Codes and number systems

Introduction to Computer
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with slides by Nisan & Schocken (www.nand2tetris.org*) and Harris & Harris (DDCA)*

Coding

- Assume that you want to communicate with your friend with a flashlight in a night, what will you do?

light painting?
What's the problem?

Solution #1

- A: 1 blink
- B: 2 blinks
- C: 3 blinks

  :
- Z: 26 blinks

What's the problem?

- How are you? = 131 blinks

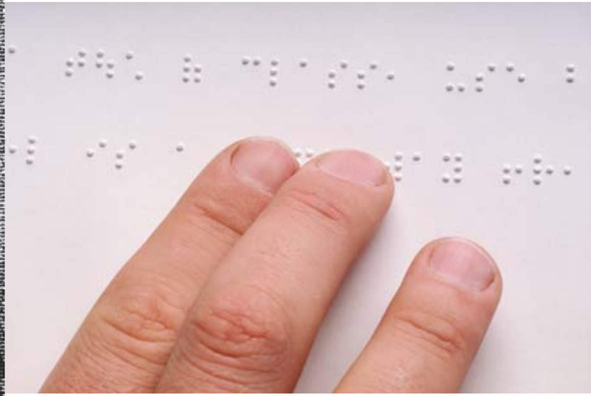
Solution #2: Morse code

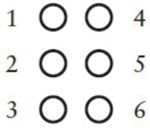
A	•▬	J	•▬▬▬	S	•••
B	▬•••	K	▬•▬	T	▬
C	▬•▬•	L	•▬••	U	••▬
D	▬••	M	▬▬	V	•••▬
E	•	N	▬•	W	•▬▬
F	••▬•	O	▬▬▬	X	▬••▬
G	▬▬•	P	•▬▬•	Y	▬•▬▬
H	••••	Q	▬▬•▬	Z	▬▬••
I	••	R	•▬•		

•••• • •▬•• •▬•• ▬▬▬

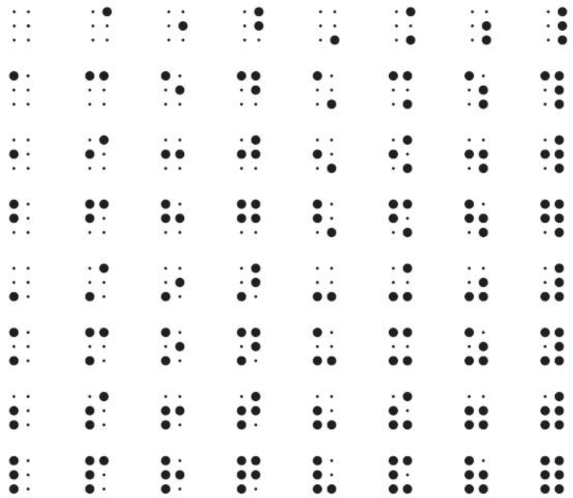
Hello

Braille





Braille

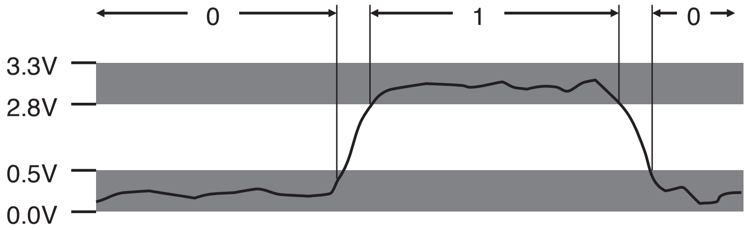


What’s common in these codes?

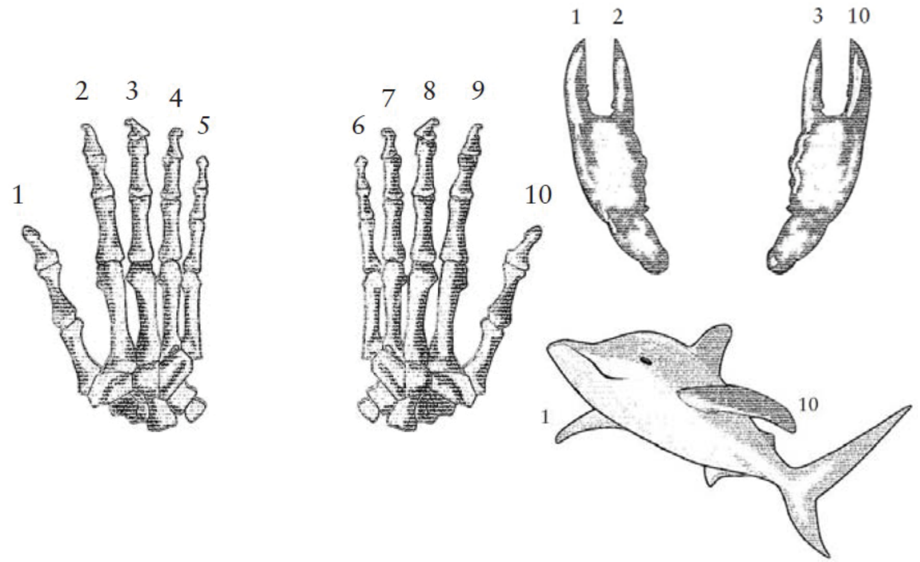
- They are both binary codes.

Binary representations

- Electronic Implementation
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



Number systems



Number Systems

- Decimal numbers

1's column
10's column
100's column
1000's column

$5374_{10} =$

- Binary numbers

1's column
2's column
4's column
8's column

$1101_2 =$

Number Systems

- Decimal numbers

1's column
10's column
100's column
1000's column

$5374_{10} = 5 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 4 \times 10^0$

five thousands, three hundreds, seven tens, four ones

- Binary numbers

1's column
2's column
4's column
8's column

$1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13_{10}$

one eight, one four, no two, one one

Binary numbers

- Digits are 1 and 0
  (a binary digit is called a bit)
  1 = true
  0 = false
- MSB –most significant bit
- LSB –least significant bit
- Bit numbering:

MSB LSB
1 0 1 1 0 0 1 0 1 0 0 1 1 1 0 0
15 0

- A bit string could have different interpretations

Powers of Two

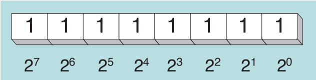
- $2^0 =$
- $2^1 =$
- $2^2 =$
- $2^3 =$
- $2^4 =$
- $2^5 =$
- $2^6 =$
- $2^7 =$
- $2^8 =$
- $2^9 =$
- $2^{10} =$
- $2^{11} =$
- $2^{12} =$
- $2^{13} =$
- $2^{14} =$
- $2^{15} =$

Powers of Two

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$
- $2^7 = 128$
- $2^8 = 256$
- $2^9 = 512$
- $2^{10} = 1024$
- $2^{11} = 2048$
- $2^{12} = 4096$
- $2^{13} = 8192$
- $2^{14} = 16384$
- $2^{15} = 32768$
- Handy to memorize up to 2^9

Unsigned binary integers

- Each digit (bit) is either 1 or 0
- Each bit represents a power of 2:



Every binary number is a sum of powers of 2

Table 1-3 Binary Bit Position Values.

2^n	Decimal Value	2^n	Decimal Value
2^0	1	2^8	256
2^1	2	2^9	512
2^2	4	2^{10}	1024
2^3	8	2^{11}	2048
2^4	16	2^{12}	4096
2^5	32	2^{13}	8192
2^6	64	2^{14}	16384
2^7	128	2^{15}	32768

Translating binary to decimal

Weighted positional notation shows how to calculate the decimal value of each binary bit:

$$dec = (D_{n-1} \times 2^{n-1}) + (D_{n-2} \times 2^{n-2}) + \ldots + (D_1 \times 2^1) + (D_0 \times 2^0)$$

D = binary digit

binary 00001001 = decimal 9:

$(1 \times 2^3) + (1 \times 2^0) = 9$

Translating unsigned decimal to binary

- Repeatedly divide the decimal integer by 2. Each remainder is a binary digit in the translated value:

Division	Quotient	Remainder
37 / 2	18	1
18 / 2	9	0
9 / 2	4	1
4 / 2	2	0
2 / 2	1	0
1 / 2	0	1

37 = 100101

Number Conversion

- Binary to decimal conversion:
  - Convert 10011_2 to decimal
- Decimal to binary conversion:
  - Convert 47_{10} to binary

Number Conversion

- Binary to decimal conversion:
  - Convert 10011_2 to decimal
  - $16 \times 1 + 8 \times 0 + 4 \times 0 + 2 \times 1 + 1 \times 1 = 19_{10}$
- Decimal to binary conversion:
  - Convert 47_{10} to binary
  - $32 \times 1 + 16 \times 0 + 8 \times 1 + 4 \times 1 + 2 \times 1 + 1 \times 1 = 101111_2$

Binary Values and Range

- *N*-digit decimal number
  - How many values?
  - Range?
  - Example: 3-digit decimal number:
- *N*-bit binary number
  - How many values?
  - Range:
  - Example: 3-digit binary number:

Binary Values and Range

- *N*-digit decimal number
  - How many values? **10^N**
  - Range? **[0, 10^N - 1]**
  - Example: 3-digit decimal number:
    - **10^3 = 1000 possible values**
    - **Range: [0, 999]**
- *N*-bit binary number
  - How many values? **2^N**
  - Range: **[0, 2^N - 1]**
  - Example: 3-digit binary number:
    - **2^3 = 8 possible values**
    - **Range: [0, 7] = [000_2 to 111_2]**

Integer storage sizes

Standard sizes:

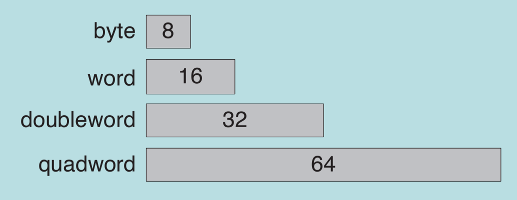


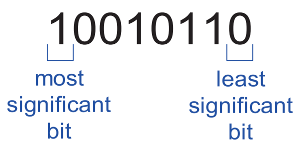
Table 1-4 Ranges of Unsigned Integers.

Storage Type	Range (low–high)	Powers of 2
Unsigned byte	0 to 255	0 to $(2^8 - 1)$
Unsigned word	0 to 65,535	0 to $(2^{16} - 1)$
Unsigned doubleword	0 to 4,294,967,295	0 to $(2^{32} - 1)$
Unsigned quadword	0 to 18,446,744,073,709,551,615	0 to $(2^{64} - 1)$

Practice: What is the largest unsigned integer that may be stored in 20 bits?

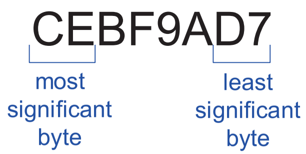
Bits, Bytes, Nibbles…

- Bits



- Bytes & Nibbles

- Bytes



Large Powers of Two

- 2^{10} = 1 kilo ≈ 1000 (1024)
- 2^{20} = 1 mega ≈ 1 million (1,048,576)
- 2^{30} = 1 giga ≈ 1 billion (1,073,741,824)

Estimating Powers of Two

- What is the value of 2^{24}?
- How many values can a 32-bit variable represent?

Estimating Powers of Two

- What is the value of 2^{24}?
  - **$2^4 \times 2^{20} \approx$ 16 million**
- How many values can a 32-bit variable represent?
  -$2^2 \times 2^{30} \approx$ 4 billion

Large measurements

- Kilobyte (KB), 2^{10} bytes
- Megabyte (MB), 2^{20} bytes
- Gigabyte (GB), 2^{30} bytes
- Terabyte (TB), 2^{40} bytes
- Petabyte
- Exabyte
- Zettabyte
- Yottabyte

Hexadecimal Numbers

Hex Digit	Decimal Equivalent	Binary Equivalent
0	0	
1	1	
2	2	
3	3	
4	4	
5	5	
6	6	
7	7	
8	8	
9	9	
A	10	
B	11	
C	12	
D	13	
E	14	
F	15	

Hexadecimal Numbers

Hex Digit	Decimal Equivalent	Binary Equivalent
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Hexadecimal Numbers

- Base 16
- Shorthand for binary

Translating binary to hexadecimal

- Each hexadecimal digit corresponds to 4 binary bits.
- Example: Translate the binary integer 000101101010011110010100 to hexadecimal:

1	6	A	7	9	4
0001	0110	1010	0111	1001	0100

Converting hexadecimal to decimal

- Multiply each digit by its corresponding power of 16:

  dec = $(D_3 \times 16^3) + (D_2 \times 16^2) + (D_1 \times 16^1) + (D_0 \times 16^0)$

- Hex 1234 equals $(1 \times 16^3) + (2 \times 16^2) + (3 \times 16^1) + (4 \times 16^0)$, or decimal 4,660.

- Hex 3BA4 equals $(3 \times 16^3) + (11 * 16^2) + (10 \times 16^1) + (4 \times 16^0)$, or decimal 15,268.

Hexadecimal to Binary Conversion

- Hexadecimal to binary conversion:
  - Convert $4AF_{16}$ (also written 0x4AF) to binary
- Hexadecimal to decimal conversion:
  - Convert 0x4AF to decimal

Hexadecimal to Binary Conversion

- Hexadecimal to binary conversion:
  - Convert $4AF_{16}$ (also written 0x4AF) to binary
  - $0100\ 1010\ 1111_2$
- Hexadecimal to decimal conversion:
  - Convert $4AF_{16}$ to decimal
  - $16^2 x4 + 16^1 x10 + 16^0 x15 = 1199_{10}$

Powers of 16

Used when calculating hexadecimal values up to 8 digits long:

16^n	Decimal Value	16^n	Decimal Value
16^0	1	16^4	65,536
16^1	16	16^5	1,048,576
16^2	256	16^6	16,777,216
16^3	4096	16^7	268,435,456

Converting decimal to hexadecimal

Division	Quotient	Remainder
422 / 16	26	6
26 / 16	1	A
1 / 16	0	1

decimal 422 = 1A6 hexadecimal

Addition

- Decimal

$$\begin{array}{r l} 11 & \leftarrow \text{carries} \\ 3734 & \\ +\ 5168 & \\ \hline 8902 & \end{array}$$

- Binary

$$\begin{array}{r l} 11 & \leftarrow \text{carries} \\ 1011 & \\ +\ 0011 & \\ \hline 1110 & \end{array}$$

Binary Addition Examples

- Add the following 4-bit binary numbers

$$\begin{array}{r} 1001 \\ +\ 0101 \\ \hline \end{array}$$

- Add the following 4-bit binary numbers

$$\begin{array}{r} 1011 \\ +\ 0110 \\ \hline \end{array}$$

Binary Addition Examples

- Add the following 4-bit binary numbers

$$\begin{array}{r} 1 \\ 1001 \\ +\ 0101 \\ \hline 1110 \end{array}$$

- Add the following 4-bit binary numbers

$$\begin{array}{r} 111 \\ 1011 \\ +\ 0110 \\ \hline 10001 \end{array}$$

Overflow!

Overflow

- Digital systems operate on a **fixed number of bits**
- Overflow: when result is too big to fit in the available number of bits
- See previous example of 11 + 6

Hexadecimal addition

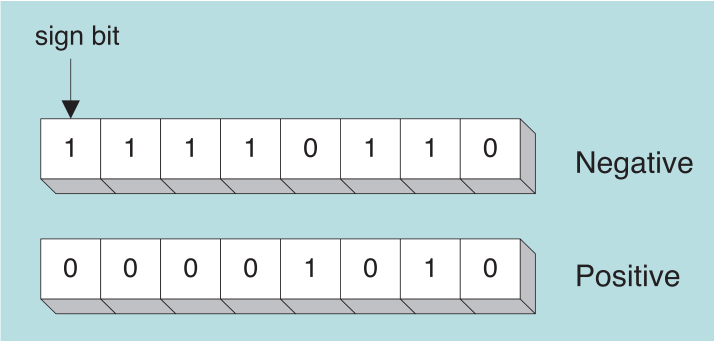
Divide the sum of two digits by the number base (16). The quotient becomes the carry value, and the remainder is the sum digit.

		1	**1**
36	28	28	6A
42	45	58	4B
78	6D	80	B5

Important skill: Programmers frequently add and subtract the addresses of variables and instructions.

Signed integers

The highest bit indicates the sign. 1 = negative, 0 = positive



If the highest digit of a hexadecmal integer is > 7, the value is negative. Examples: 8A, C5, A2, 9D

Signed Binary Numbers

- Sign/Magnitude Numbers
- Two's Complement Numbers

Sign/Magnitude Numbers

- 1 sign bit, *N*-1 magnitude bits
- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0 $A:\{a_{N-1}, a_{N-2}, \cdots a_2, a_1, a_0\}$
  - Negative number: sign bit = 1 $A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i$
- Example, 4-bit sign/mag representations of ± 6:

  +6 =

  - 6 =
- Range of an *N*-bit sign/magnitude number:

Sign/Magnitude Numbers

- 1 sign bit, *N*-1 magnitude bits
- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0
  - Negative number: sign bit = 1

$$A: \{a_{N-1}, a_{N-2}, \cdots a_2, a_1, a_0\}$$

$$A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i$$

- Example, 4-bit sign/mag representations of ± 6:

  +6 = **0110**

  - 6 = **1110**

- Range of an *N*-bit sign/magnitude number:

  [-(2^N-1^-1), 2^N-1^-1]

Sign/Magnitude Numbers

- Problems:
  - Addition doesn't work, for example -6 + 6:

$$\begin{array}{r} 1110 \\ +\ 0110 \\ \hline 10100 \end{array} \text{ (wrong!)}$$

  - Two representations of 0 (± 0):

    1000

    0000

Two's Complement Numbers

- Don't have same problems as sign/magnitude numbers:
  - Addition works
  - Single representation for 0

Two's complement notation

Steps:

- Complement (reverse) each bit
- Add 1

Starting value	00000001
Step 1: reverse the bits	11111110
Step 2: add 1 to the value from Step 1	11111110 +00000001
Sum: two's complement representation	11111111

Note that 00000001 + 11111111 = 00000000

“Taking the Two’s Complement”

- Flip the sign of a two’s complement number
- Method:
  1. Invert the bits
  2. Add 1
- Example: Flip the sign of $3_{10} = 0011_2$

“Taking the Two’s Complement”

- Flip the sign of a two’s complement number
- Method:
  1. Invert the bits
  2. Add 1
- Example: Flip the sign of $3_{10} = 0011_2$

1. 1100

2. + 1

1101 = -3_{10}

Two’s Complement Examples

- Take the two’s complement of $6_{10} = 0110_2$
- What is the decimal value of 1001_2?

Two’s Complement Examples

- Take the two’s complement of $6_{10} = 0110_2$

1. 1001

2. + 1

$1010_2 = -6_{10}$

- What is the decimal value of the two’s complement number 1001_2?

1. 0110

2. + 1

$0111_2 = 7_{10}$, so $1001_2 = -7_{10}$

Binary subtraction

- When subtracting A – B, convert B to its two's complement
- Add A to (–B)

$$\begin{array}{r} 0\,1\,0\,1\,0 \\ -\;0\,1\,0\,1\,1 \\ \hline \end{array} \longrightarrow \begin{array}{r} 0\,1\,0\,1\,0 \\ 1\,0\,1\,0\,1 \\ \hline 1\,1\,1\,1\,1 \end{array}$$

Advantages for 2's complement:

- No two 0's
- Sign bit
- Remove the need for separate circuits for add and sub

Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

$$\begin{array}{r} 0110 \\ +\;\;1010 \\ \hline \end{array}$$

- Add -2 + 3 using two's complement numbers

$$\begin{array}{r} 1110 \\ +\;\;0011 \\ \hline \end{array}$$

Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

$$\begin{array}{r} 111\; \\ 0110 \\ +\;\;1010 \\ \hline 10000 \end{array}$$

- Add -2 + 3 using two's complement numbers

$$\begin{array}{r} 111\; \\ 1110 \\ +\;\;0011 \\ \hline 10001 \end{array}$$

Two's Complement

- $\overline{b} = 1\text{-}b$
- $\overline{a} = 111\ldots1 - a = (2^N\text{-}1)\text{-}a$
- $\overline{a} + 1 = 2^N\text{-}a$
- In 2's complement, 2^N-a is used to represent –a since $2^N\text{-}a \equiv \text{-}a \pmod{2^N}$

Increasing Bit Width

- **Extend number from N to M bits ($M > N$) :**
  - Sign-extension
  - Zero-extension

Sign-Extension

- Sign bit copied to msb's
- Number value is same

- **Example 1:**
  - 4-bit representation of 3 = 0011
  - 8-bit sign-extended value: 00000011
- **Example 2:**
  - 4-bit representation of -5 = 1011
  - 8-bit sign-extended value: 11111011

Zero-Extension

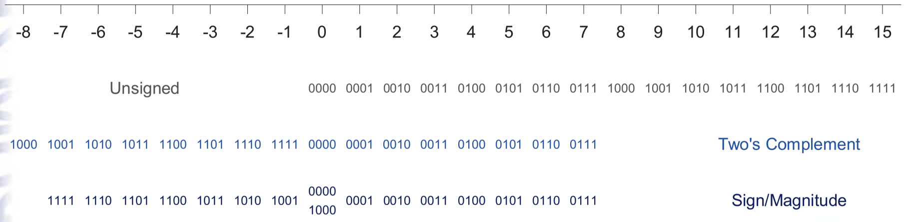
- Zeros copied to msb's
- Value changes for negative numbers

- **Example 1:**
  - 4-bit value = $0011_2 = 3_{10}$
  - 8-bit zero-extended value: $00000011 = 3_{10}$
- **Example 2:**
  - 4-bit value = $1011 = -5_{10}$
  - 8-bit zero-extended value: $00001011 = 11_{10}$

Number System Comparison

Number System	Range
Unsigned	$[0, 2^N-1]$
Sign/Magnitude	$[-(2^{N-1}-1), 2^{N-1}-1]$
Two's Complement	$[-2^{N-1}, 2^{N-1}-1]$

For example, 4-bit representation:



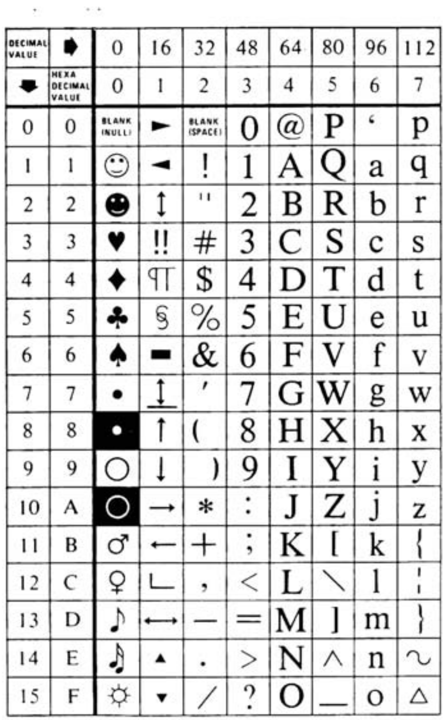
Ranges of signed integers

The highest bit is reserved for the sign. This limits the range:

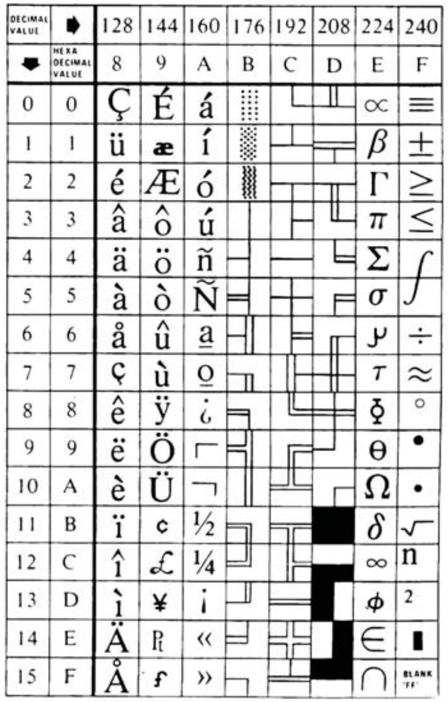
Storage Type	Range (low–high)	Powers of 2
Signed byte	−128 to +127	-2^7 to $(2^7 - 1)$
Signed word	−32,768 to +32,767	-2^{15} to $(2^{15} - 1)$
Signed doubleword	−2,147,483,648 to 2,147,483,647	-2^{31} to $(2^{31} - 1)$
Signed quadword	−9,223,372,036,854,775,808 to +9,223,372,036,854,775,807	-2^{63} to $(2^{63} - 1)$

Character

- Character sets
  - Standard ASCII(0 – 127)
  - Extended ASCII (0 – 255)
  - ANSI (0 – 255)
  - Unicode (0 – 65,535)
- Null-terminated String
  - Array of characters followed by a *null byte*
- Using the ASCII table
  - back inside cover of book



DECIMAL VALUE	➡	0	16	32	48	64	80	96	112
⬇	HEXA DECIMAL VALUE	0	1	2	3	4	5	6	7
0	0	BLANK (NULL)	►	BLANK (SPACE)	0	@	P	‘	p
1	1	☺	◄	!	1	A	Q	a	q
2	2	☻	↕	"	2	B	R	b	r
3	3	♥	‼	#	3	C	S	c	s
4	4	♦	¶	$	4	D	T	d	t
5	5	♣	§	%	5	E	U	e	u
6	6	♠	▬	&	6	F	V	f	v
7	7	•	↨	'	7	G	W	g	w
8	8	◘	↑	(	8	H	X	h	x
9	9	○	↓	)	9	I	Y	i	y
10	A	◙	→	*	:	J	Z	j	z
11	B	♂	←	+	;	K	[	k	{
12	C	♀	∟	,	<	L	\	l	¦
13	D	♪	↔	—	=	M	]	m	}
14	E	♫	▲	.	>	N	^	n	~
15	F	☼	▼	/	?	O	_	o	△



DECIMAL VALUE	➡	128	144	160	176	192	208	224	240
⬇	HEXA DECIMAL VALUE	8	9	A	B	C	D	E	F
0	0	Ç	É	á	░	└	╨	α	≡
1	1	ü	æ	í	▒	┴	╤	β	±
2	2	é	Æ	ó	▓	┬	╥	Γ	≥
3	3	â	ô	ú	│	├	╙	π	≤
4	4	ä	ö	ñ	┤	─	╘	Σ	⌠
5	5	à	ò	Ñ	╡	┼	╒	σ	⌡
6	6	å	û	ª	╢	╞	╓	µ	÷
7	7	ç	ù	º	╖	╟	╫	τ	≈
8	8	ê	ÿ	¿	╕	╚	╪	Φ	°
9	9	ë	Ö	⌐	╣	╔	┘	Θ	•
10	A	è	Ü	¬	║	╩	┌	Ω	·
11	B	ï	¢	½	╗	╦	█	δ	√
12	C	î	£	¼	╝	╠	▄	∞	ⁿ
13	D	ì	¥	¡	╜	═	▌	φ	²
14	E	Ä	₧	«	╛	╬	▐	∈	■
15	F	Å	ƒ	»	┐	╧	▀	∩	BLANK 'FF'

Representing Instructions

```
int sum(int x, int y)
{
   return x+y;
}
```

- For this example, Alpha & Sun use two 4-byte instructions
  - Use differing numbers of instructions in other cases
- PC uses 7 instructions with lengths 1, 2, and 3 bytes
  - Same for NT and for Linux
  - NT / Linux not fully binary compatible

Alpha **sum**	***Sun*** **sum**	***PC*** **sum**
00	81	55
00	C3	89
30	E0	E5
42	08	8B
01	90	45
80	02	0C
FA	00	03
6B	09	45
		08
		89
		EC
		5D
		C3

Different machines use totally different instructions and encodings