Codes and number systems

Introduction to Computer

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*with slides by Nisan & Schocken (*www.nand2tetris.org*) and Harris & Harris (DDCA)*

Coding

- Assume that you want to communicate with your friend with a flashlight in a night, what will you do?

light painting?
What's the problem?

Solution #1

- A: 1 blink
- B: 2 blinks
- C: 3 blinks

  :
- Z: 26 blinks

What's the problem?

- How are you? = 131 blinks

Solution #2: Morse code

A	•–	J	•–––	S	•••
B	–•••	K	–•–	T	–
C	–•–•	L	•–••	U	••–
D	–••	M	––	V	•••–
E	•	N	–•	W	•––
F	••–•	O	–––	X	–••–
G	––•	P	•––•	Y	–•––
H	••••	Q	––•–	Z	––••
I	••	R	•–•		

•••• • •–•• •–•• –––

Hello

Lookup

- It is easy to translate into Morse code than reverse. Why?

Lookup

•••• • •−•• •−•• −−−

•	E
−	T

••	I	−•	N
•−	A	−−	M

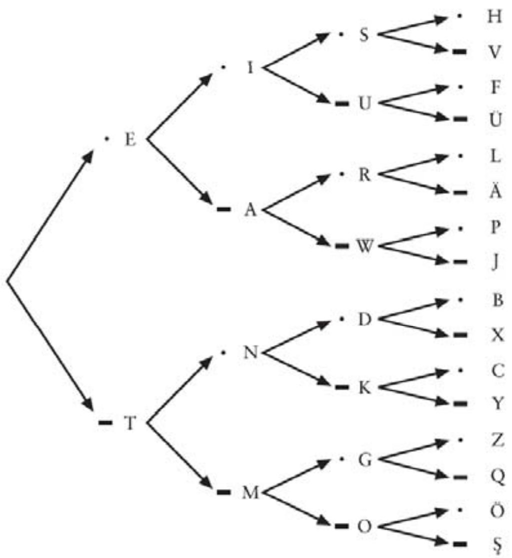
•••	S	−••	D
••−	U	−•−	K
•−•	R	−−•	G
•−−	W	−−−	O

••••	H	−•••	B
•••−	V	−••−	X
••−•	F	−•−•	C
••−−	Ü	−•−−	Y
•−••	L	−−••	Z
•−•−	Ä	−−•−	Q
•−−•	P	−−−•	Ö
•−−−	J	−−−−	Ş

Number of Dots and Dashes	Number of Codes
1	2
2	4
3	8
4	16

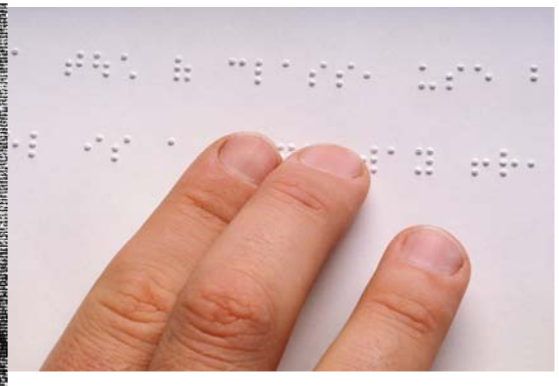
number of codes = $2^{\text{number of dots and dashes}}$

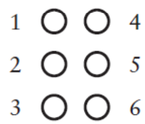
Lookup



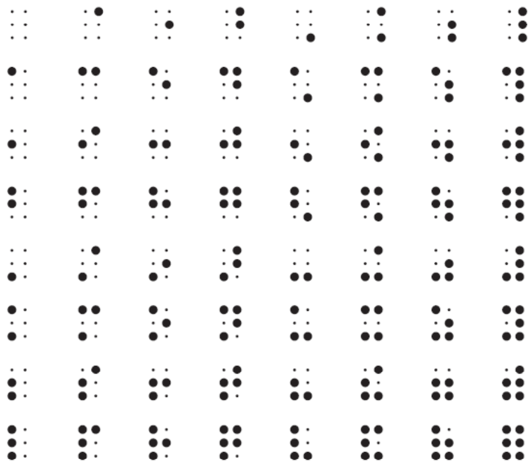
Useful for checking the correctness/ redundancy

Braille





Braille

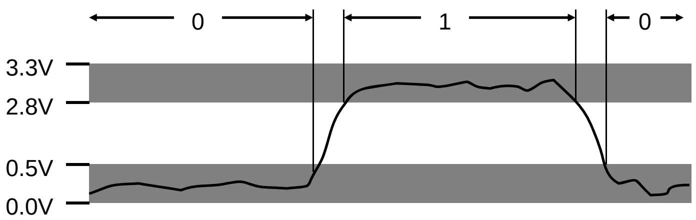


What’s common in these codes?

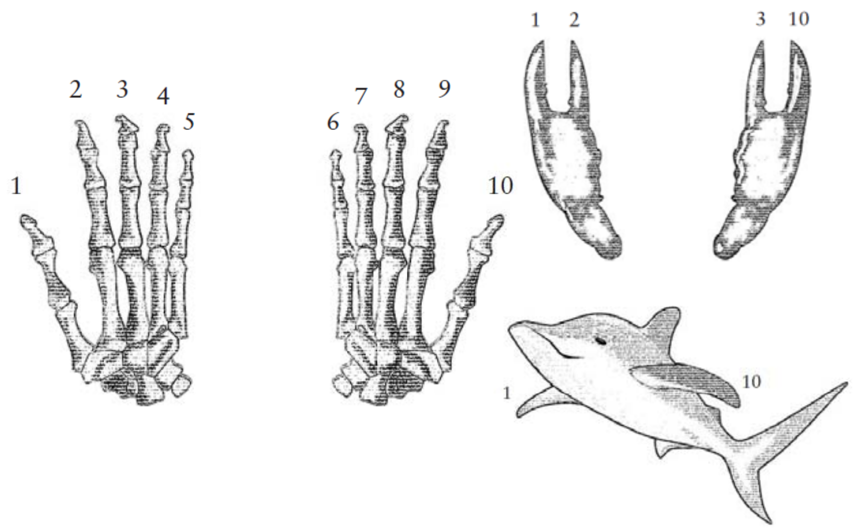
- They are both binary codes.

Binary representations

- Electronic Implementation
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



Number systems



Number Systems

- Decimal numbers

1000's column
100's column
10's column
1's column

$5374_{10} =$

- Binary numbers

8's column
4's column
2's column
1's column

$1101_2 =$

Number Systems

- Decimal numbers

1000's column
100's column
10's column
1's column

$5374_{10} = 5 ? 10^3 + 3 ? 10^2 + 7 ? 10^1 + 4 ? 10^0$

five thousands | three hundreds | seven tens | four ones

- Binary numbers

8's column
4's column
2's column
1's column

$1101_2 = 1 ? 2^3 + 1 ? 2^2 + 0 ? 2^1 + 1 ? 2^0 = 13_{10}$

one eight | one four | no two | one one

Binary numbers

- Digits are 1 and 0
  (a binary digit is called a bit)
  1 = true
  0 = false
- MSB –most significant bit
- LSB –least significant bit
- Bit numbering:

MSB LSB
1 0 1 1 0 0 1 0 1 0 0 1 1 1 0 0
15 0

- A bit string could have different interpretations

Powers of Two

- $2^0 =$
- $2^1 =$
- $2^2 =$
- $2^3 =$
- $2^4 =$
- $2^5 =$
- $2^6 =$
- $2^7 =$
- $2^8 =$
- $2^9 =$
- $2^{10} =$
- $2^{11} =$
- $2^{12} =$
- $2^{13} =$
- $2^{14} =$
- $2^{15} =$

Powers of Two

FROM ZERO TO ONE

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$
- $2^7 = 128$
- $2^8 = 256$
- $2^9 = 512$
- $2^{10} = 1024$
- $2^{11} = 2048$
- $2^{12} = 4096$
- $2^{13} = 8192$
- $2^{14} = 16384$
- $2^{15} = 32768$
- Handy to memorize up to 2^9

Unsigned binary integers

- Each digit (bit) is either 1 or 0
- Each bit represents a power of 2:

1	1	1	1	1	1	1	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Every binary number is a sum of powers of 2

Table 1-3 Binary Bit Position Values.

2^n	Decimal Value	2^n	Decimal Value
2^0	1	2^8	256
2^1	2	2^9	512
2^2	4	2^{10}	1024
2^3	8	2^{11}	2048
2^4	16	2^{12}	4096
2^5	32	2^{13}	8192
2^6	64	2^{14}	16384
2^7	128	2^{15}	32768

Translating binary to decimal

Weighted positional notation shows how to calculate the decimal value of each binary bit:

$$dec = (D_{n-1} \times 2^{n-1}) + (D_{n-2} \times 2^{n-2}) + ... + (D_1 \times 2^1) + (D_0 \times 2^0)$$

D = binary digit

binary 00001001 = decimal 9:

$(1 \times 2^3) + (1 \times 2^0) = 9$

Translating unsigned decimal to binary

- Repeatedly divide the decimal integer by 2. Each remainder is a binary digit in the translated value:

Division	Quotient	Remainder
37 / 2	18	1
18 / 2	9	0
9 / 2	4	1
4 / 2	2	0
2 / 2	1	0
1 / 2	0	1

37 = 100101

Number Conversion

- Decimal to binary conversion:
  - Convert 10011_2 to decimal

- Decimal to binary conversion:
  - Convert 47_{10} to binary

Number Conversion

- Decimal to binary conversion:
  - Convert 10011_2 to decimal
  - $16 \times 1 + 8 \times 0 + 4 \times 0 + 2 \times 1 + 1 \times 1 = 19_{10}$

- Decimal to binary conversion:
  - Convert 47_{10} to binary
  - $32 \times 1 + 16 \times 0 + 8 \times 1 + 4 \times 1 + 2 \times 1 + 1 \times 1 = 101111_2$

Binary Values and Range

- N-digit decimal number
  - How many values?
  - Range?
  - Example: 3-digit decimal number:

- N-bit binary number
  - How many values?
  - Range:
  - Example: 3-digit binary number:

Binary Values and Range

- N-digit decimal number
  - How many values? **10^N**
  - Range? **$[0, 10^N - 1]$**
  - Example: 3-digit decimal number:
    - **$10^3 = 1000$ possible values**
    - **Range: [0, 999]**

- N-bit binary number
  - How many values? **2^N**
  - Range: **$[0, 2^N - 1]$**
  - Example: 3-digit binary number:
    - **$2^3 = 8$ possible values**
    - **Range: $[0, 7] = [000_2$ to $111_2]$**

Integer storage sizes

Standard sizes:

byte	8
word	16
doubleword	32
quadword	64

Table 1-4 Ranges of Unsigned Integers.

Storage Type	Range (low–high)	Powers of 2
Unsigned byte	0 to 255	0 to $(2^8 - 1)$
Unsigned word	0 to 65,535	0 to $(2^{16} - 1)$
Unsigned doubleword	0 to 4,294,967,295	0 to $(2^{32} - 1)$
Unsigned quadword	0 to 18,446,744,073,709,551,615	0 to $(2^{64} - 1)$

Practice: What is the largest unsigned integer that may be stored in 20 bits?

FROM ZERO TO ONE

Bits, Bytes, Nibbles…

- Bits

10010110

most significant bit

least significant bit

- Bytes & Nibbles

byte

10010110

nibble

- Bytes

CEBF9AD7

most significant byte

least significant byte

ELSEVIER

FROM ZERO TO ONE

Large Powers of Two

- 2^{10} = 1 kilo ≈ 1000 (1024)
- 2^{20} = 1 mega ≈ 1 million (1,048,576)
- 2^{30} = 1 giga ≈ 1 billion (1,073,741,824)

ELSEVIER

FROM ZERO TO ONE

Estimating Powers of Two

- What is the value of 2^{24}?
- How many values can a 32-bit variable represent?

ELSEVIER

Estimating Powers of Two

- What is the value of 2^{24}?
  - $2^4 \times 2^{20} \approx$ 16 million

- How many values can a 32-bit variable represent?
  - $2^2 \times 2^{30} \approx$ 4 billion

Large measurements

- Kilobyte (KB), 2^{10} bytes
- Megabyte (MB), 2^{20} bytes
- Gigabyte (GB), 2^{30} bytes
- Terabyte (TB), 2^{40} bytes
- Petabyte
- Exabyte
- Zettabyte
- Yottabyte

Hexadecimal Numbers

Hex Digit	Decimal Equivalent	Binary Equivalent
0	0	
1	1	
2	2	
3	3	
4	4	
5	5	
6	6	
7	7	
8	8	
9	9	
A	10	
B	11	
C	12	
D	13	
E	14	
F	15	

Hexadecimal Numbers

Hex Digit	Decimal Equivalent	Binary Equivalent
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Hexadecimal Numbers

- Base 16
- Shorthand for binary

Translating binary to hexadecimal

- Each hexadecimal digit corresponds to 4 binary bits.
- Example: Translate the binary integer 000101101010011110010100 to hexadecimal:

1	6	A	7	9	4
0001	0110	1010	0111	1001	0100

Converting hexadecimal to decimal

- Multiply each digit by its corresponding power of 16:

  dec = $(D_3 \times 16^3) + (D_2 \times 16^2) + (D_1 \times 16^1) + (D_0 \times 16^0)$

- Hex 1234 equals $(1 \times 16^3) + (2 \times 16^2) + (3 \times 16^1) + (4 \times 16^0)$, or decimal 4,660.

- Hex 3BA4 equals $(3 \times 16^3) + (11 * 16^2) + (10 \times 16^1) + (4 \times 16^0)$, or decimal 15,268.

Hexadecimal to Binary Conversion

- Hexadecimal to binary conversion:
  - Convert $4AF_{16}$ (also written 0x4AF) to binary

- Hexadecimal to decimal conversion:
  - Convert 0x4AF to decimal

Hexadecimal to Binary Conversion

- Hexadecimal to binary conversion:
  - Convert $4AF_{16}$ (also written 0x4AF) to binary
  - $0100\ 1010\ 1111_2$
- Hexadecimal to decimal conversion:
  - Convert $4AF_{16}$ to decimal
  - $16^2 \times 4 + 16^1 \times 10 + 16^0 \times 15 = 1199_{10}$

Powers of 16

Used when calculating hexadecimal values up to 8 digits long:

16^n	Decimal Value	16^n	Decimal Value
16^0	1	16^4	65,536
16^1	16	16^5	1,048,576
16^2	256	16^6	16,777,216
16^3	4096	16^7	268,435,456

Converting decimal to hexadecimal

Division	Quotient	Remainder
422 / 16	26	6
26 / 16	1	A
1 / 16	0	1

decimal 422 = 1A6 hexadecimal

Addition

- Decimal

```
   11  ← carries
  3734
+ 5168
  ----
  8902
```

- Binary

```
   11  ← carries
  1011
+ 0011
  ----
  1110
```

Binary Addition Examples

- Add the following 4-bit binary numbers

```
  1001
+ 0101
```

- Add the following 4-bit binary numbers

```
  1011
+ 0110
```


Binary Addition Examples

- Add the following 4-bit binary numbers

```
     1
  1001
+ 0101
  1110
```

- Add the following 4-bit binary numbers

```
  111
  1011
+ 0110
 10001
```

Overflow!

Overflow

- Digital systems operate on a **fixed number of bits**
- Overflow: when result is too big to fit in the available number of bits
- See previous example of 11 + 6

Hexadecimal addition

Divide the sum of two digits by the number base (16). The quotient becomes the carry value, and the remainder is the sum digit.

		1	**1**
36	28	28	6A
42	45	58	4B
78	6D	80	B5

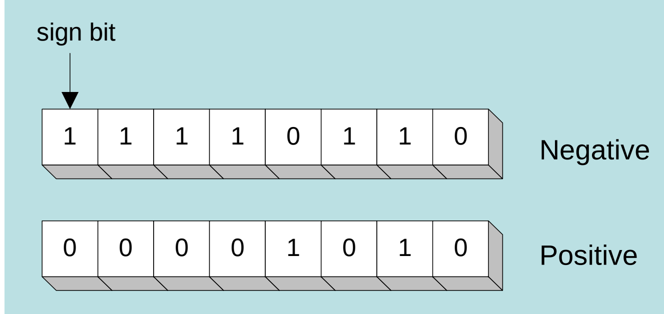
Important skill: Programmers frequently add and subtract the addresses of variables and instructions.

Signed Binary Numbers

- Sign/Magnitude Numbers
- Two's Complement Numbers

Signed integers

The highest bit indicates the sign. 1 = negative, 0 = positive



If the highest digit of a hexadecmal integer is > 7, the value is negative. Examples: 8A, C5, A2, 9D

Sign/Magnitude Numbers

- 1 sign bit, N-1 magnitude bits
- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0 $A:\{a_{N-1}, a_{N-2}, \cdots a_2, a_1, a_0\}$
  - Negative number: sign bit = 1 $A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i$
- Example, 4-bit sign/mag representations of ± 6:
  +6 =
  - 6 =
- Range of an N-bit sign/magnitude number:

Sign/Magnitude Numbers

- 1 sign bit, N-1 magnitude bits
- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0 $A:\{a_{N-1}, a_{N-2}, \cdots a_2, a_1, a_0\}$
  - Negative number: sign bit = 1 $A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i$
- Example, 4-bit sign/mag representations of ± 6:
  +6 = **0110**
  - 6 = **1110**
- Range of an N-bit sign/magnitude number:
  $[-(2^{N-1}-1), 2^{N-1}-1]$

Sign/Magnitude Numbers

- Problems:
  - Addition doesn't work, for example -6 + 6:

    1110
    + 0110
    10100 (wrong!)

  - Two representations of 0 (± 0):

    1000
    0000

Two's Complement Numbers

- Don't have same problems as sign/magnitude numbers:
  - Addition works
  - Single representation for 0

Two's complement notation

Steps:
- Complement (reverse) each bit
- Add 1

Starting value	00000001
Step 1: reverse the bits	11111110
Step 2: add 1 to the value from Step 1	11111110 +00000001
Sum: two's complement representation	11111111

Note that 00000001 + 11111111 = 00000000

"Taking the Two's Complement"

- Flip the sign of a two's complement number
- Method:
  1. Invert the bits
  2. Add 1
- Example: Flip the sign of $3_{10} = 0011_2$

"Taking the Two's Complement"

- Flip the sign of a two's complement number
- Method:
  1. Invert the bits
  2. Add 1
- Example: Flip the sign of $3_{10} = 0011_2$
  1. **1100**
  2. **+ 1**

  $1101 = -3_{10}$

Two's Complement Examples

- Take the two's complement of $6_{10} = 0110_2$
- What is the decimal value of 1001_2?

Two's Complement Examples

- Take the two's complement of $6_{10} = 0110_2$
  1. 1001
  2. + 1

  $1010_2 = -6_{10}$
- What is the decimal value of the two's complement number 1001_2?
  1. 0110
  2. + 1

  $0111_2 = 7_{10}$, so $1001_2 = -7_{10}$

Binary subtraction

- When subtracting A – B, convert B to its two's complement
- Add A to (–B)

$$\begin{array}{r} 0\,1\,0\,1\,0 \\ -\ 0\,1\,0\,1\,1 \\ \hline \end{array} \longrightarrow \begin{array}{r} 0\,1\,0\,1\,0 \\ 1\,0\,1\,0\,0 \\ \hline 1\,1\,1\,1\,1 \end{array}$$

Advantages for 2's complement:

- No two 0's
- Sign bit
- Remove the need for separate circuits for add and sub

Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

```
   0110
+  1010
```

- Add -2 + 3 using two's complement numbers

```
   1110
+  0011
```


Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

```
  111
   0110
+  1010
  10000
```

- Add -2 + 3 using two's complement numbers

```
  111
   1110
+  0011
  10001
```


Increasing Bit Width

- **Extend number from *N* to *M* bits (*M* > *N*) :**
  - Sign-extension
  - Zero-extension

Sign-Extension

- Sign bit copied to msb's
- Number value is same

- **Example 1:**
  - 4-bit representation of 3 = 0011
  - 8-bit sign-extended value: 00000011
- **Example 2:**
  - 4-bit representation of -5 = 1011
  - 8-bit sign-extended value: 11111011

Zero-Extension

- Zeros copied to msb’s
- Value changes for negative numbers

- **Example 1:**
  - 4-bit value = $0011_2 = 3_{10}$
  - 8-bit zero-extended value: 00000011 = 3_{10}
- **Example 2:**
  - 4-bit value = 1011 = -5_{10}
  - 8-bit zero-extended value: 00001011 = 11_{10}

Number System Comparison

Number System	Range
Unsigned	$[0, 2^N-1]$
Sign/Magnitude	$[-(2^{N-1}-1), 2^{N-1}-1]$
Two’s Complement	$[-2^{N-1}, 2^{N-1}-1]$

For example, 4-bit representation:

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Unsigned									0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
Two's Complement	1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111								
Sign/Magnitude		1111	1110	1101	1100	1011	1010	1001	0000 1000	0001	0010	0011	0100	0101	0110	0111								

Ranges of signed integers

The highest bit is reserved for the sign. This limits the range:

Storage Type	Range (low–high)	Powers of 2
Signed byte	−128 to +127	-2^7 to $(2^7 - 1)$
Signed word	−32,768 to +32,767	-2^{15} to $(2^{15} - 1)$
Signed doubleword	−2,147,483,648 to 2,147,483,647	-2^{31} to $(2^{31} - 1)$
Signed quadword	−9,223,372,036,854,775,808 to +9,223,372,036,854,775,807	-2^{63} to $(2^{63} - 1)$

Character

- Character sets
  - Standard ASCII (0 – 127)
  - Extended ASCII (0 – 255)
  - ANSI (0 – 255)
  - Unicode (0 – 65,535)
- Null-terminated String
  - Array of characters followed by a *null byte*
- Using the ASCII table
  - back inside cover of book

DECIMAL VALUE	→	0	16	32	48	64	80	96	112
↓	HEXA DECIMAL VALUE	0	1	2	3	4	5	6	7
0	0	BLANK (NULL)	►	BLANK (SPACE)	0	@	P	`	p
1	1	☺	◄	!	1	A	Q	a	q
2	2	☻	↕	"	2	B	R	b	r
3	3	♥	‼	#	3	C	S	c	s
4	4	♦	¶	$	4	D	T	d	t
5	5	♣	§	%	5	E	U	e	u
6	6	♠	▬	&	6	F	V	f	v
7	7	•	↨	'	7	G	W	g	w
8	8	◘	↑	(	8	H	X	h	x
9	9	○	↓	)	9	I	Y	i	y
10	A	◙	→	*	:	J	Z	j	z
11	B	♂	←	+	;	K	[	k	{
12	C	♀	∟	,	<	L	\	l	¦
13	D	♪	↔	-	=	M	]	m	}
14	E	♫	▲	.	>	N	^	n	~
15	F	☼	▼	/	?	O	_	o	△

DECIMAL VALUE	→	128	144	160	176	192	208	224	240
↓	HEXA DECIMAL VALUE	8	9	A	B	C	D	E	F
0	0	Ç	É	á	░	└	╨	∝	≡
1	1	ü	æ	í	▒	┴	╤	β	±
2	2	é	Æ	ó	▓	┬	╥	Γ	≥
3	3	â	ô	ú	│	├	╙	π	≤
4	4	ä	ö	ñ	┤	─	╘	Σ	⌠
5	5	à	ò	Ñ	╡	┼	╒	σ	⌡
6	6	å	û	ª	╢	╞	╓	µ	÷
7	7	ç	ù	º	╖	╟	╫	τ	≈
8	8	ê	ÿ	¿	╕	╚	╪	Φ	°
9	9	ë	Ö	⌐	╣	╔	┘	Θ	•
10	A	è	Ü	¬	║	╩	┌	Ω	·
11	B	ï	¢	½	╗	╦	█	δ	√
12	C	î	£	¼	╝	╠	▄	∞	ⁿ
13	D	ì	¥	¡	╜	═	▌	φ	²
14	E	Ä	₧	«	╛	╬	▐	∈	■
15	F	Å	ƒ	»	┐	╧	▀	∩	BLANK 'FF'

Representing Instructions

```
int sum(int x, int y)
{
   return x+y;
}
```

- For this example, Alpha & Sun use two 4-byte instructions
  - Use differing numbers of instructions in other cases
- PC uses 7 instructions with lengths 1, 2, and 3 bytes
  - Same for NT and for Linux
  - NT / Linux not fully binary compatible

Alpha sum	Sun sum	PC sum
00	81	55
00	C3	89
30	E0	E5
42	08	8B
01	90	45
80	02	0C
FA	00	03
6B	09	45
		08
		89
		EC
		5D
		C3

Different machines use totally different instructions and encodings