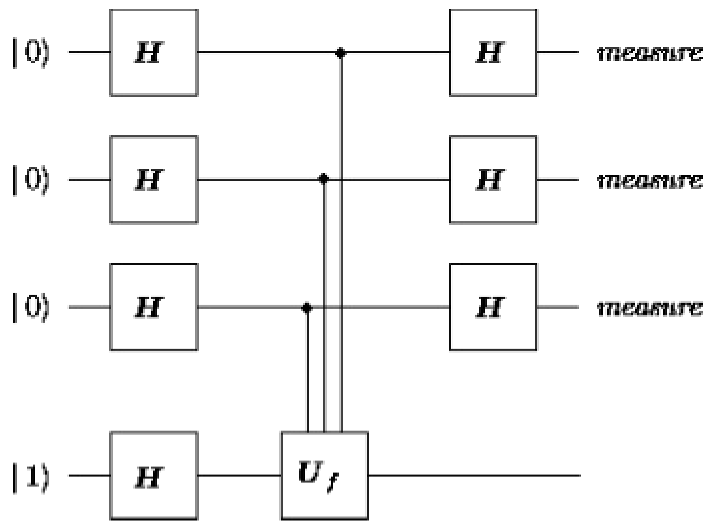
# Deutsch-Jozsa algorighm

- Question: **f: $\{0,1\}^N \rightarrow \{0,1\}$, f constant or balanced?**
- **Circuit (N=3):**




- Output: **All bits = 0 ⇔ f is constant**

## Mechanism

- Initialization: $|y_0> = |0>^{\otimes n}|1>$
- Apply H on first N bits: $|y_1> = \sum_{x\in\{0,1\}^n} \frac{|x>}{\sqrt{2^n}}\left[\frac{|0> - |1>}{\sqrt{2}}\right]$
- Apply U: $|y_2> = \sum_x \frac{|x>}{\sqrt{2^n}}\left[\frac{|0+f(x)> - |1+f(x)>}{\sqrt{2}}\right] = \sum_x \frac{(-1)^{f(x)}|x>}{\sqrt{2^n}}\left[\frac{|0> - |1>}{\sqrt{2}}\right]$
- Apply H on first N bits: $|y_3> = \sum_z \sum_x \frac{(-1)^{x\cdot z + f(x)}|z>}{2^n}\left[\frac{|0> - |1>}{\sqrt{2}}\right]$ $\left(\because H^{\otimes n}|x_1,\ldots,x_n> = \frac{\sum_z (-1)^{x\cdot z}|z>}{\sqrt{2^n}}\right)$
- Measure z:

  (a) f constant --- amplitude of $|0>^{\otimes n}$ = +1 or -1, and $y_3$ unit length => other amplitudes = 0.

  (b) f balanced --- amplitude of $|0>^{\otimes n}$ = 0 by cancellation => other amplitudes $\neq 0$.

# Appendix

$$\begin{aligned} \boldsymbol{H} \mid 0\rangle &= \frac{1}{\sqrt{2}}(\mid 0\rangle + \mid 1\rangle) = \frac{1}{\sqrt{2}}\left((-1)^{0\cdot 0} \mid 0\rangle + (-1)^{0\cdot 1} \mid 1\rangle\right) \\ \boldsymbol{H} \mid 1\rangle &= \frac{1}{\sqrt{2}}(\mid 0\rangle - \mid 1\rangle) = \frac{1}{\sqrt{2}}\left((-1)^{1\cdot 0} \mid 0\rangle + (-1)^{1\cdot 1} \mid 1\rangle\right) \end{aligned}$$

In summary:

$$\boldsymbol{H} \mid x\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} (-1)^{x\cdot y} \mid y\rangle$$

Apply it to a tensor product of *n* qubits:

$$\begin{aligned} &\boldsymbol{H} \mid x_1\rangle \otimes \boldsymbol{H} \mid x_2\rangle \otimes \ldots \otimes \boldsymbol{H} \mid x_n\rangle \\ &= \left(\frac{1}{\sqrt{2}} \sum_{y_1=0}^{1} (-1)^{x_1\cdot y_1} \mid y_1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{y_2=0}^{1} (-1)^{x_2\cdot y_2} \mid y_2\rangle\right) \otimes \ldots \\ &\quad \otimes \left(\frac{1}{\sqrt{2}} \sum_{y_n=0}^{1} (-1)^{x_n\cdot y_n} \mid y_n\rangle\right) \\ &= \frac{1}{2^{n/2}} \sum_{y_1 y_2 \cdots y_n} (-1)^{x_1\cdot y_1}(-1)^{x_2\cdot y_2} \cdots (-1)^{x_n\cdot y_n} \mid y_1 y_2 \ldots y_n\rangle \\ &= \frac{1}{2^{n/2}} \sum_{\boldsymbol{y}=\boldsymbol{0}}^{2^n-1} (-1)^{\boldsymbol{x}\cdot\boldsymbol{y}} \mid \boldsymbol{y}\rangle \end{aligned}$$