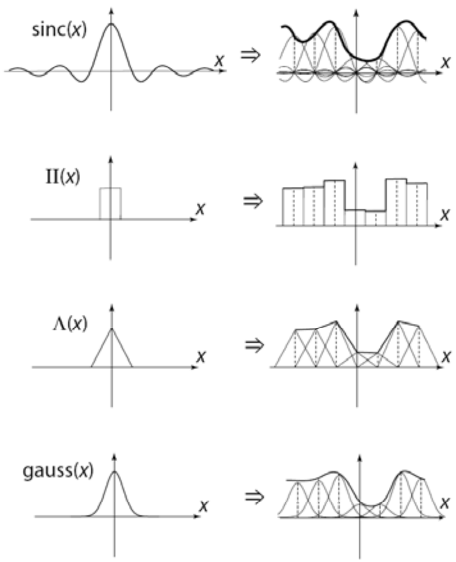
$$\tilde{f}(x) = \sum_{i=-\infty}^{\infty} \mathrm{sinc}(x - i) f(i)$$

Reconstruction filters

The sinc filter, while ideal, has two drawbacks:

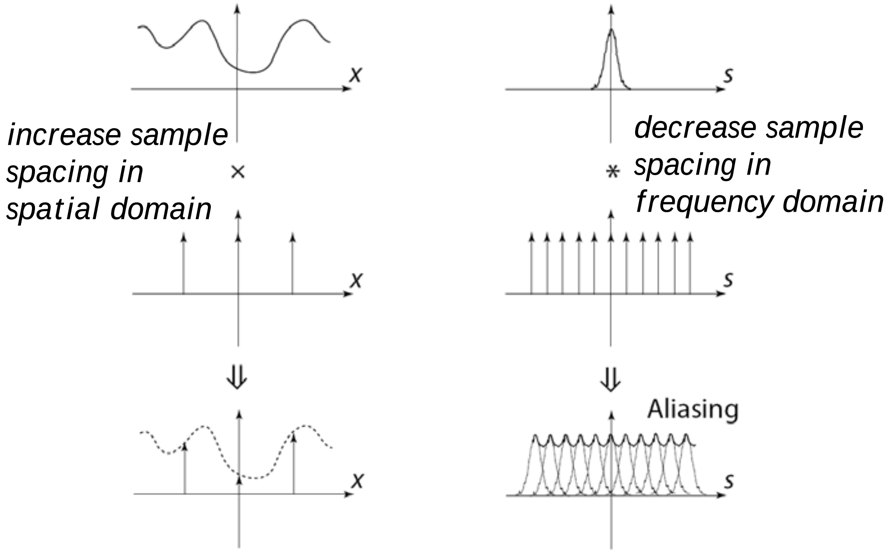
- It has a large support (slow to compute)
- It introduces ringing in practice



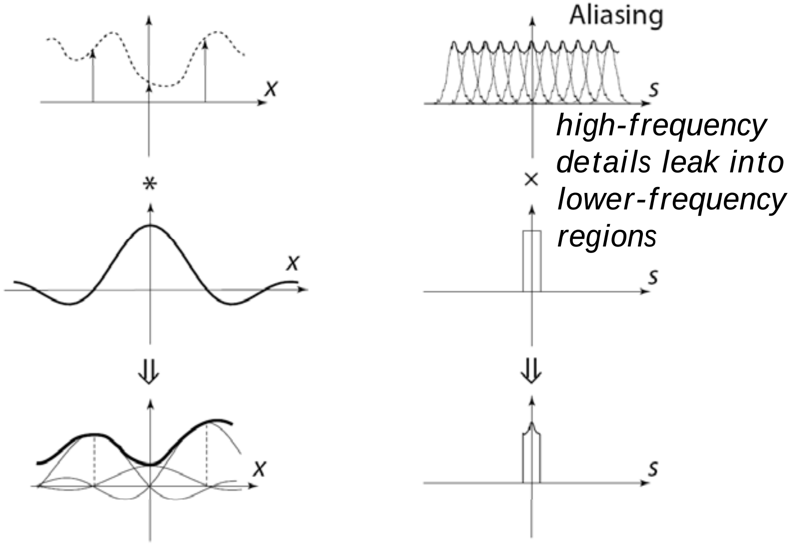


The box filter is bad because its Fourier transform is a sinc filter which includes high frequency contribution from the infinite series of other copies.

Aliasing



Aliasing



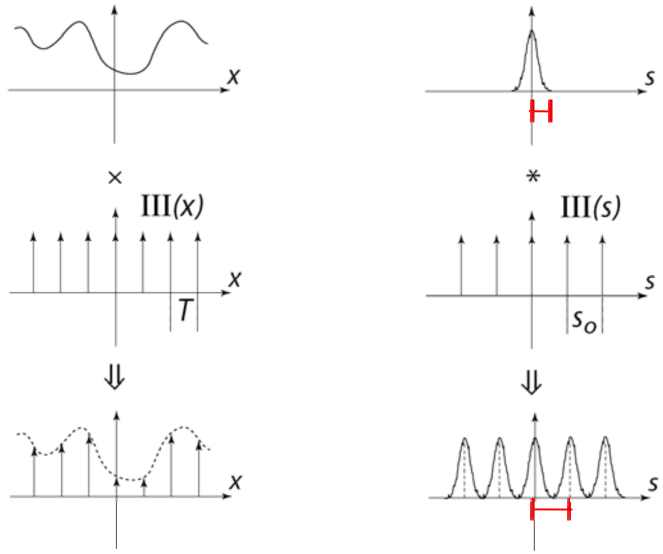
Sampling theorem

This result is known as the **Sampling Theorem** and is due to Claude Shannon who first discovered it in 1949:

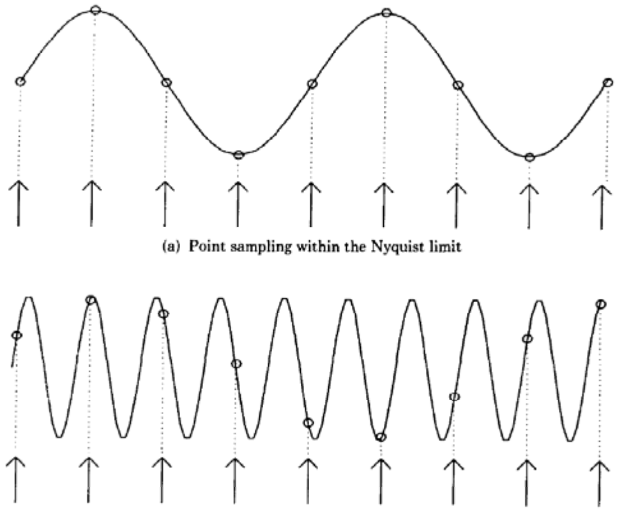
> A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above ½ the sampling frequency.

For a given **bandlimited** function, the minimum rate at which it must be sampled is the **Nyquist frequency**.

Sampling theorem



Aliasing due to under-sampling



(a) Point sampling within the Nyquist limit

(b) Point sampling beyond the Nyquist limit

Sampling theorem

- For band limited functions, we can just increase the sampling rate
- However, few of interesting functions in computer graphics are band limited, in particular, functions with discontinuities.
- It is mostly because the discontinuity always falls between two samples and the samples provides no information about this discontinuity.

Aliasing

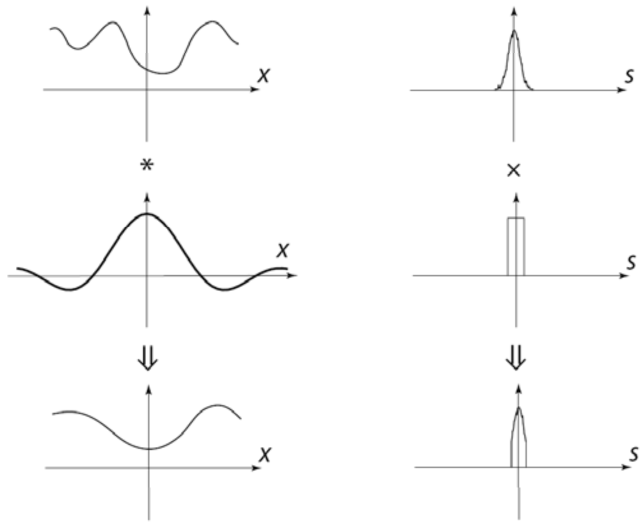
- Prealiasing: due to sampling under Nyquist rate
- Postaliasing: due to use of imperfect reconstruction filter

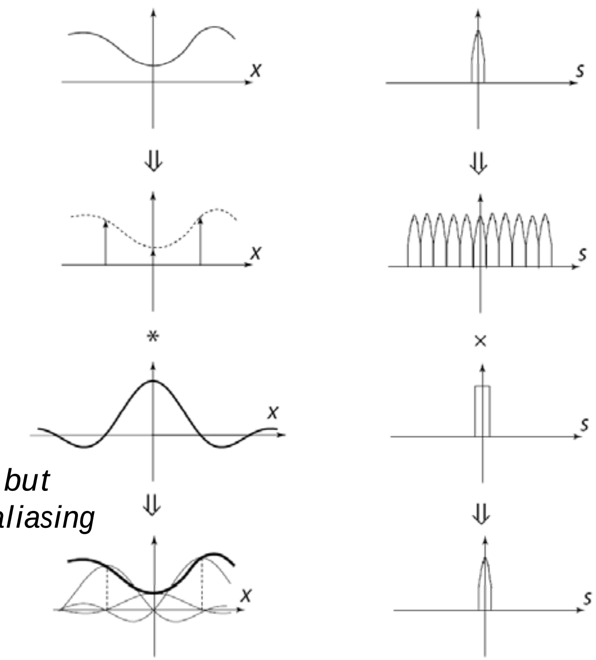
Antialiasing

- Antialiasing = Preventing aliasing

1. Analytically prefilter the signal
   - Not solvable in general
2. Uniform supersampling and resample
3. Nonuniform or stochastic sampling

Antialiasing (Prefiltering)

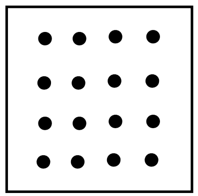




It is blurred, but better than aliasing

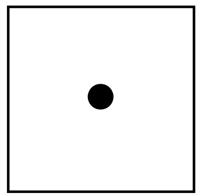
Uniform supersampling

- Increasing the sampling rate moves each copy of the spectra further apart, potentially reducing the overlap and thus aliasing
- Resulting samples must be resampled (filtered) to image sampling rate



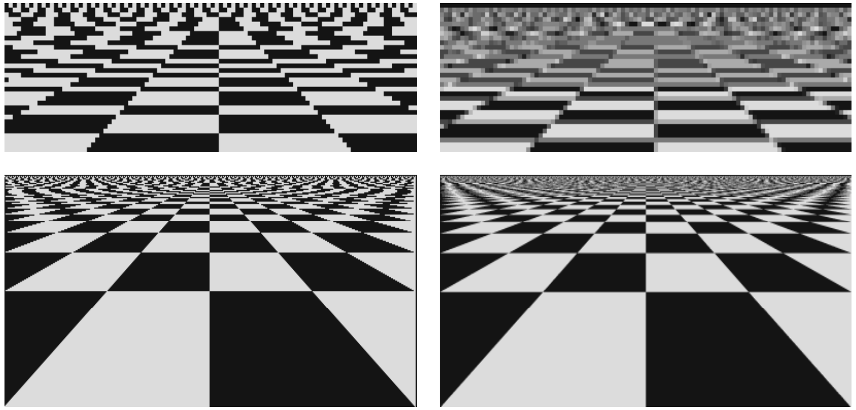
Samples

$$Pixel = \sum_s w_s \cdot Sample_s$$



Pixel

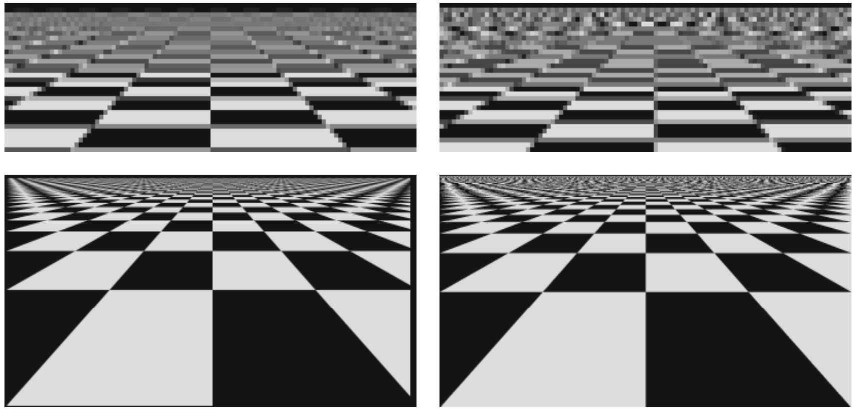
Point vs. Supersampled



Point **4x4 Supersampled**

Checkerboard sequence by Tom Duff

Analytic vs. Supersampled



Exact Area **4x4 Supersampled**

Non-uniform sampling

- Uniform sampling
  - The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
  - Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
  - Aliases are coherent (structured), and very noticeable
- Non-uniform sampling
  - Samples at non-uniform locations have a different spectrum; a single spike plus noise
  - Sampling a signal in this way converts aliases into broadband noise
  - Noise is incoherent (structurelss), and much less objectionable
- Aliases can't be removed, but an be made less noticeable.

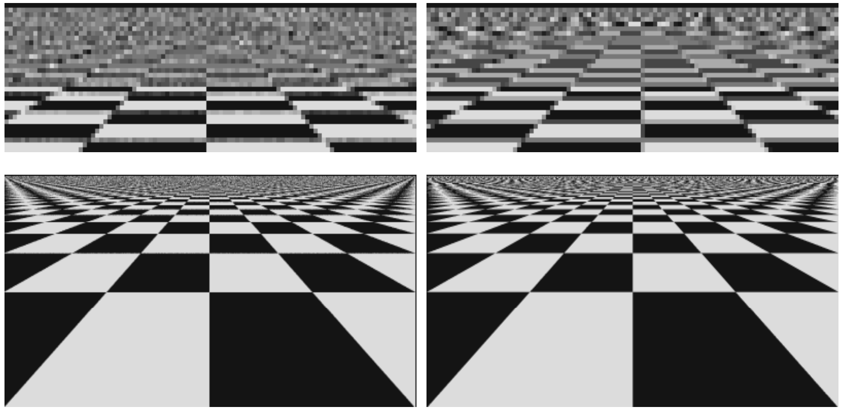
Antialiasing (nonuniform sampling)

- The impulse train is modified as

$$\sum_{i=-\infty}^{\infty}\delta\left(x-\left(iT+\frac{1}{2}-\xi\right)\right)$$

- It turns regular aliasing into noise. But random noise is less distracting than coherent aliasing.

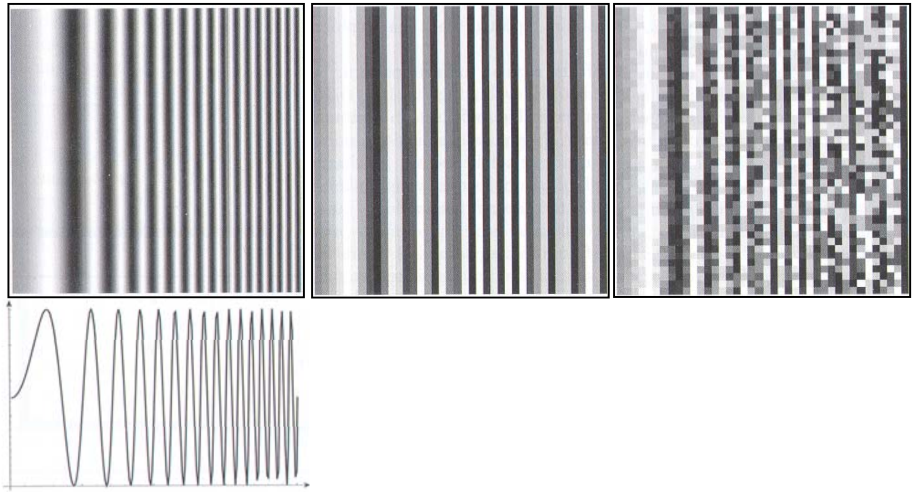
Jittered vs. Uniform Supersampling



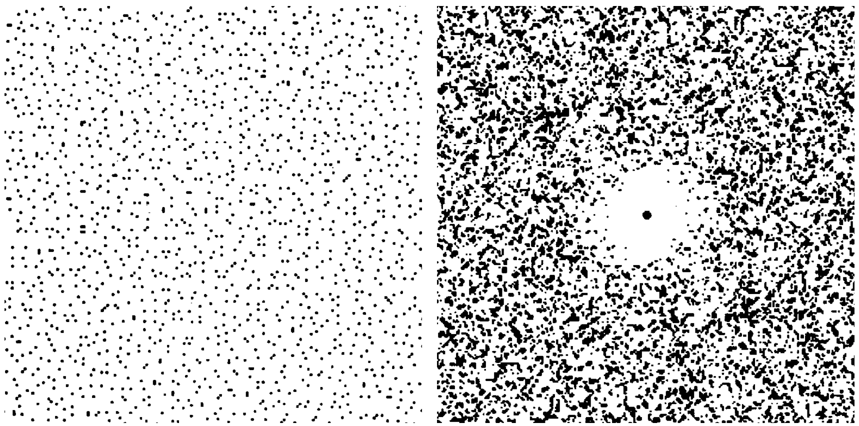
4x4 Jittered Sampling **4x4 Uniform**

Prefer noise over aliasing

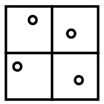
reference *aliasing* *noise*



Jittered sampling



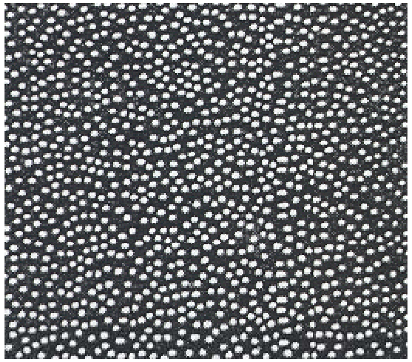
Add uniform random jitter to each sample



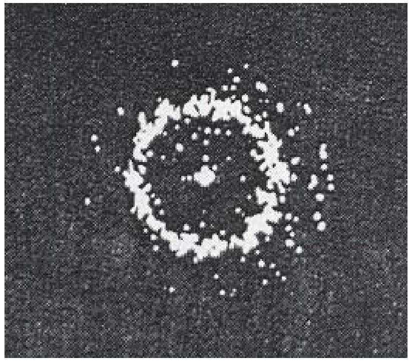
Poisson disk noise (Yellott)

- Blue noise
- Spectrum should be noisy and lack any concentrated spikes of energy (to avoid coherent aliasing)
- Spectrum should have deficiency of low-frequency energy (to hide aliasing in less noticeable high frequency)

Distribution of extrafoveal cones



Monkey eye
cone distribution

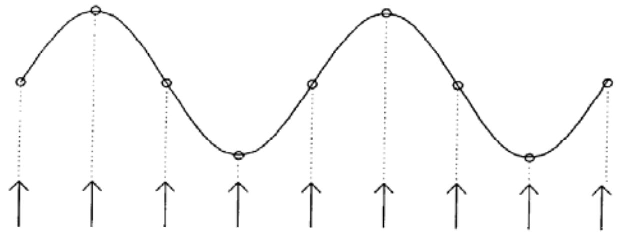


Fourier transform

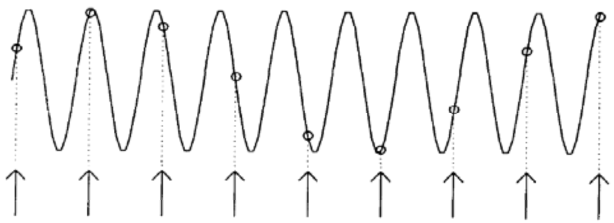
Yellott theory

- **Aliases replaced by noise**
- **Visual system less sensitive to high freq noise**

Example



(a) Point sampling within the Nyquist limit



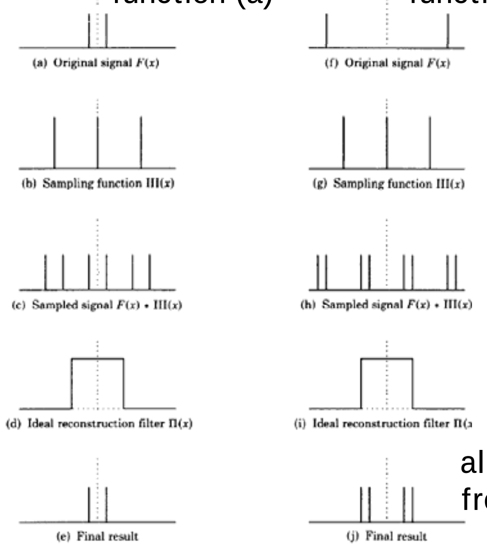
(b) Point sampling beyond the Nyquist limit

Aliasing

frequency
domain

function (a)

function (b)



alias=false
frequency

Stochastic sampling

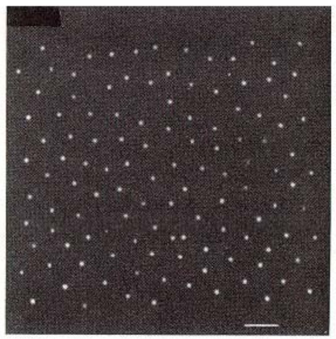


Fig. 3a. Monkey eye photoreceptor distribution.

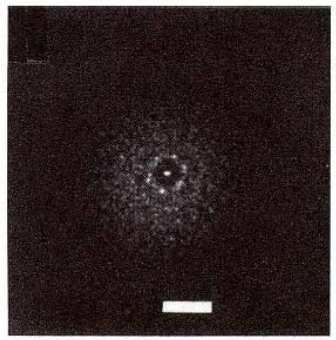
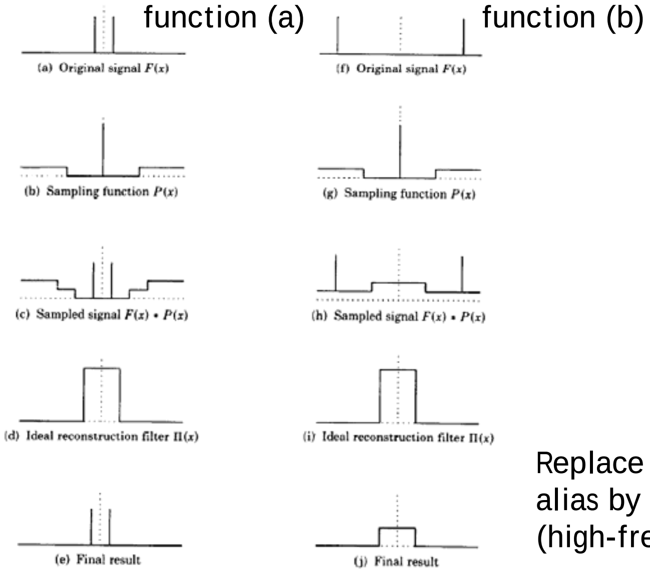


Fig. 3b. Optical transform of monkey eye.

Stochastic sampling



Replace structure alias by structureless (high-freq) noise

Antialiasing (adaptive sampling)

- Take more samples only when necessary. However, in practice, it is hard to know where we need supersampling. Some heuristics could be used.
- It only makes a less aliased image, but may not be more efficient than simple supersampling particular for complex scenes.

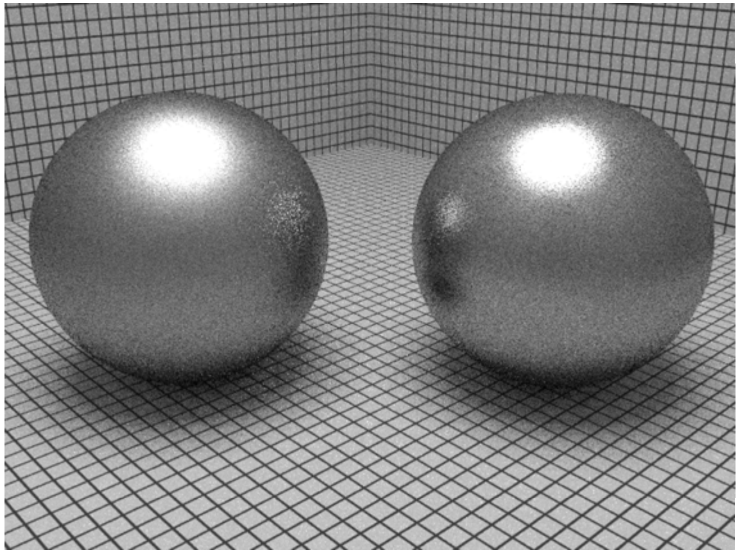
Application to ray tracing

- Sources of aliasing: object boundary, small objects, textures and materials
- Good news: we can do sampling easily
- Bad news: we can't do prefiltering (because we do not have the whole function)
- Key insight: we can never remove all aliasing, so we develop techniques to mitigate its impact on the quality of the final image.

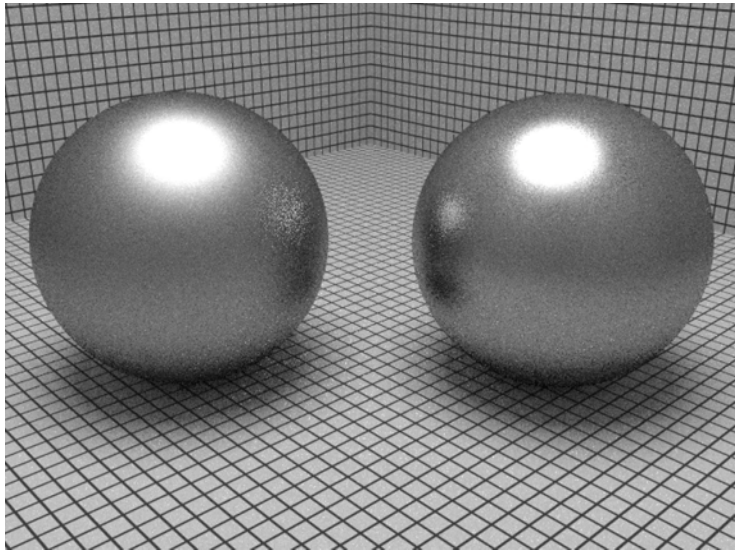
`pbrt` sampling interface

- Creating good sample patterns can substantially improve a ray tracer's efficiency, allowing it to create a high-quality image with fewer rays.
- Because evaluating radiance is costly, it pays to spend time on generating better sampling.
- `core/sampling.*, samplers/*`
- `random.cpp, stratified.cpp, bestcandidate.cpp, lowdiscrepancy.cpp,`

An ineffective sampler



A more effective sampler



Main rendering loop

```
void Scene::Render() {
  Sample *sample = new Sample(surfaceIntegrator,
                              volumeIntegrator,
                              this);
  ...
  while (sampler->GetNextSample(sample)) {
      RayDifferential ray;
      float rW = camera->GenerateRay(*sample, &ray);
      <Generate ray differentials for camera ray>
      float alpha;
      Spectrum Ls = 0.f;
      if (rW > 0.f)
            Ls = rW * Li(ray, sample, &alpha);
      ...
      camera->film->AddSample(*sample,ray,Ls,alpha);
      ...
  }
  ...
  camera->film->WriteImage();
}
```

fill in eye ray info and other samples for integrator

Sample

```
struct Sample {
  Sample(SurfaceIntegrator *surf,
         VolumeIntegrator *vol,
         const Scene *scene);
  ...
  float imageX, imageY;
  float lensU, lensV;
  float time;
  // Integrator Sample Data
  vector<u_int> n1D, n2D;
  float **oneD, **twoD;
  ...
}
```

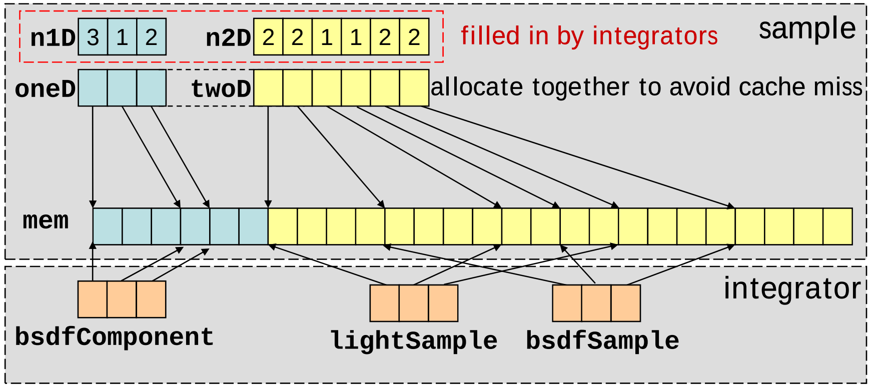
store required information for one eye ray sample

Note that it stores all samples required for one eye ray. That is, it may depend on depth.

Sample is allocated once in Render(). Sampler is called to fill in the information for each eye ray. The integrator can ask for multiple 1D and/or 2D samples, each with an arbitrary number of entries, e.g. depending on #lights. For example, WhittedIntegrator does not need samples. DirectLighting needs samples proportional to #lights.

Data structure

- Different types of lights require different numbers of samples, usually 2D samples.
- Sampling BRDF requires 2D samples.
- Selection of BRDF components requires 1D samples.



Sample

```
Sample::Sample(SurfaceIntegrator *surf,
      VolumeIntegrator *vol, const Scene *scene) {
  // calculate required number of samples
  // according to integration strategy
  surf->RequestSamples(this, scene);
  vol->RequestSamples(this, scene);

  // Allocate storage for sample pointers
  int nPtrs = n1D.size() + n2D.size();
  if (!nPtrs) {
      oneD = twoD = NULL;
      return;
  }
  oneD=(float **)AllocAligned(nPtrs*sizeof(float *));
  twoD = oneD + n1D.size();
```

Sample

```
  // Compute total number of sample values needed
  int totSamples = 0;
  for (u_int i = 0; i < n1D.size(); ++i)
      totSamples += n1D[i];
  for (u_int i = 0; i < n2D.size(); ++i)
      totSamples += 2 * n2D[i];
  // Allocate storage for sample values
  float *mem = (float *)AllocAligned(totSamples *
      sizeof(float));
  for (u_int i = 0; i < n1D.size(); ++i) {
      oneD[i] = mem;
      mem += n1D[i];
  }
  for (u_int i = 0; i < n2D.size(); ++i) {
      twoD[i] = mem;
      mem += 2 * n2D[i];
  }
}
```

DirectLighting::RequestSamples

```
void RequestSamples(Sample *sample, Scene *scene) {
  if (strategy == SAMPLE_ALL_UNIFORM) {
    u_int nLights = scene->lights.size();
    lightSampleOffset = new int[nLights];
    bsdfSampleOffset = new int[nLights];
    bsdfComponentOffset = new int[nLights];
    for (u_int i = 0; i < nLights; ++i) {
      const Light *light = scene->lights[i];
      int lightSamples =
        scene->sampler->RoundSize(light->nSamples);
      lightSampleOffset[i] =
        sample->Add2D(lightSamples);
      bsdfSampleOffset[i] =
        sample->Add2D(lightSamples);
      bsdfComponentOffset[i] =
        sample->Add1D(lightSamples);
    }
    lightNumOffset = -1;
  }
```

DirectLighting::RequestSamples

```
else {
  // Allocate and request samples for sampling one
   light
    lightNumOffset = sample->Add1D(1);
    lightSampleOffset = new int[1];
    lightSampleOffset[0] = sample->Add2D(1);

    bsdfComponentOffset = new int[1];
    bsdfComponentOffset[0] = sample->Add1D(1);
    bsdfSampleOffset = new int[1];
    bsdfSampleOffset[0] = sample->Add2D(1);
  }
}
```

PathIntegrator::RequestSamples

```
void PathIntegrator::RequestSamples(Sample *sample,
      const Scene *scene)
{
  for (int i = 0; i < SAMPLE_DEPTH; ++i) {
      lightNumOffset[i] = sample->Add1D(1);
      lightPositionOffset[i] = sample->Add2D(1);

      bsdfComponentOffset[i] = sample->Add1D(1);
      bsdfDirectionOffset[i] = sample->Add2D(1);

      outgoingComponentOffset[i] = sample->Add1D(1);
      outgoingDirectionOffset[i] = sample->Add2D(1);
  }
}
```

Sampler

```
Sampler(int xstart, int xend,        range of pixels
        int ystart, int yend, int spp);
bool GetNextSample(Sample *sample);
int  TotalSamples()                 sample per pixel
        samplesPerPixel *
       (xPixelEnd - xPixelStart) *
       (yPixelEnd - yPixelStart);
```

Random sampler

```
RandomSampler::RandomSampler(…) {  Just for illustration; does
  ...                              not work well in practice
  // Get storage for a pixel's worth of stratified
  samples imageSamples = (float *)AllocAligned(5 *
      xPixelSamples * yPixelSamples * sizeof(float));
  lensSamples = imageSamples +
                2 * xPixelSamples * yPixelSamples;
  timeSamples = lensSamples +
                2 * xPixelSamples * yPixelSamples;

  // prepare samples for the first pixel
  for (i=0; i<5*xPixelSamples*yPixelSamples; ++i)
      imageSamples[i] = RandomFloat();
  // Shift image samples to pixel coordinates
  for (o=0; o<2*xPixelSamples*yPixelSamples; o+=2) {
      imageSamples[o]   += xPos;  private copy of the
      imageSamples[o+1] += yPos; } current pixel position
  samplePos = 0;  #samples for current pixel
}
```

Random sampler

```
bool RandomSampler::GetNextSample(Sample *sample) {
  if (samplePos == xPixelSamples * yPixelSamples) {
    // Advance to next pixel for sampling
    if (++xPos == xPixelEnd) {
      xPos = xPixelStart;
      ++yPos; }
    if (yPos == yPixelEnd)
      return false;

    for (i=0; i < 5*xPixelSamples*yPixelSamples; ++i)
      imageSamples[i] = RandomFloat();

    // Shift image samples to pixel coordinates
    for (o=0; o<2*xPixelSamples*yPixelSamples; o+=2)
    { imageSamples[o]   += xPos;
      imageSamples[o+1] += yPos; }
    samplePos = 0;
  }
```

number of generated samples in this pixel

generate all samples for one pixel at once

Random sampler

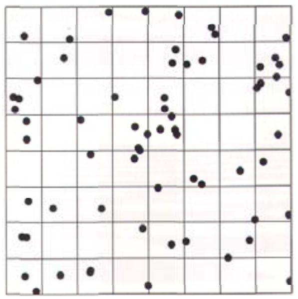
```
  // Return next sample point according to samplePos
  sample->imageX = imageSamples[2*samplePos];
  sample->imageY = imageSamples[2*samplePos+1];
  sample->lensU = lensSamples[2*samplePos];
  sample->lensV = lensSamples[2*samplePos+1];
  sample->time = timeSamples[samplePos];

  // Generate samples for integrators
  for (u_int i = 0; i < sample->n1D.size(); ++i)
      for (u_int j = 0; j < sample->n1D[i]; ++j)
            sample->oneD[i][j] = RandomFloat();
  for (u_int i = 0; i < sample->n2D.size(); ++i)
      for (u_int j = 0; j < 2*sample->n2D[i]; ++j)
            sample->twoD[i][j] = RandomFloat();

  ++samplePos;
  return true;
}
```

Random sampling

a pixel

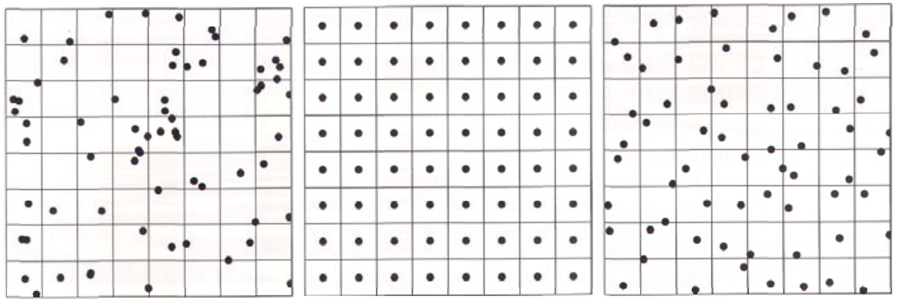


completely random

Stratified sampling

- Subdivide the sampling domain into non-overlapping regions (*strata*) and take a single sample from each one so that it is less likely to miss important features.

Stratified sampling



completely random *stratified uniform* *stratified jittered*

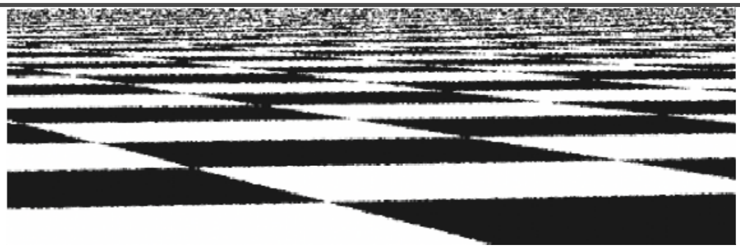
turns aliasing into noise

Comparison of sampling methods

256 samples per pixel as reference

1 sample per pixel (no jitter)

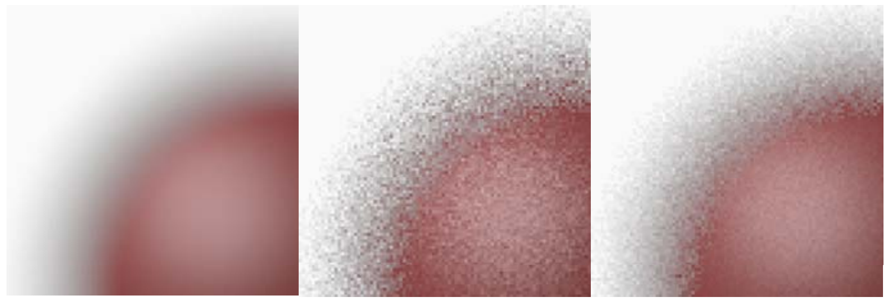
Comparison of sampling methods



1 sample per pixel (jittered)

4 samples per pixel (jittered)

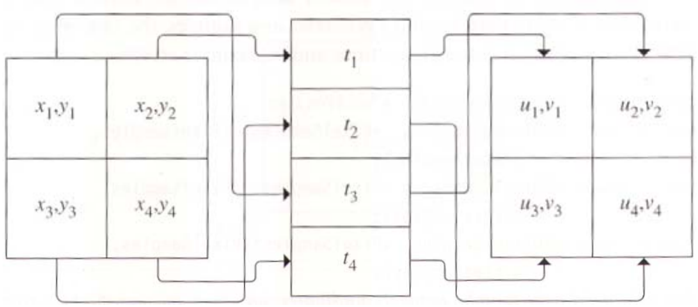
Stratified sampling



reference *random* *stratified jittered*

High dimension

- D dimension means N^D cells.
- Solution: make strata separately and associate them randomly, also ensuring good distributions.



Stratified sampler

```
if (samplePos == xPixelSamples * yPixelSamples) {
  // Advance to next pixel for stratified sampling
  ...
  // Generate stratified samples for (xPos, yPos)
  StratifiedSample2D(imageSamples,
    xPixelSamples, yPixelSamples, jitterSamples);
  StratifiedSample2D(lensSamples,
    xPixelSamples, yPixelSamples, jitterSamples);
  StratifiedSample1D(timeSamples,
    xPixelSamples*yPixelSamples, jitterSamples);

  // Shift stratified samples to pixel coordinates
  ...
  // Decorrelate sample dimensions
  Shuffle(lensSamples,xPixelSamples*yPixelSamples,2);
  Shuffle(timeSamples,xPixelSamples*yPixelSamples,1);
  samplePos = 0;
}
```

Stratified sampling

n stratified samples within [0..1]

```
void StratifiedSample1D(float *samp, int nSamples,
                        bool jitter) {
   float invTot = 1.f / nSamples;
   for (int i = 0;  i < nSamples; ++i) {
       float delta = jitter ? RandomFloat() : 0.5f;
       *samp++ = (i + delta) * invTot;
   }
}
```

nx*ny stratified samples within [0..1]X[0..1]

```
void StratifiedSample2D(float *samp, int nx, int ny,
                        bool jitter) {
   float dx = 1.f / nx, dy = 1.f / ny;
   for (int y = 0; y < ny; ++y)
       for (int x = 0; x < nx; ++x) {
             float jx = jitter ? RandomFloat() : 0.5f;
             float jy = jitter ? RandomFloat() : 0.5f;
             *samp++ = (x + jx) * dx;
             *samp++ = (y + jy) * dy;
       }
}
```

Shuffle

```
void Shuffle(float *samp, int count, int dims) {
  for (int i = 0; i < count; ++i) {
    u_int other = RandomUInt() % count;
    for (int j = 0; j < dims; ++j)
      swap(samp[dims*i + j], samp[dims*other + j]);
  }
}
```

d-dimensional vector swap

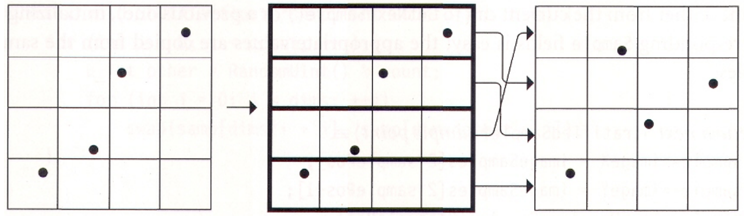
Stratified sampler

```
// Return next _StratifiedSampler_ sample point
sample->imageX = imageSamples[2*samplePos];
sample->imageY = imageSamples[2*samplePos+1];
sample->lensU = lensSamples[2*samplePos];
sample->lensV = lensSamples[2*samplePos+1];
sample->time = timeSamples[samplePos];
// what if integrator asks for 7 stratified 2D samples
// Generate stratified samples for integrators
for (u_int i = 0; i < sample->n1D.size(); ++i)
  LatinHypercube(sample->oneD[i], sample->n1D[i], 1);
for (u_int i = 0; i < sample->n2D.size(); ++i)
  LatinHypercube(sample->twoD[i], sample->n2D[i], 2);

++samplePos;
return true;
```

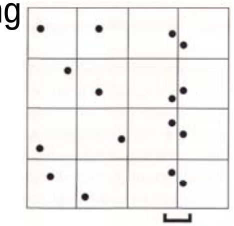
Latin hypercube sampling

- Integrators could request an arbitrary n samples. nx1 or 1xn doesn't give a good sampling pattern.



A worst case for stratified sampling

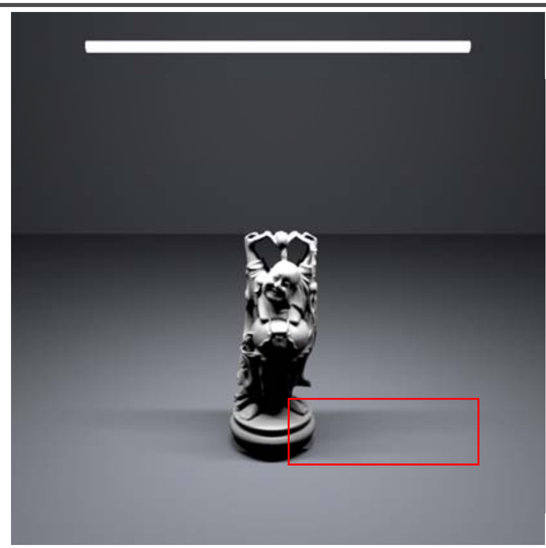
LHS can prevent this to happen



Latin Hypercube

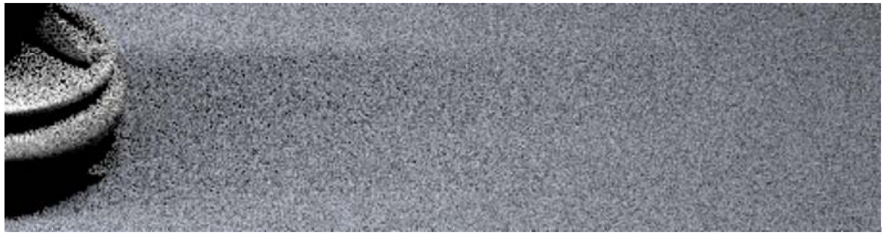
```
void LatinHypercube(float *samples,
                    int nSamples, int nDim)
{
  // Generate LHS samples along diagonal
  float delta = 1.f / nSamples;
  for (int i = 0; i < nSamples; ++i)
    for (int j = 0; j < nDim; ++j)
      samples[nDim*i+j] = (i+RandomFloat())*delta;
  note the difference with shuffle
  // Permute LHS samples in each dimension
  for (int i = 0; i < nDim; ++i) {
    for (int j = 0; j < nSamples; ++j) {
      u_int other = RandomUInt() % nSamples;
      swap(samples[nDim * j + i],
           samples[nDim * other + i]);
    }
  }
}
```

Stratified sampling



Stratified sampling

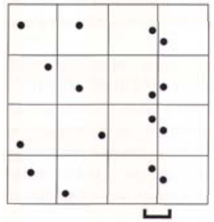
1 camera sample and 16 shadow samples per pixel



16 camera samples and each with 1 shadow sample per pixel

Low discrepancy sampling

- A possible problem with stratified sampling



- Discrepancy can be used to evaluate the quality of patterns

Low discrepancy sampling

a family of shapes

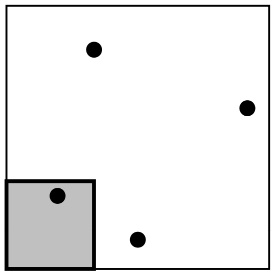
set of N sample points

maximal difference

$$D_N(B,P) = \sup_{b\in B}\left|\boxed{\frac{\#\{x_i \in b\}}{N}} - \boxed{Vol(b)}\right|$$

volume estimated by sample number

real volume



When B is the set of AABBs with a corner at the origin, this is called star discrepancy $D_N^*(P)$

1D discrepancy

$$x_i = \frac{i}{N} \quad \Rightarrow \quad D_N^*(x_1,\ldots,x_n) = \frac{1}{N}$$

$$x_i = \frac{i-0.5}{N} \quad \Rightarrow \quad D_N^*(x_1,\ldots,x_n) = \frac{1}{2N}$$

$$x_i = general \quad \Rightarrow \quad D_N^*(x_1,\ldots,x_n) = \frac{1}{2N} + \max_{1\le i\le N}\left|x_i - \frac{2i-1}{2N}\right|$$

Uniform is optimal! However, we have learnt that Irregular patterns are perceptually superior to uniform samples. Fortunately, for higher dimension, the low-discrepancy patterns are less uniform and works reasonably well as sample patterns in practice.

Next, we introduce methods specifically designed for generating low-discrepancy sampling patterns.

Radical inverse

- A positive number n can be expressed in a base b as

$$n = a_k \ldots a_2 a_1 = a_1 b^0 + a_2 b^1 + a_3 b^2 + \ldots$$

- A radical inverse function in base b converts a nonnegative integer n to a floating-point number in $[0,1)$

$$\Phi_b(n) = 0.a_1 a_2 \ldots a_k = a_1 b^{-1} + a_2 b^{-2} + a_3 b^{-3} + \ldots$$

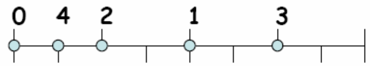
```
inline double RadicalInverse(int n, int base) {
  double val = 0;
  double invBase = 1. / base, invBi = invBase;
  while (n > 0) {
    int d_i = (n % base);
    val += d_i * invBi;
    n /= base;
    invBi *= invBase;
  }
  return val;
}
```

$a_k \ldots a_2 a_1$ $\qquad$ $0.a_1 a_2 \ldots a_k$

van der Corput sequence

- The simplest sequence $x_i = \Phi_2(i)$
- Recursively split 1D line in half, sample centers
- Achieve minimal possible discrepancy

$$D_N^*(P) = O\left(\frac{\log N}{N}\right)$$



i	binary form of i	radical inverse	x_i
0	0	0.0	0
1	1	0.1	0.5
2	10	0.01	0.25
3	11	0.11	0.75
4	100	0.001	0.125
5	101	0.101	0.625
6	110	0.011	0.375

High-dimensional sequence

- Two well-known low-discrepancy sequences
  - Halton
  - Hammersley

Halton sequence

- Use relatively prime numbers as bases for each dimension

recursively split the dimension into p_d parts, sample centers

$$x_i = (\Phi_2(i), \Phi_3(i), \Phi_5(i), \ldots, \Phi_{p_d}(i))$$

- Achieve best possible discrepancy for N-D

$$D_N^*(P) = O\left(\frac{(\log N)^d}{N}\right)$$

- Can be used if N is not known in advance
- All prefixes of a sequence are well distributed so as additional samples are added to the sequence, low discrepancy will be maintained

Hammersley sequence

- Similar to Halton sequence.
- Slightly better discrepancy than Halton.
- Needs to know N in advance.

$$x_i = \left(\frac{i-1/2}{N}, \Phi_{b_1}(i), \Phi_{b_2}(i), \ldots, \Phi_{b_{d-1}}(i)\right)$$

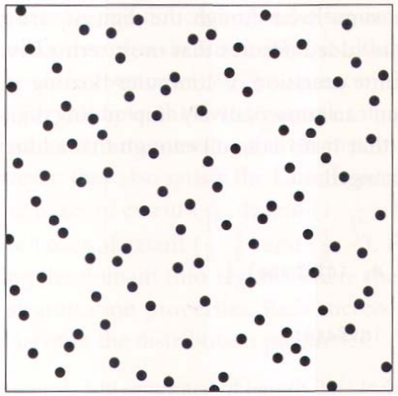
Folded radical inverse

- Add the offset i to the ith digit d_i and take the modulus b.

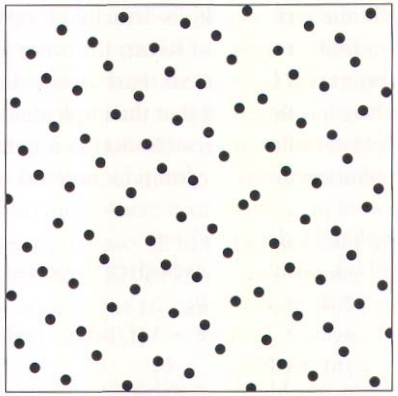
$$\Phi_b(n) = \sum_{i=1}^{\infty} ((a_i + i - 1) \bmod b) \frac{1}{b^i}$$

- It can be used to improve Hammersley and Halton, called Hammersley-Zaremba and Halton-Zaremba.

Radial inverse



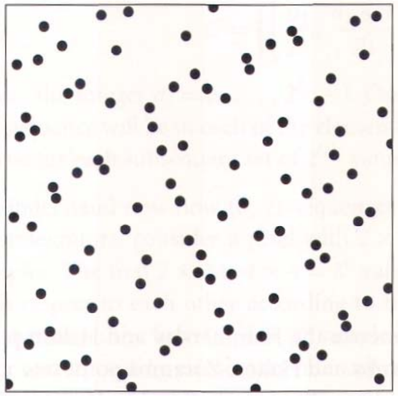
Halton



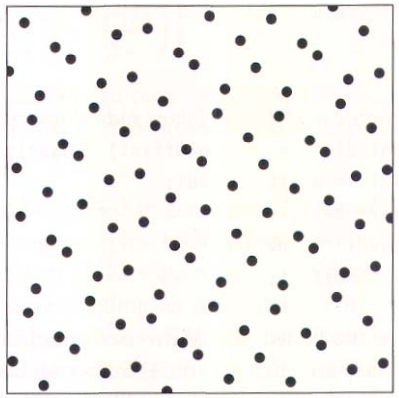
Hammersley

Better for that there are fewer clumps.

Folded radial inverse



Halton

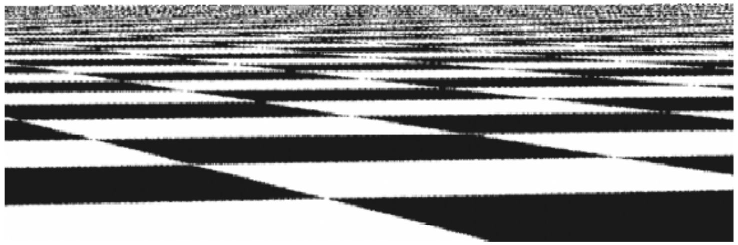


Hammersley

The improvement is more obvious

Low discrepancy sampling

stratified jittered, 1 sample/pixel

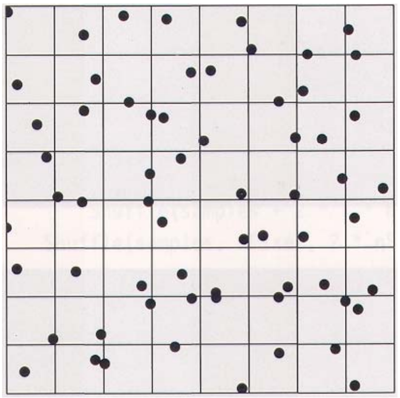


Hammersley sequence, 1 sample/pixel

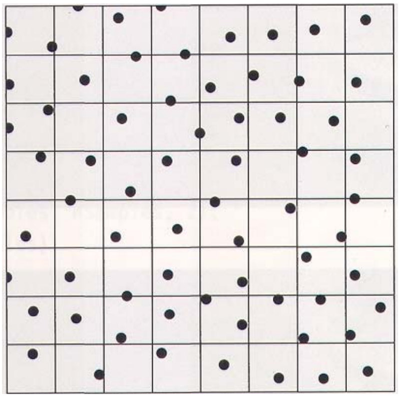
Best candidate sampling

- Stratified sampling doesn't guarantee good sampling across pixels.
- *Poisson disk pattern* addresses this issue. The Poisson disk pattern is a group of points with no two of them closer to each other than some specified distance.
- It can be generated by *dart throwing*. It is time-consuming.
- *Best-candidate* algorithm by Dan Mitchell. It randomly generates many candidates but only inserts the one farthest to all previous samples.

Best candidate sampling



stratified jittered



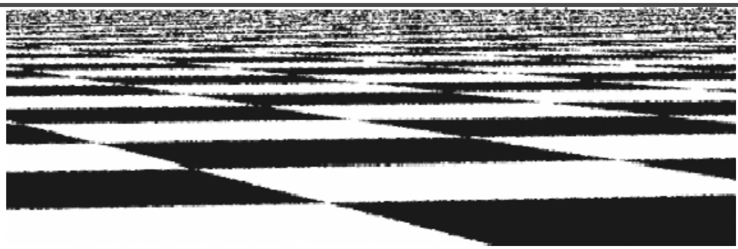
best candidate

It avoids holes and clusters.

Best candidate sampling

- Because of it is costly to generate best candidate pattern, pbrt computes a "tilable pattern" offline (by treating the square as a rolled torus).
- **`tools/samplepat.cpp`→`sampler/sampledata.cpp`**

Best candidate sampling



stratified jittered, 1 sample/pixel

best candidate, 1 sample/pixel

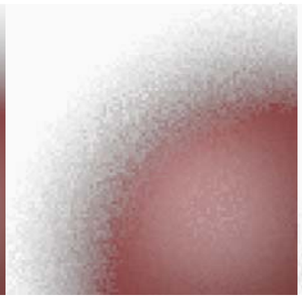
Best candidate sampling

stratified jittered, 4 sample/pixel

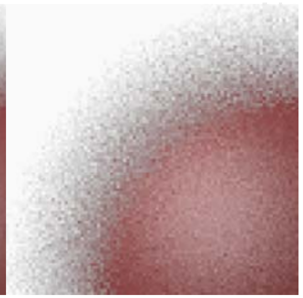
best candidate, 4 sample/pixel

Comparisons

reference



low-discrepancy



best candidate

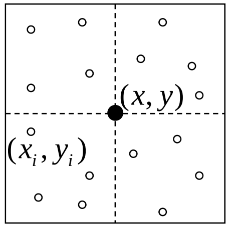
Reconstruction filters

- Given the *chosen* image samples, we can do the following to compute pixel values.
    1. reconstruct a continuous function L' from samples
    2. prefilter L' to remove frequency higher than Nyquist limit
    3. sample L' *at pixel locations*
- Because we will only sample L' at pixel locations, we do not need to explicitly reconstruct L's. Instead, we combine the first two steps.

Reconstruction filters

- Ideal reconstruction filters do not exist because of discontinuity in rendering. We choose nonuniform sampling, trading off noise for aliasing. There is no theory about ideal reconstruction for nonuniform sampling yet.
- Instead, we consider an interpolation problem

$$\underbrace{I(x,y)}_{\text{final value}} = \frac{\sum_i \overbrace{f(x-x_i,y-y_i)}^{\text{filter}}\,\overbrace{L(x_i,y_i)}^{\text{sampled radiance}}}{\sum_i f(x-x_i,y-y_i)}$$



Filter

- provides an interface to *f(x,y)*
- **`Film`** stores a pointer to a filter and use it to filter the output before writing it to disk.

width, half of support

```
Filter::Filter(float xw, float yw)
Float Evaluate(float x, float y);
```

$f(x,y)$ x, y is guaranteed to be within the range; range checking is not necessary

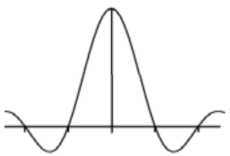
- **`filters/* (box, gaussian, mitchell, sinc, triangle)`**

Box filter

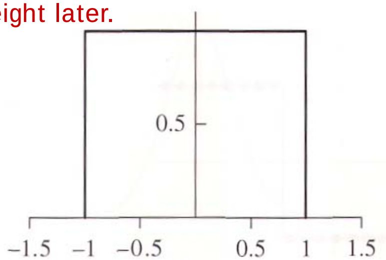
- Most commonly used in graphics. It's just about the worst filter possible, incurring postaliasing by high-frequency leakage.

```
Float BoxFilter::Evaluate(float x, float y)
{
  return 1.;
}
```

no need to normalize since the weighted sum is divided by the total weight later.

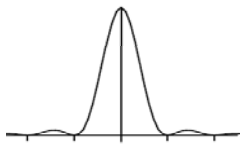


Sinc

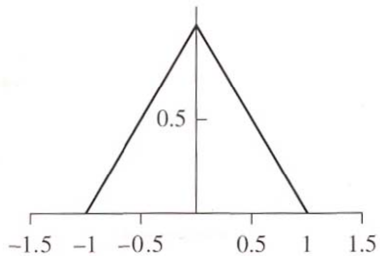


Triangle filter

```
Float TriangleFilter::Evaluate(float x, float y)
{
  return max(0.f, xWidth-fabsf(x)) *
         max(0.f, yWidth-fabsf(y));
}
```



Sinc^2

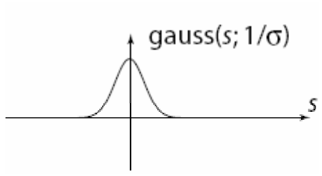


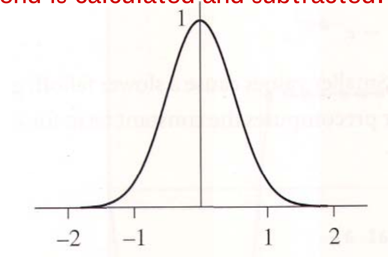
Gaussian filter

- Gives reasonably good results in practice

```
Float GaussianFilter::Evaluate(float x, float y)
{
  return Gaussian(x, expX)*Gaussian(y, expY);
}
```

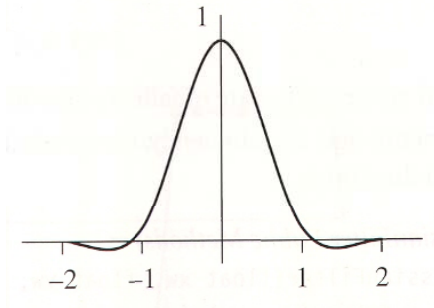
Gaussian essentially has a infinite support; to compensate this, the value at the end is calculated and subtracted.





Mitchell filter

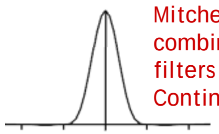
- parametric filters, tradeoff between ringing and blurring
- Negative lobes improve sharpness; ringing starts to enter the image if they become large.



Mitchell filter

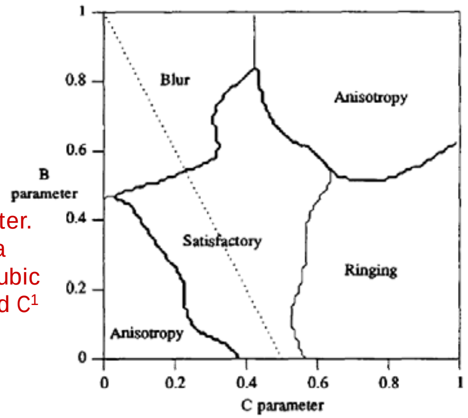
$$h(x)=\frac{1}{6}\begin{cases}(12-9B-6C)x^3+(-18+12B+6C)x^2+(6-2B) & |x|<1\\ (-B-6C)x^3+(6B+30C)x^2+(-12B-48C)x+(8B+24C) & 1<|x|<2\\ 0 & otherwise\end{cases}$$

- Separable filter
- Two parameters, B and C, B+2C=1 suggested



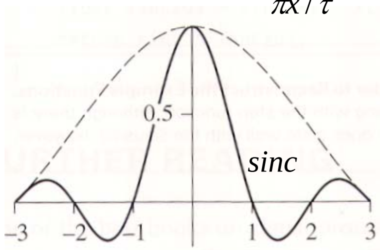
Sinc[4]

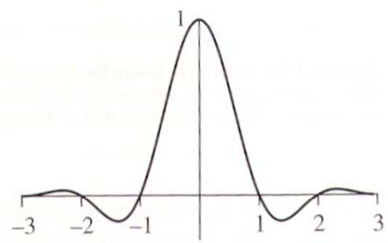
FFT of a cubic filter. Mitchell filter is a combination of cubic filters with C^0 and C^1 Continuity.



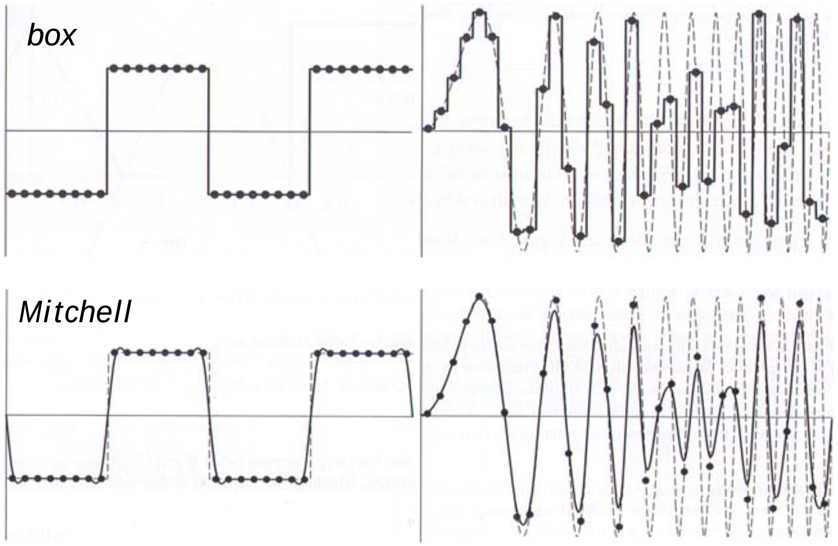
Windowed sinc filter

$$Lanczos\quad w(x)=\frac{\sin \pi x/\tau}{\pi x/\tau}$$

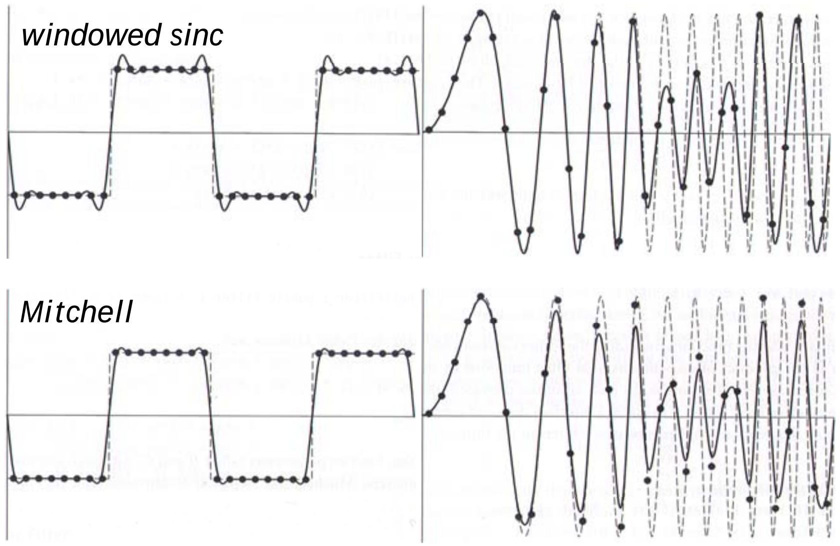




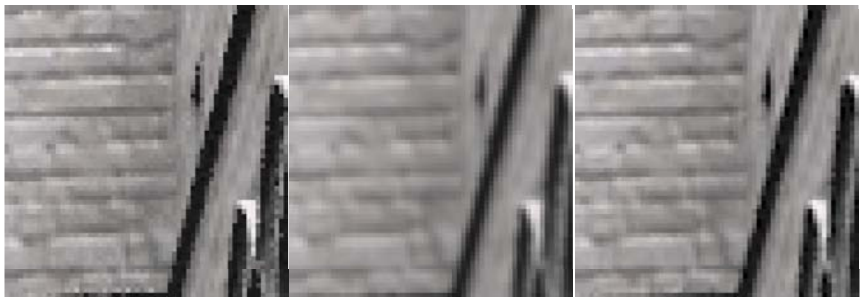
Comparisons



Comparisons



Comparisons



box *Gaussian* *Mitchell*