The First Postulate of Quantum Mechanics

Associated with any isolated physical system is a complex vector space with inner product (a **Hilbert space**) known as the **state space** of the system. The physical state of the system is completely described by its **state vector**, which is a unit vector in the system's state space.

The simplest quantum mechanical system is the **qubit**, which has a 2-d state space. With an orthonormal basis $\{|0\rangle, |1\rangle\}$ the most general state vector of a qubit can be expressed as

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$

where $a, b \in \mathbf{C}$ and $\langle\psi|\psi\rangle = 1$ (or $|a|^2 + |b|^2 = 1$).

Phase

The two states $e^{i\theta}|\psi\rangle$ and $|\psi\rangle$ are considered the same physical state.

$$\langle\psi|e^{-i\theta}\mathbf{E}e^{i\theta}|\psi\rangle = \langle\psi|\mathbf{E}|\psi\rangle$$

The factor $e^{i\theta}$ is called the **global phase**, which has no physical significance.

The two qubit states $a|0\rangle + be^{i\theta}|1\rangle$ and $a|0\rangle + b|1\rangle$ differ by **relative phase** between $|0\rangle$ and $|1\rangle$, and are not the same physical state. For example, the two states $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ are orthogonal.

Bloch Sphere

The most general state vector of a qubit can be expressed as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle,$$

where θ is the polar angle and φ is the azimuthal angle of a unit vector $\widehat{n}$ in 3-d real vector space.

Correspondence between 3-d real unit vectors and qubit states

$\widehat{n}$	θ	φ	$\|\psi\rangle$
$\widehat{z}$	0	0	$\|0\rangle$
$-\widehat{z}$	π	0	$\|1\rangle$
$\widehat{x}$	$\frac{\pi}{2}$	0	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$-\widehat{x}$	$\frac{\pi}{2}$	π	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$
$\widehat{y}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$
$-\widehat{y}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$

The Second Postulate of Quantum Mechanics

The evolution of a closed quantum system is described by a unitary transformation. The state vector $|\psi\rangle$ of a system at time t_1 is related to its state vector $|\psi'\rangle$ at time t_2 by a unitary operator $\mathbf{U}$ which depends on t_1, t_2,

$$|\psi'\rangle = \mathbf{U}|\psi\rangle.$$

The Pauli matrices and the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ are all 2×2 unitary matrices, and so describe valid qubit transformations.

The evolution of a closed quantum system in continuous time is described by the **Schrödinger equation**,

$$i\frac{d|\psi\rangle}{dt} = \mathbf{H}|\psi\rangle.$$

$\mathbf{H}$ is a fixed self-adjoint operator known as the **Hamiltonian** of the system. The state at time t is $|\psi(t)\rangle = \mathbf{U}|\psi(0)\rangle = e^{-i\mathbf{H}t}|\psi(0)\rangle$.

The Third Postulate of Quantum Mechanics

Quantum measurements are described by a set $\{\mathbf{M}_m\}$ of **measurement operators** acting on the state vector of the measured system. If the state of the system before measurement is $|\psi\rangle$, then outcome m occurs with probability

$$p(m) = \langle\psi|\mathbf{M}_m^\dagger \mathbf{M}_m|\psi\rangle,$$

and the new state in this case is

$$\frac{\mathbf{M}_m|\psi\rangle}{\sqrt{\langle\psi|\mathbf{M}_m^\dagger \mathbf{M}_m|\psi\rangle}}.$$

The probabilities of all outcomes sum to one:

$$1 = \sum_m p(m) = \sum_m \langle\psi|\mathbf{M}_m^\dagger \mathbf{M}_m|\psi\rangle,$$

so that

$$\sum_m \mathbf{M}_m^\dagger \mathbf{M}_m = \mathbf{1}.$$

Measuring a qubit to be $|0\rangle$ or $|1\rangle$ is thus a measurement with

$$\mathbf{M}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The state $|\psi\rangle = a|0\rangle + b|1\rangle$ then has probability

$$p(0) = \langle\psi|\mathbf{M}_0^\dagger \mathbf{M}_0|\psi\rangle = \langle\psi|\mathbf{M}_0|\psi\rangle = |a|^2$$

of yielding $|0\rangle$, and probability

$$p(1) = \langle\psi|\mathbf{M}_1^\dagger \mathbf{M}_1|\psi\rangle = \langle\psi|\mathbf{M}_1|\psi\rangle = |b|^2$$

of yielding $|1\rangle$ after the measurement, with new states

$$\frac{\mathbf{M}_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle, \frac{\mathbf{M}_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle$$

respectively.

Distinguishing Quantum States by Measurement

For mutually orthogonal states $|\psi_i\rangle$, performing measurement with $\mathbf{M}_i = |\psi_i\rangle\langle\psi_i|$ and $\mathbf{M}_0$ satisfying

$$\mathbf{M}_0^\dagger \mathbf{M}_0 = \mathbf{1} - \sum_{i \neq 0} |\psi_i\rangle\langle\psi_i|,$$

then for each state $|\psi_i\rangle$ outcome i occurs with probability $p(i) = \langle\psi_i|\mathbf{M}_i|\psi_i\rangle = 1$. So the states can be distinguished from each other.

But when the states are not orthogonal, then $|\psi_j\rangle = a|\psi_i\rangle + b|\psi_i^\perp\rangle$ where $a \neq 0$, measurement may yield $|\psi_i\rangle$ for both states.

Projective Measurements

A projective measurement is described by an observable $\mathbf{M}$, which is a self-adjoint operator on the state space of the measured system, and has spectral decomposition

$$\mathbf{M} = \sum_m \lambda_m \mathbf{P}_m,$$

where $\mathbf{P}_m$ is the projection onto the eigenspace of $\mathbf{M}$ with eigenvalue λ_m. When $|\psi\rangle$ is measured, outcome m occurs with probability

$$p(m) = \langle\psi|\mathbf{P}_m|\psi\rangle,$$

and the new state is

$$\frac{\mathbf{P}_m|\psi\rangle}{\sqrt{p(m)}}.$$

This is the special case of postulate 3 where the $\mathbf{M}_m$'s are orthogonal projectors. That is, the $\mathbf{M}_m$'s are self-adjoint, and $\mathbf{M}_m\mathbf{M}_n = \delta_{mn}\mathbf{M}_m$.

When outcome m occurs, we say the measured value is λ_m. So the expected value of the projective measurement $\mathbf{M}$ is

$$\begin{aligned}
\langle \mathbf{M} \rangle &= \sum_m \lambda_m p(m) \\
&= \sum_m \lambda_m \langle \psi | \mathbf{P}_m | \psi \rangle \\
&= \langle \psi | \left(\sum_m \lambda_m \mathbf{P}_m \right) | \psi \rangle \\
&= \langle \psi | \mathbf{M} | \psi \rangle .
\end{aligned}$$

The variance is

$$\begin{aligned}
(\Delta \mathbf{M})^2 &= \left\langle (\mathbf{M} - \langle \mathbf{M} \rangle)^2 \right\rangle \\
&= \left\langle \mathbf{M}^2 \right\rangle - \langle \mathbf{M} \rangle^2 .
\end{aligned}$$

The Fourth Postulate of Quantum Mechanics

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. For systems 1 through n, if system i has state space $\mathcal{H}_i$, then the state space of their composite system is

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}2 \otimes \ldots \otimes \mathcal{H}_n,$$

which has dimension

$$\text{dim}(\mathcal{H}) = \text{dim}(\mathcal{H}_1) \times \text{dim}(\mathcal{H}_2) \times \ldots \times \text{dim}(\mathcal{H}_n);$$

if system i is in state $|\psi_i\rangle$, then the state of the composite system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle.$$

The Measurement Process I

Quantum system Q in the state $|\psi\rangle$ is to be measured by operators $\mathbf{M}_m$. We use an **ancilla system** A with orthogonal states $|m\rangle$ corresponding to the measurement results m. The initial state of A is $|0\rangle$, a fixed state. We perform a unitary operation $\mathbf{U}$ on the composite system QA

$$\mathbf{U}|\psi\rangle|0\rangle \equiv \sum_m \mathbf{M}_m|\psi\rangle|m\rangle.$$

Then the projective measurement $\mathbf{P}_m \equiv \mathbf{1}_Q \otimes |m\rangle\langle m|$ is performed, outcome m occurs with probability

$$\begin{aligned}
p(m) &= \langle\psi|\langle 0|\mathbf{U}^\dagger \mathbf{P}_m \mathbf{U}|\psi\rangle|0\rangle \\
&= \sum_{m'm''} \langle\psi|\mathbf{M}^\dagger_{m'}\langle m'| \left(\mathbf{1}_Q \otimes |m\rangle\langle m|\right) \mathbf{M}_{m''}|\psi\rangle|m''\rangle \\
&= \sum_{m'm''} \langle\psi|\mathbf{M}^\dagger_{m'}\mathbf{M}_{m''}|\psi\rangle \times \langle m'|m\rangle\langle m|m''\rangle \\
&= \langle\psi|\mathbf{M}^\dagger_m \mathbf{M}_m|\psi\rangle,
\end{aligned}$$

and the new state is

$$\frac{\mathbf{P}_m\mathbf{U}|\psi\rangle|0\rangle}{\sqrt{\langle\psi|\langle 0|\mathbf{U}^\dagger\mathbf{P}_m\mathbf{U}|\psi\rangle|0\rangle}} = \frac{\mathbf{M}_m|\psi\rangle|m\rangle}{\sqrt{\langle\psi|\mathbf{M}_m^\dagger\mathbf{M}_m|\psi\rangle}}.$$

where system A is in state $|m\rangle$ and the state of system Q is

$$\frac{\mathbf{M}_m|\psi\rangle}{\sqrt{\langle\psi|\mathbf{M}_m^\dagger\mathbf{M}_m|\psi\rangle}}.$$

So after the unitary evolution of QA the measurement result can be obtained by measuring the ancilla A with projective measurement $\mathbf{P}_m \equiv |m\rangle\langle m|$ and $\mathbf{P}_0 = \mathbf{1} - \sum_m |m\rangle\langle m|$. In this way all measurements can be accomplished with projective measurements.

Review

Postulate 1 The state of a physical system is a unit vector in a Hilbert space (its state space).

Postulate 2 The evolution of physical systems is unitary.

Postulate 3 Quantum measurement of $|\psi\rangle$ by operators $\mathbf{M}_m$ yields result m with

$$p(m) = \langle\psi|\mathbf{M}_m^\dagger \mathbf{M}_m|\psi\rangle,$$

and new state

$$\frac{\mathbf{M}_m|\psi\rangle}{\sqrt{\langle\psi|\mathbf{M}_m^\dagger \mathbf{M}_m|\psi\rangle}}.$$

Postulate 4 The state space of a composite system is the tensor product of its component systems' state spaces.

All information is stored in the state vector.

There is no passive observation.