Positive Operator-Valued Measure (POVM)

From the third postulate of quantum mechanics, if we define

$$\mathbf{E}_m \equiv \mathbf{M}_m^\dagger \mathbf{M}_m,$$

then

$$\sum_m \mathbf{E}_m = \mathbf{1} \text{ and } p(m) = \langle \psi | \mathbf{E}_m | \psi \rangle.$$

The positive operators $\mathbf{E}_m$ are the POVM elements associated with the measurement, and the set $\{\mathbf{E}_m\}$ is a POVM.

Since the POVM elements sum to identity, they provide a partition of identity. In the special case of projective measurements, the $\mathbf{E}_m$'s are orthogonal projections, and the direct sum of the subspaces they project onto is the Hilbert space of the measured system.

Schmidt Decomposition

For any state $|\psi\rangle_{AB}$ of a composite system AB, there exists orthonormal states $|i\rangle_A$ and $|i\rangle_B$ for systems A and B respectively such that

$$|\psi\rangle_{AB} = \sum_i \lambda_i |i\rangle_A |i\rangle_B,$$

where the λ_i's are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$.

So for any pure state $|\psi\rangle_{AB}$, the density operator for systems A and B are

$$\begin{aligned} \rho_A = \mathrm{tr}_B\,(|\psi\rangle_{AB}) = \textstyle\sum_i \lambda_i^2 |i\rangle_{AA}\langle i| \\ \rho_B = \mathrm{tr}_A\,(|\psi\rangle_{AB}) = \textstyle\sum_i \lambda_i^2 |i\rangle_{BB}\langle i| \end{aligned}$$

They have the same eigenvalues.

The λ_i's are the **Schmidt coefficients**, the bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are called the **Schmidt bases** for A and B respectively, and the number of non-zero λ_i's is called the **Schmidt number**. The Schmidt number provide a definition and measure for entanglement.

Entanglement between A and B can be defined as when the Schmidt number of $|\psi\rangle_{AB}$ is larger than one, since then the states are not separable. The value of the Schmidt number also reflect the degree of entanglement in that local operations cannot change its value. If unitary operation $\mathbf{U}$ is performed on A only, then the new composite state is

$$|\psi\rangle_{AB} = \sum_i \lambda_i (\mathbf{U}|i\rangle_A)|i\rangle_B,$$

the Schmidt number remains constant.

Purifications

Given a state ρ_A in system A, we can introduce a system R and define a pure state $|\Psi\rangle_{AR}$ of the composite system AR such that

$$\rho_A = \text{tr}_R\left(|\Psi\rangle_{AR\,AR}\langle\Psi|\right).$$

That is, the mixed state ρ_A becomes the pure state $|\Psi\rangle_{AR\,AR}\langle\Psi|$. This is a purely mathematical procedure known as **purification**. System R is called the reference system, and has no physical reality.

Given an arbitrary state ρ_A of system A, according to the basic properties of density matrices, we can diagonalize it:

$$\rho_A = \sum_i p_i |i\rangle_{A\,A}\langle i|,$$

where the $|i\rangle_A$'s are mutually orthogonal and $p_i \geq 0$, $\sum_i p_i = 1$.

Take system R to have the same state space as A, and define an orthonormal basis $\{|i\rangle_R\}$, then the pure state

$$|\Psi\rangle_{AR} = \sum_i \sqrt{p_i}|i\rangle_A|i\rangle_R$$

is a purification for ρ_A.

Taking the partial trace

$$\begin{aligned}
& \mathrm{tr}_R\left(|\Psi\rangle_{AR\,AR}\langle\Psi|\right) \\
= \; & \mathrm{tr}_R\left(\left(\sum_i \sqrt{p_i}|i\rangle_A|i\rangle_R\right)\left(\sum_j(\sqrt{p_j})^*{}_A\langle j|_R\langle j|\right)\right) \\
= \; & \mathrm{tr}_R\left(\sum_{ij}\sqrt{p_i p_j}\left(|i\rangle_A \otimes |i\rangle_R\right)\left({}_A\langle j| \otimes {}_R\langle j|\right)\right) \\
= \; & \sum_{ij}\sqrt{p_i p_j}|i\rangle_{A\,A}\langle j|\mathrm{tr}\left(|i\rangle_{R\,R}\langle j|\right) \\
= \; & \sum_{ij}\sqrt{p_i p_j}|i\rangle_{A\,A}\langle j|\delta_{ij} \\
= \; & \sum_i p_i|i\rangle_{A\,A}\langle i| \\
= \; & \rho_A
\end{aligned}$$

The purification process defines a pure state by its Schmidt decomposition, that is $|i\rangle_A$ and $|i\rangle_R$ are the Schmidt bases for A and R respectively. The Schmidt coefficients are $\sqrt{p_i}$.

So the pure state produced by purification using the same reference system R is unique up to a local unitary transformation. That is, for

$$|\Psi'\rangle_{AR} = \sum_i \sqrt{p_i}|i\rangle_A \otimes \mathbf{U}|i\rangle_R,$$

$$\begin{aligned}
& \mathrm{tr}_R\left(|\Psi'\rangle_{AR\,AR}\langle\Psi'|\right) \\
= \ & \mathrm{tr}_R\left(\left(\sum_i \sqrt{p_i}|i\rangle_A \otimes \mathbf{U}|i\rangle_R\right)\right. \\
& \left.\left(\sum_j (\sqrt{p_j})^*{}_A\langle j| \otimes {}_R\langle j|\mathbf{U}^\dagger\right)\right) \\
= \ & \rho_A.
\end{aligned}$$

Relations Between the Physical States of Two Systems

For systems A and B, if their states are classically correlated, then their state vectors have some algebraic relation to each other; if they are entangled, then the relation between their state vectors becomes inseparable.

If system A is in the state $|x\rangle_A = a|0\rangle_A + b|1\rangle_A$, then the states of the composite system when there are some classical corrrelations with system B are (for equality and reverse in the Bloch sphere):

$$|x\rangle_A|y\rangle_B = (a|0\rangle_A + b|1\rangle_A) \otimes (a|0\rangle_B + b|1\rangle_B),$$
$$|x\rangle_A|y\rangle_B = (a|0\rangle_A + b|1\rangle_A) \otimes (b|0\rangle_B + a|1\rangle_B).$$

If they are entangled, then the composite states are:

$$a|0\rangle_A|0\rangle_B + b|1\rangle_A|1\rangle_B,$$
$$a|0\rangle_A|1\rangle_B + b|1\rangle_A|0\rangle_B.$$

There is no unitary transformation on the composite system AB that produces classical correlations between the states of A and B for arbitrary states of A. The best one can do is to produce an entangled state. This is the basis of the no-cloning theorem.

The purpose of cloning is to produce an *independent* copy of data. Entanglement causes the copy to be dependent on future values of the source data, this defeats the purpose of cloning, yet it can be seen as the ultimate copy mechanism.

Bell Inequality

Two particles are prepared, particle 1 is measured for physical properties P_Q or P_R by random, while particle 2 is measured for physical properties P_S or P_T by random. All measurement results are either 1 or -1. The measurement results are denoted Q, R, S, and T for P_Q, P_R, P_S, and P_T respectively.

Since

$$Q, R, S, T = \pm 1$$

we have

$$\begin{aligned} & QS + RS + RT - QT \\ = \; & (Q+R)S + (R-Q)T = \pm 2. \end{aligned}$$

Let $p(q, r, s, t)$ denote the probability that before measurement the particles have value

$$Q = q, R = r, S = s, T = t,$$

then

$$\begin{aligned} & \langle QS + RS + RT - QT \rangle \\ = \ & \sum_{qrst} p(q, r, s, t)(qs + rs + rt - qt) \\ \leq \ & \sum_{qrst} 2p(q, r, s, t) \\ = \ & 2. \end{aligned}$$

So we have

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle \leq 2.$$

This is an instance of Bell inequality.

Now suppose the two particles sent are in the qubit state

$$\frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle),$$

and the measurements are projective measurements where

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \quad S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Simple calculation shows that

$$\langle QS\rangle = \frac{1}{\sqrt{2}}, \langle RS\rangle = \frac{1}{\sqrt{2}}, \langle RT\rangle = \frac{1}{\sqrt{2}}, \langle QT\rangle = -\frac{1}{\sqrt{2}},$$

thus

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}.$$

The Bell inequality is violated.

Projective Measurements of the Qubit

In the qubit system, a projective measurement of the spin on axis $\widehat{n}$ (on the Bloch sphere) is represented by

$$M = \widehat{n} \cdot \vec{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 = \sum_{m=1}^{2} \lambda_m \mathbf{P}_m,$$

where

$$\lambda_1 = 1, \quad \mathbf{P}_1 = \mathbf{E}(\widehat{n}, +) = \tfrac{1}{2}(\mathbf{1} + \widehat{n} \cdot \vec{\sigma}),$$
$$\lambda_2 = -1, \quad \mathbf{P}_2 = \mathbf{E}(\widehat{n}, -) = \tfrac{1}{2}(\mathbf{1} - \widehat{n} \cdot \vec{\sigma}),$$

and the probabilities for obtaining spin up and spin down are

$$p(1) = \langle\psi|\mathbf{P}_1|\psi\rangle,$$
$$p(2) = \langle\psi|\mathbf{P}_2|\psi\rangle,$$

respectively.

Another Bell Inequality

The Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$ satisfies

$$((\hat{n}\cdot\vec{\sigma})\otimes\mathbf{1} + \mathbf{1}\otimes(\hat{n}\cdot\vec{\sigma}))|\psi^-\rangle = 0.$$

This means that if we measure the first qubit along $\hat{n}$ and the second along $-\hat{n}$, their results will always be anticorrelated:

$$\begin{aligned}
& \langle(\hat{n}\cdot\vec{\sigma})\otimes\mathbf{1}\rangle \\
= \ & \langle\psi^-|((\hat{n}\cdot\vec{\sigma})\otimes\mathbf{1})|\psi^-\rangle \\
= \ & -\langle\psi^-|(\mathbf{1}\otimes(\hat{n}\cdot\vec{\sigma}))|\psi^-\rangle \\
= \ & \langle\psi^-|(\mathbf{1}\otimes(-\hat{n}\cdot\vec{\sigma}))|\psi^-\rangle \\
= \ & \langle\mathbf{1}\otimes(-\hat{n}\cdot\vec{\sigma})\rangle
\end{aligned}$$

If we measure the spin of the first qubit along $\hat{n}$, and the spin of the second qubit along $\hat{m}$ (The measurement $(\hat{n} \cdot \vec{\sigma}) \otimes (\hat{m} \cdot \vec{\sigma})$), then the probability of obtaining the same or different results for the two qubits is

$$\begin{aligned} & \langle\psi^-| \left(\mathbf{E}(\hat{n},+) \otimes \mathbf{E}(\hat{m},+)\right) |\psi^-\rangle \\ = & \langle\psi^-| \left(\mathbf{E}(\hat{n},-) \otimes \mathbf{E}(\hat{m},-)\right) |\psi^-\rangle \\ = & \frac{1}{4}(1 - \cos\theta) \\ & \langle\psi^-| \left(\mathbf{E}(\hat{n},+) \otimes \mathbf{E}(\hat{m},-)\right) |\psi^-\rangle \\ = & \langle\psi^-| \left(\mathbf{E}(\hat{n},-) \otimes \mathbf{E}(\hat{m},+)\right) |\psi^-\rangle \\ = & \frac{1}{4}(1 + \cos\theta) \end{aligned}$$

So we have probabilities $\frac{1}{2}(1 - \cos\theta)$ of obtaining the same result and $\frac{1}{2}(1 + \cos\theta)$ of obtaining different results.

Consider measuring along three co-plane axes 60° apart, measuring the first qubit along $\widehat{n}_1$ with result r_1 and the second along $-\widehat{n}_2$ with result r_2 we can conclude that if we could somehow measure the first qubit along $\widehat{n}_1$ and $\widehat{n}_2$ we would obtain the results r_1 and r_2. The sum of the probabilities that the same result is obtained for any two axes is

$$P_{same}(\widehat{n}_1, \widehat{n}_2) + P_{same}(\widehat{n}_2, \widehat{n}_3) + P_{same}(\widehat{n}_1, \widehat{n}_3)$$
$$= 3 \cdot \frac{1}{2}(1 - \cos 60^\circ) = \frac{3}{4}.$$

Since there are only two possible results, the Bell inequality in this case is

$$P_{same}(\widehat{n}_1, \widehat{n}_2) + P_{same}(\widehat{n}_2, \widehat{n}_3) + P_{same}(\widehat{n}_1, \widehat{n}_3) \geq 1,$$

which is violated in this particular instance.

Two assumptions made in the proof of the Bell inequality need to be reconsidered:

1. Realism. The physical properties have definite values that exist independent of observation.

2. Locality. The measurement of the two particles does not influence each other's result.

That is, the world may not be locally realistic.

The concept of entanglement is the key to understanding non-locality.

The Collapse of the State Vector

In quantum measurement, we let the measured system Q interact with the measurement device A, the state of the composite system becomes

$$\sum_n a_n |\psi_n\rangle_Q |\varphi_n\rangle_A.$$

The $\{|\varphi_n\rangle_A\}$'s represent values that could be read out on the measurement device. The classical nature of the measurement device implies that after measurement, the state (readings) of the measurement device takes on a well defined value. The state of the composite system would then be

$$|\psi_n\rangle_Q |\varphi_n\rangle_A$$

for a particular value of n, or

$$\sum_n |a_n|^2 \left(|\psi_n\rangle_Q |\varphi_n\rangle_A\right) \left({}_Q\langle\psi_n|_A\langle\varphi_n|\right),$$

the state has collapsed (become mixed).

The Copenhagen interpretation of measurement is based on the state collapse of a quantum system due to interaction with a macroscopic, *classical*, measurement device. But interaction with a classical system cannot be part of a consistent quantum theory.

Von Neumann separates the measurement process into two stages. The first stage ("Von Neumann measurement") describes the interaction between Q and A that gives rise to the entangled composite state. The second stage ("observation") is the state collapse after which a definite read out on the measurement device is obtained.

In our formulation the realization of general measurements as projective measurements is combined with the first stage of quantum measurement according to von Neumann. (See The Measurement Process I.)

Views on the Collapse of the State Vector

During the collapse, a completely known state vector seems to have evolved into one of several possible outcome states (a mixed state when the result is not known). Yet such a process is not describable with unitary evolution, giving rise to the third postulate of quantum mechanics and two distinct ways for a system to evolve; one deterministic and linear, and the other probabilistic.

Von Neumann divided the measurement process in an attempt to resolve the paradox, yet the observation (interaction with a classical system) must occur at some stage, the paradox still exists. Von Neumann brings the human consciousness into the picture.

The Everett-DeWitt interpretation suggests that the observation stage never takes place, only one of each possibility is available for one (conscious or mechanical) observer, each observer records a possible version of reality. Thus this unprovable proposition states that the universe's state branches at each quantum event. The universe observes itself?

Environment decoherence is also used to explain the paradox. But while irreversibility is accounted for by the interpretation, the collapse is still not explained, thus the cat is still a paradox.

Decoherence I: Depolarization

Decoherence is the process in which a pure (coherent) state becomes mixed due to interaction with another system (environment). That is, it becomes entangled with another system.

In the depolarization of a qubit system, such an entanglement is manifested in the form of qubit errors:

1. Bit flip error: $\begin{array}{c} |0\rangle \to |1\rangle \\ |1\rangle \to |0\rangle \end{array}$ (σ_1).

2. Phase flip error: $\begin{array}{c} |0\rangle \to |0\rangle \\ |1\rangle \to -|1\rangle \end{array}$ (σ_3).

3. Both errors: $\begin{array}{c} |0\rangle \to +i|1\rangle \\ |1\rangle \to -i|0\rangle \end{array}$ (σ_2).

Suppose the qubit system A becomes entangled with the environment E, since there are four situations of interest (no error and three kinds of errors), we can set the state space of E to four dimensions with basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$, representing no error, σ_1 (error 1), σ_2 (error 3), and σ_3 (error 2) respectively.

For initial state $|\psi\rangle_A|0\rangle_E = (a|0\rangle_A + b|1\rangle_A)|0\rangle_E$, and probability of error p with each kind of error equally possible, then the state of system AE evolves as

$$
\begin{aligned}
& |\psi\rangle_A|0\rangle_E \\
\rightarrow\ & \sqrt{1-p}|\psi\rangle_A|0\rangle_E + \sqrt{\frac{p}{3}}(\sigma_1|\psi\rangle_A|1\rangle_E \\
& + \sigma_2|\psi\rangle_A|2\rangle_E + \sigma_3|\psi\rangle_A|3\rangle_E)\,.
\end{aligned}
$$

The state of system A becomes mixed (entangled with system E). Measuring E puts A in a definite state and any errors can be correct by consulting the measured value.

Without loss of generality, we can assume $|\psi\rangle_A = |0\rangle_A$, which has spin pointing in the $(0, 0, 1)$ direction, the initial density matrix is $\frac{1}{2}(\mathbf{1} + \sigma_3)$, and the evolved density matrix is

$$\begin{aligned} & \begin{pmatrix} 1 - p + \frac{p}{3} & 0 \\ 0 & 2\frac{p}{3} \end{pmatrix} \\ = \; & \frac{1}{2}\left(\mathbf{1} + (1 - \frac{4p}{3})\sigma_3\right), \end{aligned}$$

which is a mixed state with Bloch ball representation $(0, 0, 1 - \frac{4p}{3})$. With enough evolution cycles the state would become $\frac{1}{2}\mathbf{1}$, which is a completely random state. In other words, when the qubit has depolarized from the state $|0\rangle_{AA}\langle 0|$ to $\frac{1}{2}\mathbf{1}$, information is completely lost to the environment.

Decoherence II: Phase-damping

In phase-damping decoherence, the state of the system is not changed after entanglement with the environment, yet the relative phase information is lost, resulting in a mixed state.

For a qubit system A and environment E, phase-damping occurs as the unitary transformation

$$\begin{aligned} |0\rangle_A|0\rangle_E &\rightarrow \sqrt{1-p}|0\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E \\ |1\rangle_A|0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|1\rangle_A|2\rangle_E \end{aligned}$$

The environment occasionally gets changed to states $|1\rangle_E$ and $|2\rangle_E$ when qubit A is in $|0\rangle_A$ and $|1\rangle_A$ respectively. The basis $\{|0\rangle_A, |1\rangle_A\}$ is preferred by the environment in this case in that they are the only states that does not change due to environment interaction, any other state will be changed in the process.

For the initial state $|\psi\rangle_A|0\rangle_E = (a|0\rangle_A + b|1\rangle_A)|0\rangle_E$ of system AE, evolution produces the new state

$$\begin{aligned} & a\sqrt{1-p}|0\rangle_A|0\rangle_E + a\sqrt{p}|0\rangle_A|1\rangle_E \\ + \; & b\sqrt{1-p}|1\rangle_A|0\rangle_E + b\sqrt{p}|1\rangle_A|2\rangle_E \end{aligned}$$

The initial density matrix of A is

$$\rho_A = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}.$$

We obtain the state of A by partial trace

$$\begin{aligned} \rho'_A &= |a|^2|0\rangle_{AA}\langle 0| + ab^*(1-p)|0\rangle_{AA}\langle 1| \\ & \quad + a^*b(1-p)|1\rangle_{AA}\langle 0| + |b|^2|1\rangle_{AA}\langle 1| \\ &= \begin{pmatrix} |a|^2 & ab^*(1-p) \\ a^*b(1-p) & |b|^2 \end{pmatrix}. \end{aligned}$$

The off-diagonal elements would decrease after evolution, with enough evolution cycles the density matrix would become

$$\rho''_A = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = |a|^2|0\rangle_{AA}\langle 0| + |b|^2|1\rangle_{AA}\langle 1|.$$

Phase information is lost and we are left with classical probabilities of the preferred basis states.

Phase-damping can be used to explain the formulation of the unnatural Schrödinger cat state, but it still remains to explain why only one of the outcome is preceived.

Initially the cat and the atom is not entangled:

$$|0\rangle_A |\text{Alive}\rangle_{Cat},$$

After interaction they are entangled:

$$\frac{1}{\sqrt{2}}|0\rangle_A |\text{Alive}\rangle_{Cat} + \frac{1}{\sqrt{2}}|1\rangle_A |\text{Dead}\rangle_{Cat},$$

The environment prefers either a live cat or a dead cat, but not any combination of these two, so the whole Atom-Cat system phase-damps to the density matrix state

$$\frac{1}{2}((|0\rangle_A|\text{Alive}\rangle_{Cat})({}_A\langle 0|_{Cat}\langle \text{Alive}|) \\ +(|1\rangle_A|\text{Dead}\rangle_{Cat})({}_A\langle 1|_{Cat}\langle \text{Dead}|))$$

The cat has probability $\frac{1}{2}$ to be dead or alive, but what does that mean?

N. J. Cerf and C. Adami's Views on the Quantum Measurement Process

The collapse of the physical state in the quantum measurement process is an illusion brought about by the observation of part of a composite system that is quantum entangled and thus *inseparable*. Rather than collapsing, the state of a measured system becomes entangled with the state of the measurement device. The fact that no state collapse or quantum jump occured is implicit in the quantum eraser phenomenon.

Due to the absence of state collapse, the unitary description of quantum measurement is *reversible*. But in a general measurement situation where the measured system is entangled with a macroscopic system, it is practically impossible to track all the atoms involved, thus it is *practically irreversible*.