Quantum Computers
Final Exam Solution

1. Define $d\mu = p(\theta,\phi)d\theta d\phi = \frac{1}{4\pi}\sin\theta d\theta d\phi$.

   (a) For $|\psi(\theta,\phi)\rangle$ and $|\varphi(\theta',\phi')\rangle$,

$$
\begin{aligned}
\langle F\rangle &= \iint |\langle\psi(\theta,\phi)|\varphi(\theta',\phi')\rangle|^2 d\mu d\mu' \\
&= \iint \left|\left(\cos\frac{\theta}{2}\langle 0| + e^{-i\phi}\sin\frac{\theta}{2}\langle 1|\right)\left(\cos\frac{\theta'}{2}|0\rangle + e^{-i\phi'}\sin\frac{\theta'}{2}|1\rangle\right)\right|^2 d\mu d\mu' \\
&= \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi}\left(\cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + 2\cos(\phi'-\phi)\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}\right. \\
&\quad \left. + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2}\right)\sin\theta\sin\theta' d\theta d\phi d\theta' d\phi' \\
&= \frac{1}{4}\int_0^{\pi}\int_0^{\pi}\left(\cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2}\right)\sin\theta\sin\theta' d\theta d\theta' \\
&= \frac{1}{4}\left[\left(\int_0^{\pi}\cos^2\frac{\theta}{2}\sin\theta d\theta\right)^2 + \left(\int_0^{\pi}\sin^2\frac{\theta}{2}\sin\theta d\theta\right)^2\right] \\
&= \frac{1}{2}.
\end{aligned}
$$

   Or, using density operators $\rho(\theta,\phi)$ and $\rho(\theta',\phi')$,

$$
\begin{aligned}
\langle F\rangle &= \iint \mathrm{tr}\left(\rho(\theta,\phi)\rho(\theta',\phi')\right) d\mu d\mu' \\
&= \iint \mathrm{tr}\left(\frac{1}{2}(\mathbf{I} + \sin\theta\cos\phi\sigma_1 + \sin\theta\sin\phi\sigma_2 + \cos\theta\sigma_3)\right. \\
&\quad \left.\frac{1}{2}(\mathbf{I} + \sin\theta'\cos\phi'\sigma_1 + \sin\theta'\sin\phi'\sigma_2 + \cos\theta'\sigma_3)\right) d\mu d\mu' \\
&= \frac{1}{16}\int_0^{\pi}\int_0^{\pi}\mathrm{tr}\left((\mathbf{I} + \cos\theta\sigma_3)(\mathbf{I} + \cos\theta'\sigma_3)\right)\sin\theta\sin\theta' d\theta d\theta' \\
&= \frac{1}{8}\int_0^{\pi}\int_0^{\pi}(1 + \cos\theta\cos\theta')\sin\theta\sin\theta' d\theta d\theta' \\
&= \frac{1}{8}\left(\int_0^{\pi}\sin\theta d\theta\right)^2 \\
&= \frac{1}{2}.
\end{aligned}
$$

   In the third equality terms involving $\cos\phi$, $\sin\phi$, $\cos\phi'$, or $\sin\phi'$ vanish after integration. Also note that $\mathrm{tr}(\sigma_3) = 0$ and $\sigma_3\sigma_3 = \mathbf{I}$.

Notice that the chosen $|\psi\rangle$ and $|\varphi\rangle$ is independent, hence $\langle \rho_\psi \rho_\varphi \rangle = \langle \rho_\psi \rangle \langle \rho_\varphi \rangle$, that is, we can consider the two states separately. Since the probability of $\rho(\theta, \phi)$ is $p(\theta, \phi)$, we have

$$
\begin{aligned}
\langle \rho_\psi \rangle &= \frac{1}{4\pi} \iint \frac{1}{2}(\mathbf{I} + \sin\theta\cos\phi\sigma_1 + \sin\theta\sin\phi\sigma_2 + \cos\theta\sigma_3)\sin\theta d\theta d\phi \\
&= \frac{1}{4}\int_0^\pi (\mathbf{I} + \cos\theta\sigma_3)\sin\theta d\theta \\
&= \frac{1}{2}\mathbf{I}.
\end{aligned}
$$

So

$$
\langle F \rangle = \mathrm{tr}\left(\frac{1}{2}\mathbf{I}\frac{1}{2}\mathbf{I}\right) = \frac{1}{2}.
$$

(b)

$$
\begin{aligned}
\langle F \rangle &= \iint \mathrm{tr}\left(\rho_\psi \rho\right) \frac{1}{4\pi} \sin\theta d\theta d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi |\langle\psi|0\rangle|^4 + |\langle\psi|1\rangle|^4 \sin\theta d\theta d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right)\sin\theta d\theta d\phi \\
&= \frac{1}{2}\int_0^\pi \left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right)\sin\theta d\theta \\
&= \frac{1}{2}\int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos^2\theta\right)\sin\theta d\theta \\
&= \frac{2}{3}.
\end{aligned}
$$

(c) The difference represents the information gained by making a measurement in terms of fidelity.

2. The state after rearrangement is

$$
|\Phi\rangle_{AB} = \frac{1}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle,
$$

(a)

$$
\rho_A = \rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}.
$$

(b) The eigenvalues are $\lambda_+ = \frac{1}{2} + \frac{\sqrt{3}}{4}$ and $\lambda_- = \frac{1}{2} - \frac{\sqrt{3}}{4}$, with corresponding eigenstates $|\uparrow_x\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\downarrow_x\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(c)

$$|\Phi\rangle_{AB} = \sqrt{\lambda_+}|\uparrow_x\rangle_A|\uparrow_x\rangle_B + \sqrt{\lambda_-}|\downarrow_x\rangle_A|\downarrow_x\rangle_B.$$

3. (a) $\rho'_A = \begin{pmatrix} a & (1-p)b \\ (1-p)c & d \end{pmatrix}$.

(b) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

(c) $\vec{p'} = ((1-p)p_1, (1-p)p_2, p_3)$.

(d) $(0, 0, p_3)$.

4. (a)

$$\begin{array}{c} \mathbf{U}|\varphi_1\rangle_A|\beta\rangle_B = |\varphi_1\rangle_A|\beta_1\rangle_B \\ \mathbf{U}|\varphi_2\rangle_A|\beta\rangle_B = |\varphi_2\rangle_A|\beta_2\rangle_B \end{array},$$

since $\mathbf{U}$ is unitary, we have

$$\begin{aligned} & \left({}_A\langle\varphi_1|_B\langle\beta|\right)\left(|\varphi_2\rangle_A|\beta\rangle_B\right) = \left({}_A\langle\varphi_1|_B\langle\beta_1|\right)\left(|\varphi_2\rangle_A|\beta_2\rangle_B\right) \\ \Rightarrow\ & {}_A\langle\varphi_1|\varphi_2\rangle_A = {}_A\langle\varphi_1|\varphi_2\rangle_{AB}\langle\beta_1|\beta_2\rangle_B \\ \Rightarrow\ & 1 = {}_B\langle\beta_1|\beta_2\rangle_B \end{aligned}$$

because ${}_A\langle\varphi_1|\varphi_2\rangle_A \neq 0$. That is, $|\beta_1\rangle_B = |\beta_2\rangle_B$.

(b) With ${}_A\langle\varphi_1|\varphi_2\rangle_A = 0$, there is no restriction on ${}_B\langle\beta_1|\beta_2\rangle_B$. This reflects the fact that orthogonal states can be distinguished with perfect accuracy.

5. (a)

$$\begin{aligned} p_{\text{error}} &= p_1 \text{tr}\left(\rho_1 \mathbf{E}_2\right) + p_2 \text{tr}\left(\rho_2 \mathbf{E}_1\right) \\ &= \text{tr}\left(p_1\rho_1(\mathbf{I} - \mathbf{E}_1) + p_2\rho_2\mathbf{E}_1\right) \\ &= \text{tr}\left(p_1\rho_1 + (p_2\rho_2 - p_1\rho_1)\mathbf{E}_1\right) \\ &= \text{tr}\left(p_1\rho_1\right) + \text{tr}\left((p_2\rho_2 - p_1\rho_1)\mathbf{E}_1\right) \\ &= p_1 + \text{tr}\left(\sum_i \lambda_i |i\rangle\langle i|\mathbf{E}_1\right) \\ &= p_1 + \sum_i \lambda_i \langle i|\mathbf{E}_1|i\rangle \end{aligned}$$

(b)

$$\mathbf{E}_1 = \sum_{i:\lambda_i<0} |i\rangle\langle i|.$$

(c) $p_2 - p_1 = 1 - 2p_1 = \sum_i \lambda_i$, so

$$p_1 = \frac{1}{2} - \frac{1}{2}\sum_i \lambda_i.$$

$$\begin{aligned}
(p_{\text{error}})_{\text{optimal}} &= p_1 + \sum_{i:\lambda_i<0} \lambda_i \\
&= \frac{1}{2} - \frac{1}{2}\sum_i \lambda_i + \sum_{i:\lambda_i<0} \lambda_i \\
&= \frac{1}{2} - \frac{1}{2}\sum_{i:\lambda_i>0} \lambda_i + \frac{1}{2}\sum_{i:\lambda_i<0} \lambda_i \\
&= \frac{1}{2} - \frac{1}{2}\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}.
\end{aligned}$$

6. (a) Since ρ is a density operator, all its eigenvalues $\lambda_i' > 0$. Then

$$p_2\rho_2 - p_1\rho_1 = (p_2 - p_1)\rho = (p_2 - p_1)\sum_i \lambda_i'|i\rangle\langle i| = \sum_i \lambda_i|i\rangle\langle i|.$$

So $\lambda_i = (p_2 - p_1)\lambda_i'$ for all i.
When $p_2 \geq p_1$, the λ_i's are all non-negative, so

$$(p_{\text{error}})_{\text{optimal}} = p_1.$$

When $p_2 < p_1$, the λ_i's are all negative, so

$$(p_{\text{error}})_{\text{optimal}} = p_1 + \sum_i \lambda_i = p_1 + (p_2 - p_1) = p_2.$$

(b)

$$\begin{aligned}
\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} &= \text{tr}\left(\left[(p_2\rho_2 - p_1\rho_1)^\dagger(p_2\rho_2 - p_1\rho_1)\right]^{\frac{1}{2}}\right) \\
&= \text{tr}\left([(p_2\rho_2 - p_1\rho_1)(p_2\rho_2 - p_1\rho_1)]^{\frac{1}{2}}\right) \\
&= \text{tr}\left(\left[(p_2^2\rho_2^2 - p_1\rho_1p_2\rho_2 - p_2\rho_2p_1\rho_1 + p_1^2\rho_1^2)\right]^{\frac{1}{2}}\right) \\
&= \text{tr}\left(\left[(p_2^2\rho_2^2 + p_1^2\rho_1^2)\right]^{\frac{1}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr}\left(\left[(p_2^2\rho_2^2 + p_1\rho_1 p_2\rho_2 + p_2\rho_2 p_1\rho_1 + p_1^2\rho_1^2)\right]^{\frac{1}{2}}\right) \\
&= \operatorname{tr}\left([(p_2\rho_2 + p_1\rho_1)(p_2\rho_2 + p_1\rho_1)]^{\frac{1}{2}}\right) \\
&= \operatorname{tr}\left(p_2\rho_2 + p_1\rho_1\right) \\
&= p_2 + p_1 \\
&= 1
\end{aligned}
$$

So we have

$$(p_{\text{error}})_{\text{optimal}} = \frac{1}{2} - \frac{1}{2}\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} = \frac{1}{2} - \frac{1}{2} = 0.$$

(c) When $\rho_1 = \rho_2$, there is no way to distinguish between the two, hence the best we can do is guess the one with the higher probability, so

$$p_{\text{error}} = \min(p_1, p_2).$$

When ρ_1 and ρ_2 are orthogonal, they can be distinguished with no error, hence

$$p_{\text{error}} = 0.$$