

Monte Carlo Integration

Digital Image Synthesis

Yung-Yu Chuang

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with slides by Pat Hanrahan and Torsten Moller

Introduction



- The integral equations generally don't have analytic solutions, so we must turn to numerical methods.
- Standard methods like Trapezoidal integration or Gaussian quadrature are not effective for high-dimensional and discontinuous integrals.

$$L_o(\mathbf{p}, \omega_o) = L_e(\mathbf{p}, \omega_o) + \int_{s^2} f(\mathbf{p}, \omega_o, \omega_i) L_i(\mathbf{p}, \omega_i) |\cos \theta_i| d\omega_i$$

Numerical quadrature

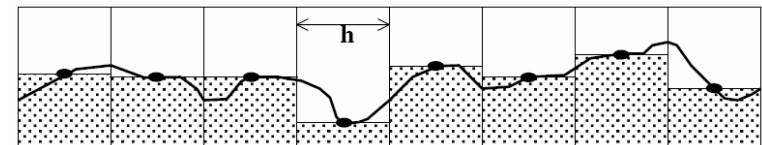


- Suppose we want to calculate $I = \int_a^b f(x) dx$, but can't solve it analytically. The approximations through quadrature rules have the form

$$\hat{I} = \sum_{i=1}^n w_i f(x_i)$$

which is essentially the weighted sum of samples of the function at various points

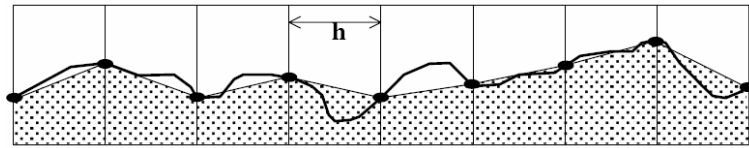
Midpoint rule



$$\begin{aligned} \hat{I} &= h \sum_{i=1}^n f(a + (i - \frac{1}{2})h) \\ &= h \left[f(a + \frac{h}{2}) + f(a + \frac{3h}{2}) + \dots + f(b - \frac{h}{2}) \right] \end{aligned}$$

convergence $\hat{I} - I = -\frac{(b-a)^3}{24n^2} f''(\xi) = O(n^{-2})$

Trapezoid rule



$$\begin{aligned}\hat{I} &= \sum_{i=1}^n \frac{h}{2} [f(a + (i-1)h) + f(a + ih)] \\ &= h \left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \dots + f(b-h) + \frac{1}{2}f(b) \right]\end{aligned}$$

convergence $\hat{I} - I = \frac{(b-a)^3}{12n^2} f''(\xi^*) = O(n^{-2})$

Simpson's rule



- Similar to trapezoid but using a quadratic polynomial approximation

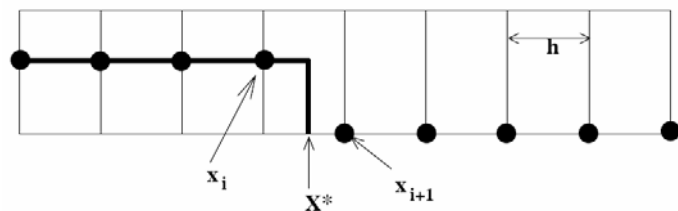
$$\begin{aligned}\hat{I} &= h \left[\frac{1}{3}f(a) + \frac{4}{3}f(a+h) + \frac{2}{3}f(a+2h) + \frac{4}{3}f(a+3h) + \frac{2}{3}f(a+4h) + \dots \right. \\ &\quad \left. + \frac{4}{3}f(b-h) + \frac{1}{3}f(b) \right]\end{aligned}$$

convergence $|\hat{I} - I| = \frac{(b-a)^5}{180(2n)^4} f^{(4)}(\xi) = O(n^{-4})$
assuming f has a continuous fourth derivative.

Curse of dimensionality and discontinuity



- For a sd function f , $\hat{I} = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_s=1}^n w_{i_1} w_{i_2} \dots w_{i_s} f(x_{i_1}, x_{i_2}, \dots, x_{i_s})$
- If the 1d rule has a convergence rate of $O(n^{-r})$, the sd rule would require a much larger number (n^s) of samples to work as well as the 1d one. Thus, the convergence rate is only $O(n^{-r/s})$.
- If f is discontinuous, convergence is $O(n^{-1/s})$ for sd .



Randomized algorithms



- *Las Vegas* v.s. *Monte Carlo*
- *Las Vegas*: gives the right answer by using randomness.
- *Monte Carlo*: gives the right answer *on the average*. Results depend on random numbers used, but statistically likely to be close to the right answer.

Monte Carlo integration



- Monte Carlo integration: uses sampling to estimate the values of integrals. It only requires to be able to evaluate the integrand at arbitrary points, making it *easy to implement* and *applicable to many problems*.
- If n samples are used, it converges at the rate of $O(n^{-1/2})$. That is, to cut the error in half, it is necessary to evaluate four times as many samples.
- Images by Monte Carlo methods are often noisy.

Monte Carlo methods



- Advantages
 - Easy to implement
 - Easy to think about (but be careful of statistical bias)
 - Robust when used with complex integrands and domains (shapes, lights, ...)
 - Efficient for high dimensional integrals
- Disadvantages
 - Noisy
 - Slow (many samples needed for convergence)

Basic concepts



- X is a random variable
- Applying a function to a random variable gives another random variable, $Y=f(X)$.
- CDF (cumulative distribution function)

$$P(x) \equiv \Pr\{X \leq x\}$$

- PDF (probability density function): nonnegative, sum to 1

$$p(x) \equiv \frac{dP(x)}{dx}$$

- canonical uniform random variable ξ (provided by standard library and easy to transform to other distributions)

Discrete Probability Distributions



- Discrete events X_i with probability p_i

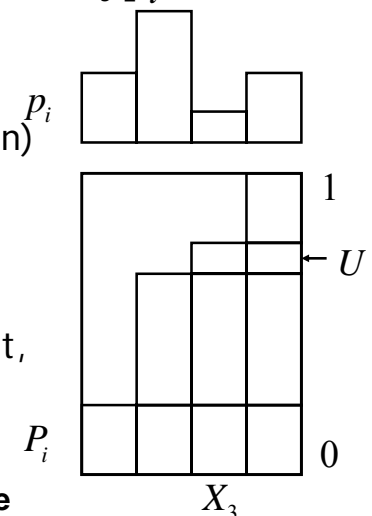
$$p_i \geq 0 \quad \sum_{i=1}^n p_i = 1$$

- Cumulative PDF (distribution)

$$P_j = \sum_{i=1}^j p_i$$

- Construction of samples
- To randomly select an event,
- Select X_i if $P_{i-1} < U \leq P_i$

Uniform random variable



Continuous Probability Distributions



- PDF $p(x)$

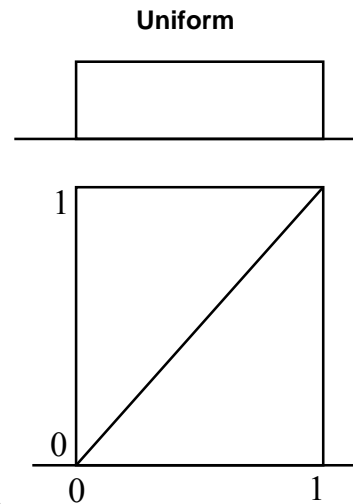
$$p(x) \geq 0$$

- CDF $P(x)$

$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x) \quad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) dx \\ = P(\beta) - P(\alpha)$$



Expected values

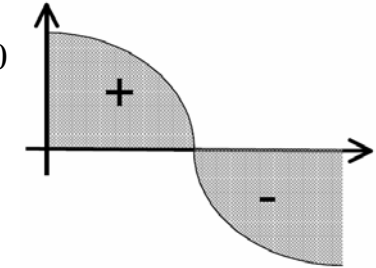


- Average value of a function $f(x)$ over some distribution of values $p(x)$ over its domain D

$$E_p[f(x)] = \int_D f(x) p(x) dx$$

- Example: cos function over $[0, \pi]$, p is uniform

$$E_p[\cos(x)] = \int_0^{\pi} \cos x \frac{1}{\pi} dx = 0$$



Variance



- Expected deviation from the expected value
- Fundamental concept of quantifying the error in Monte Carlo methods

$$V[f(x)] = E[(f(x) - E[f(x)])^2]$$

Properties



$$E[af(x)] = aE[f(x)]$$

$$E\left[\sum_i f(X_i)\right] = \sum_i E[f(X_i)]$$

$$V[af(x)] = a^2V[f(x)]$$

$$\longrightarrow V[f(x)] = E[(f(x))^2] - E[f(x)]^2$$

Monte Carlo estimator



- Assume that we want to evaluate the integral of $f(x)$ over $[a,b]$ $\int_a^b f(x)dx$
- Given a uniform random variable X_i over $[a,b]$, Monte Carlo estimator

$$F_N = \frac{b-a}{N} \sum_{i=1}^N f(X_i)$$

says that the expected value $E[F_N]$ of the estimator F_N equals the integral

$$\begin{aligned} E[F_N] &= E\left[\frac{b-a}{N} \sum_{i=1}^N f(X_i)\right] \\ &= \frac{b-a}{N} \sum_{i=1}^N E[f(X_i)] \\ &= \frac{b-a}{N} \sum_{i=1}^N \int_a^b f(x)p(x)dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b f(x)dx \\ &= \int_a^b f(x)dx \end{aligned}$$

General Monte Carlo estimator



- Given a random variable X drawn from an arbitrary PDF $p(x)$, then the estimator is

$$\begin{aligned} F_N &= \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)} & E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}\right] \\ & & &= \frac{1}{N} \sum_{i=1}^N \int_a^b \frac{f(x)}{p(x)} p(x)dx \\ & & &= \int_a^b f(x)dx \end{aligned}$$

- Although the converge rate of MC estimator is $O(N^{1/2})$, slower than other integral methods, its converge rate is independent of the dimension, making it the only practical method for high dimensional integral

Convergence of Monte Carlo



- Chebyshev's inequality: let X be a random variable with expected value μ and variance σ^2 . For any real number $k > 0$,

$$\Pr\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

- For example, for $k = \sqrt{2}$, it shows that at least half of the value lie in the interval $(\mu - \sqrt{2}\sigma, \mu + \sqrt{2}\sigma)$
- Let $Y_i = f(X_i)/p(X_i)$, the MC estimate F_N becomes

$$F_N = \frac{1}{N} \sum_{i=1}^N Y_i$$

Convergence of Monte Carlo



- According to Chebyshev's inequality,

$$\Pr\{|F_N - E[F_N]| \geq \left(\frac{V[F_N]}{\delta}\right)^{1/2}\} \leq \delta$$

$$V[F_N] = V\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] = \frac{1}{N^2} V\left[\sum_{i=1}^N Y_i\right] = \frac{1}{N^2} \sum_{i=1}^N V[Y_i] = \frac{1}{N} V[Y]$$

- Plugging into Chebyshev's inequality,

$$\Pr\{|F_N - I| \geq \frac{1}{\sqrt{N}} \left(\frac{V[Y]}{\delta}\right)^{1/2}\} \leq \delta$$

So, for a fixed threshold, the error decreases at the rate $N^{-1/2}$.

Properties of estimators



- An estimator F_N is called unbiased if for all N

$$E[F_N] = Q$$

That is, the expected value is independent of N .

- Otherwise, the bias of the estimator is defined as

$$\beta[F_N] = E[F_N] - Q$$

- If the bias goes to zero as N increases, the estimator is called consistent

$$\lim_{N \rightarrow \infty} \beta[F_N] = 0$$

$$\lim_{N \rightarrow \infty} E[F_N] = Q$$

Example of a biased consistent estimator



- Suppose we are doing antialiasing on a 1d pixel, to determine the pixel value, we need to evaluate $I = \int_0^1 w(x)f(x)dx$, where $w(x)$ is the filter function with $\int_0^1 w(x)dx = 1$
- A common way to evaluate this is

$$F_N = \frac{\sum_{i=1}^N w(X_i)f(X_i)}{\sum_{i=1}^N w(X_i)}$$

- When $N=1$, we have

$$E[F_1] = E\left[\frac{w(X_1)f(X_1)}{w(X_1)}\right] = E[f(X_1)] = \int_0^1 f(x)dx \neq I$$

Example of a biased consistent estimator



- When $N=2$, we have

$$E[F_2] = \int_0^1 \int_0^1 \frac{w(x_1)f(x_1) + w(x_2)f(x_2)}{w(x_1) + w(x_2)} dx_1 dx_2 \neq I$$

- However, when N is very large, the bias approaches to zero

$$F_N = \frac{\frac{1}{N} \sum_{i=1}^N w(X_i)f(X_i)}{\frac{1}{N} \sum_{i=1}^N w(X_i)}$$

$$\lim_{N \rightarrow \infty} E[F_N] = \frac{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N w(X_i)f(X_i)}{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N w(X_i)} = \frac{\int_0^1 w(x)f(x)dx}{\int_0^1 w(x)dx} = \int_0^1 w(x)f(x)dx = I$$

Choosing samples



- $F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$
- Carefully choosing the PDF from which samples are drawn is an important technique to reduce variance. We want the f/p to have a low variance. Hence, it is necessary to be able to draw samples from the chosen PDF.
- How to sample an arbitrary distribution from a variable of uniform distribution?
 - Inversion
 - Transform
 - Rejection

Inversion method



- Cumulative probability distribution function

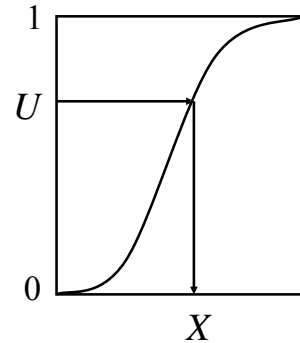
$$P(x) = \Pr(X < x)$$

- Construction of samples

$$\text{Solve for } X = P^{-1}(U)$$

- Must know:

- The integral of $p(x)$
- The inverse function $P^{-1}(x)$



Proof for the inversion method



- Let U be an uniform random variable and its CDF is $P_u(x)=x$. We will show that $Y=P^{-1}(U)$ has the CDF $P(x)$.

$$\Pr\{Y \leq x\} = \Pr\{P^{-1}(U) \leq x\} = \Pr\{U \leq P(x)\} = P_u(P(x)) = P(x)$$

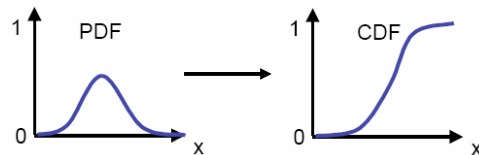
because P is monotonic

Thus, Y 's CDF is exactly $P(x)$.

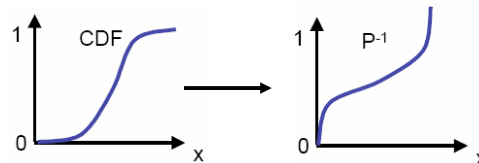
Inversion method



- Compute CDF $P(x)$



- Compute $P^{-1}(x)$



- Obtain ξ
- Compute $X_i = P^{-1}(\xi)$

Example: Power Function



It is used in sampling Blinn's microfacet model.

- Assume

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick (It only works for sampling power distribution)

$$Y = \max(U_1, U_2, \dots, U_n, U_{n+1})$$

Similarly, a trick to obtain a Gaussian distribution is to take average.

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U_i < x) = x^{n+1}$$

Example: exponential distribution



$p(x) = ce^{-ax}$ useful for rendering participating media.

$$\int_0^{\infty} ce^{-ax} dx = 1 \rightarrow c = a$$

- Compute CDF $P(x)$

$$P(x) = \int_0^x ae^{-as} ds = 1 - e^{-ax}$$

- Compute $P^{-1}(x)$

$$P^{-1}(x) = -\frac{1}{a} \ln(1-x)$$

- Obtain ξ

- Compute $X_i = P^{-1}(\xi)$
- $$X = -\frac{1}{a} \ln(1-\xi) = -\frac{1}{a} \ln \xi$$

Transformation of variables



- Given a random variable X from distribution $p_x(x)$ to a random variable $Y=y(X)$, where $Y=y(X)$ and y is one-to-one, i.e. monotonic. We want to derive the distribution of Y , $p_y(y)$.

- $P_y(y) = \Pr\{Y \leq y(x)\} = \Pr\{X \leq x\} = P_x(x)$

- PDF:

$$\frac{dP_y(y)}{dy} = \frac{dP_x(x)}{dx} \frac{dx}{dy}$$

$$p_y(y) \frac{dy}{dx} = \frac{dP_y(y)}{dy} \frac{dy}{dx} = p_x(x)$$

$$p_y(y) = \left(\frac{dy}{dx}\right)^{-1} p_x(x)$$

Example



$$p_x(x) = 2x$$

$$Y = \sin X$$

$$p_y(y) = (\cos x)^{-1} p_x(x) = \frac{2x}{\cos x} = \frac{2 \sin^{-1} y}{\sqrt{1-y^2}}$$

Transformation method



- A problem to apply the above method is that we usually have some PDF to sample from, not a given transformation.
- Given a source random variable X with $p_x(x)$ and a target distribution $p_y(y)$, try transform X into to another random variable Y so that Y has the distribution $p_y(y)$.
- We first have to find a transformation $y(x)$ so that $P_x(x) = P_y(y)$. Thus,

$$y(x) = P_y^{-1}(P_x(x))$$

Transformation method



- Let's prove that the above transform works.

We first prove that the random variable $Z = P_x(X)$ has a uniform distribution. If so, then $P_y^{-1}(Z)$ should have distribution $P_x(x)$ from the inversion method.

$$\Pr\{Z \leq x\} = \Pr\{P_x(X) \leq x\} = \Pr\{X \leq P_x^{-1}(x)\} = P_x(P_x^{-1}(x)) = x$$

Thus, Z is uniform and the transformation works.

- It is an obvious generalization of the inversion method, in which X is uniform and $P_x(x) = x$.

Example



$$p_x(x) = x \longrightarrow p_y(y) = e^y$$

$$P_x(x) = \frac{x^2}{2} \quad P_y(y) = e^y$$
$$P_y^{-1}(y) = \ln y$$

$$y(x) = P_y^{-1}(P_x(x)) = \ln\left(\frac{x^2}{2}\right) = 2 \ln x - \ln 2$$

Thus, if X has the distribution $p_x(x) = x$, then the random variable $Y = 2 \ln X - \ln 2$ has the distribution $p_y(y) = e^y$

Multiple dimensions



- Easily generalized - using the Jacobian of $Y = T(X)$
$$p_y(T(x)) = |J_T(x)|^{-1} p_x(x)$$

- Example - polar coordinates
$$x = r \cos \theta$$
$$y = r \sin \theta$$

$$J_T(x) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$p(x, y) = r^{-1} p(r, \theta)$$

We often need the other way around, $p(r, \theta) = r p(x, y)$

Spherical coordinates



- The spherical coordinate representation of directions is

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$|J_T| = r^2 \sin \theta$$

$$p(r, \theta, \phi) = r^2 \sin \theta p(x, y, z)$$

Spherical coordinates



- Spherical coordinates:

$$p(r, \theta, \phi) = r^2 \sin \theta p(x, y, z)$$

- Now looking at spherical directions:
- We want to solid angle to be uniformly distributed $d\omega = \sin \theta d\theta d\phi$
- Hence the density in terms of ϕ and θ :

$$p(\theta, \phi) d\theta d\phi = p(\omega) d\omega$$

$$p(\theta, \phi) = \sin \theta p(\omega)$$

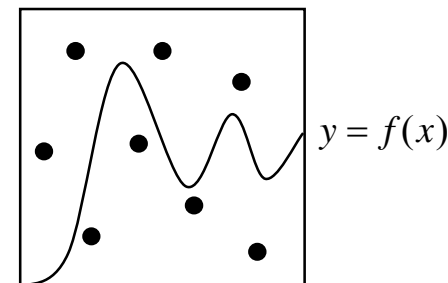
Rejection method



- Sometimes, we can't integrate into CDF or invert CDF

$$I = \int_0^1 f(x) dx$$

$$= \iint_{y < f(x)} dx dy$$



- Algorithm

Pick U_1 and U_2

Accept U_1 if $U_2 < f(U_1)$

- Wasteful? **Efficiency = Area / Area of rectangle**

Rejection method



- Rejection method is a dart-throwing method without performing the above steps

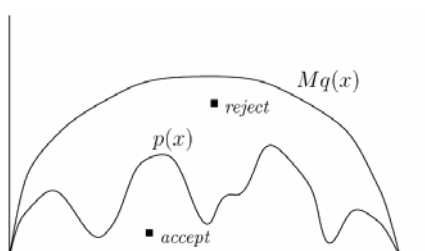
1. Find $q(x)$ so that $p(x) < Mq(x)$

2. Dart throwing

a. Choose a pair (X, ξ) , where X is sampled from $q(x)$

b. If $(\xi < p(X)/Mq(X))$ return X

- Essentially, we pick a point $(X, \xi Mq(X))$. If it lies beneath $p(X)$ then we are fine.



Why it works



- For each iteration, we generate X_i from q . The sample is returned if $\xi < p(X)/Mq(X)$, which happens with probability $p(X)/Mq(X)$.

- So, the probability to return x is

$$q(x) \frac{p(x)}{Mq(x)} = \frac{p(x)}{M}$$

- Thus, when a sample is returned (probability $1/M$), then X_i is distributed according to $p(x)$.

Example: sampling a unit sphere



```
void RejectionSampleDisk(float *x, float *y) {
    float sx, sy;
    do {
        sx = 1.f - 2.f * RandomFloat();
        sy = 1.f - 2.f * RandomFloat();
    } while (sx*sx + sy*sy > 1.f)
    *x = sx; *y = sy;
}
```

$\pi/4 \sim 78.5\%$ good samples, gets worse in higher dimensions, for example, for sphere, $\pi/8 \sim 39.3\%$

Multidimensional sampling



- Separable case - independently sample X from p_x and Y from p_y : $p(x, y) = p_x(x)p_y(y)$
- Often times this is not possible - compute the marginal density function $p(x)$ first:

$$p(x) = \int p(x, y) dy$$

- Then compute conditional density function (p of y given x) $p(y|x) = \frac{p(x, y)}{p(x)}$
- Use 1D sampling with $p(x)$ and $p(y|x)$

Sampling a hemisphere



- Uniformly, I.e. $p(\omega) = c$

$$1 = \int_{H^2} p(\omega) \quad c = \frac{1}{2\pi} \rightarrow p(\omega) = \frac{1}{2\pi}$$

- Sampling θ first:

$$p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \int_0^{2\pi} \frac{\sin\theta}{2\pi} d\phi = \sin\theta$$
$$p(\theta, \phi) = \frac{\sin\theta}{2\pi}$$

- Now sampling in ϕ :

$$p(\phi|\theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi}$$

Sampling a hemisphere



- Now we use inversion technique in order to sample the PDF's:

$$P(\theta) = \int_0^\alpha \sin\alpha d\alpha = 1 - \cos\theta$$

$$P(\phi|\theta) = \int_0^\alpha \frac{1}{2\pi} d\alpha = \frac{\phi}{2\pi}$$

- Inverting these:

$$\theta = \cos^{-1} \xi_1$$

$$\phi = 2\pi\xi_2$$

Sampling a hemisphere



- Converting these to Cartesian coords:

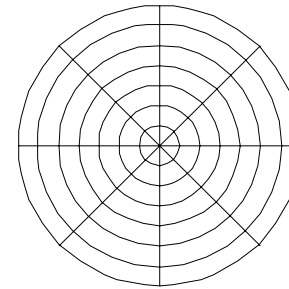
$$\begin{aligned} \theta &= \cos^{-1} \xi_1 & x &= \sin \theta \cos \phi = \cos(2\pi\xi_2) \sqrt{1 - \xi_1^2} \\ \phi &= 2\pi\xi_2 & y &= \sin \theta \sin \phi = \sin(2\pi\xi_2) \sqrt{1 - \xi_1^2} \\ & & z &= \cos \theta = \xi_1 \end{aligned}$$

- Similar derivation for a full sphere

Sampling a disk

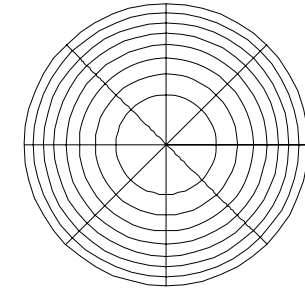


WRONG \neq Equi-Areal



$$\begin{aligned} \theta &= 2\pi U_1 \\ r &= U_2 \end{aligned}$$

RIGHT = Equi-Areal

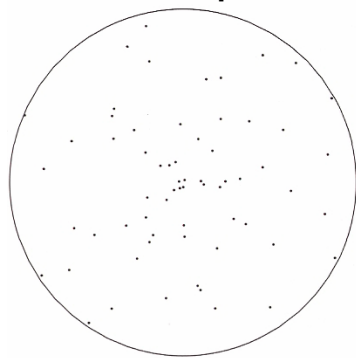


$$\begin{aligned} \theta &= 2\pi U_1 \\ r &= \sqrt{U_2} \end{aligned}$$

Sampling a disk

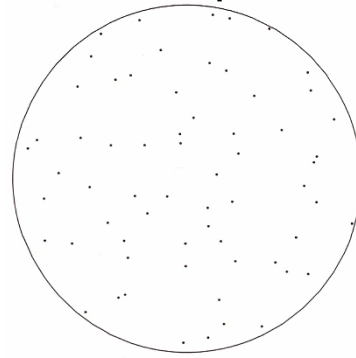


WRONG \neq Equi-Areal



$$\begin{aligned} \theta &= 2\pi U_1 \\ r &= U_2 \end{aligned}$$

RIGHT = Equi-Areal



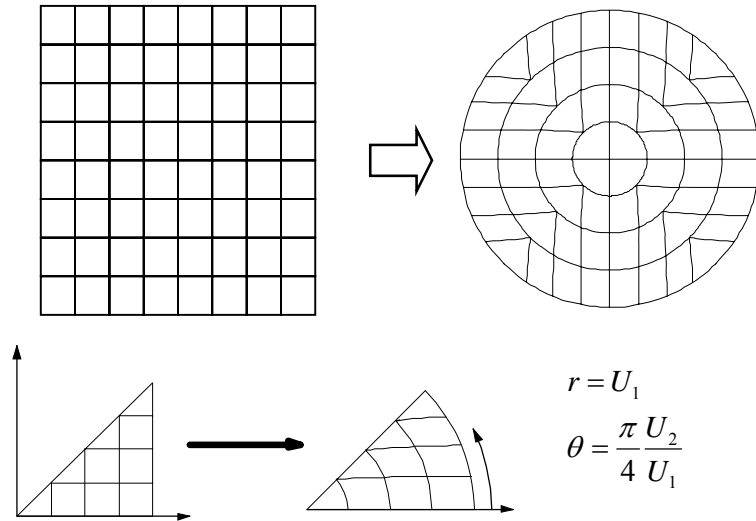
$$\begin{aligned} \theta &= 2\pi U_1 \\ r &= \sqrt{U_2} \end{aligned}$$

Sampling a disk



- Uniformly: $p(x, y) = \frac{1}{\pi}$ $p(r, \theta) = rp(x, y) = \frac{r}{\pi}$
- Sampling r first: $p(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r$
- Then sampling in θ : $p(\theta | r) = \frac{p(r, \theta)}{p(r)} = \frac{1}{2\pi}$
- Inverting the CDF: $P(r) = r^2$ $P(\theta | r) = \frac{\theta}{2\pi}$
 $r = \sqrt{\xi_1}$ $\theta = 2\pi\xi_2$

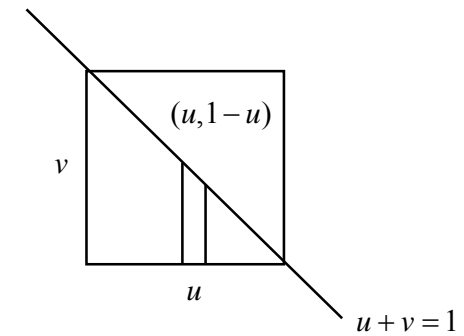
Shirley's mapping



Sampling a Triangle



$$\begin{aligned}
 u &\geq 0 \\
 v &\geq 0 \\
 u + v &\leq 1
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_0^1 \int_0^{1-u} dv du = \int_0^1 (1-u) du = -\frac{(1-u)^2}{2} \Big|_0^1 = \frac{1}{2} \\
 p(u, v) &= 2
 \end{aligned}$$

Sampling a Triangle



- Here u and v are not independent! $p(u, v) = 2$
- Conditional probability

$$p(u) \equiv \int p(u, v) dv \quad p(u | v) \equiv \frac{p(u, v)}{p(v)}$$

$$p(u) = 2 \int_0^{1-u} dv = 2(1-u)$$

$$P(u_0) = \int_0^{u_0} 2(1-u) du = (1-u_0)^2$$

$$u_0 = 1 - \sqrt{U_1}$$

$$p(v | u) = \frac{1}{(1-u)}$$

$$v_0 = \sqrt{U_1} U_2$$

$$P(v_0 | u_0) = \int_0^{v_0} p(v | u_0) dv = \int_0^{v_0} \frac{1}{(1-u_0)} dv = \frac{v_0}{(1-u_0)}$$

Cosine weighted hemisphere



$$p(\omega) \propto \cos \theta$$

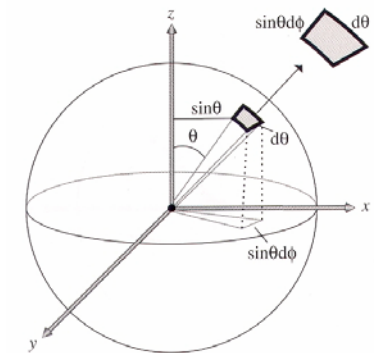
$$1 = \int_{\Omega} p(\omega) d\omega$$

$$1 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \cos \theta \sin \theta d\theta d\phi$$

$$1 = c 2\pi \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta$$

$$c = \frac{1}{\pi}$$

$$p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta$$



$$d\omega = \sin \theta d\theta d\phi$$

Cosine weighted hemisphere



$$p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta$$

$$p(\theta) = \int_0^{2\pi} \frac{1}{\pi} \cos \theta \sin \theta d\phi = 2 \cos \theta \sin \theta = \sin 2\theta$$

$$p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi}$$

$$P(\theta) = -\frac{1}{2} \cos 2\theta + \frac{1}{2} = \xi_1 \quad \theta = \frac{1}{2} \cos^{-1}(1 - 2\xi_1)$$

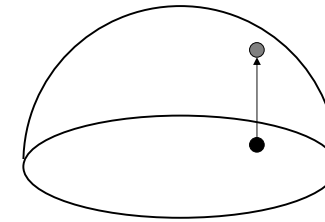
$$P(\phi | \theta) = \frac{\phi}{2\pi} = \xi_2 \quad \phi = 2\pi\xi_2$$

Cosine weighted hemisphere



- Malley's method: uniformly generates points on the unit disk and then generates directions by projecting them up to the hemisphere above it.

```
Vector CosineSampleHemisphere(float u1, float u2){
    Vector ret;
    ConcentricSampleDisk(u1, u2, &ret.x, &ret.y);
    ret.z = sqrtf(max(0.f, 1.f - ret.x*ret.x -
                    ret.y*ret.y));
    return ret;
}
```



Cosine weighted hemisphere



- Why Malley's method works?
- Unit disk sampling $p(r, \phi) = \frac{r}{\pi}$
- Map to hemisphere $(r, \phi) \rightarrow (\sin \theta, \phi)$
 $Y = T(X)$ here $Y = (r, \phi), X = (\theta, \phi)$

$$p_y(T(x)) = |J_T(x)|^{-1} p_x(x)$$

$$|J_T(x)| = \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} = \cos \theta$$

$$p(\theta, \phi) = |J_T| p(r, \phi) = \frac{\cos \theta \sin \theta}{\pi}$$

Sampling Phong lobe



$$p(\omega) \propto \cos^n \theta$$

$$p(\omega) = c \cos^n \theta \rightarrow \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} c \cos^n \theta \sin \theta d\theta d\phi = 1$$

$$\rightarrow -2\pi c \int_{\cos \theta=1}^0 \cos^n \theta d \cos \theta = 1 \rightarrow \frac{2\pi c}{n+1} = 1$$

$$\rightarrow c = \frac{n+1}{2\pi}$$

$$p(\theta, \phi) = \frac{n+1}{2\pi} \cos^n \theta \sin \theta$$

Sampling Phong lobe



$$p(\theta, \phi) = \frac{n+1}{2\pi} \cos^n \theta \sin \theta$$

$$p(\theta) = \int_{\phi=0}^{2\pi} \frac{n+1}{2\pi} \cos^n \theta \sin \theta d\phi = (n+1) \cos^n \theta \sin \theta$$

$$P(\theta') = \int_{\theta=0}^{\theta'} (n+1) \cos^n \theta \sin \theta d\theta$$

$$= -(n+1) \int_{\theta=0}^{\theta'} \cos^n \theta d \cos \theta = -(n+1) \frac{\cos^{n+1} \theta}{n+1} \Big|_{\cos \theta=1}^{\cos \theta'}$$

$$= 1 - \cos^{n+1} \theta'$$

$$\theta = \cos^{-1} \left(\sqrt[n+1]{\xi_1} \right)$$

Sampling Phong lobe



$$p(\theta, \phi) = \frac{n+1}{2\pi} \cos^n \theta \sin \theta$$

$$p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{\frac{n+1}{2\pi} \cos^n \theta \sin \theta}{(n+1) \cos^n \theta \sin \theta} = \frac{1}{2\pi}$$

$$P(\phi' | \theta) = \int_{\phi=0}^{\phi'} \frac{1}{2\pi} d\phi = \frac{\phi'}{2\pi}$$

$$\phi = 2\pi \xi_2$$

Sampling Phong lobe



When $n=1$, it is actually equivalent to cosine-weighted hemisphere

$$n=1, (\theta, \phi) = (\cos^{-1} \sqrt{\xi_1}, 2\pi \xi_2) \quad (\theta, \phi) = \left(\frac{1}{2} \cos^{-1}(1-2\xi_1), 2\pi \xi_2 \right)$$

$$P(\theta) = 1 - \cos^{n+1} \theta = 1 - \cos^2 \theta \quad P(\theta) = -\frac{1}{2} \cos 2\theta + \frac{1}{2}$$

$$-\frac{1}{2} \cos 2\theta + \frac{1}{2} = -\frac{1}{2} (1 - 2 \sin^2 \theta) + \frac{1}{2} = \sin^2 \theta = 1 - \cos^2 \theta$$