Quantum Computers
Final Exam

January 6, 2004

Do any 4 of the 6 problems.

1. Fidelity

   (a) A single qubit is in an unknown pure state $|\psi\rangle$, selected at random from an ensemble uniformly distributed over the Bloch sphere ($p(\theta, \phi) = \frac{1}{4\pi}\sin\theta$). We take a guess $|\varphi\rangle$ that is also selected randomly on the Bloch sphere. The fidelity F is defined as

   $$F \equiv |\langle\psi|\varphi\rangle|^2,\ F \equiv \mathrm{tr}\left(\rho_\psi \rho_\varphi\right),$$

   for pure states and density operators respectively. Calculate the *expected* fidelity $\langle F\rangle$ of our guess by either equations. (Hint: this problem can be solved by either integrating over the Bloch sphere (expressing the states in polar coordinates) or finding the density operator (the expected state) for both states.)

   (b) We perform a measurement of the spin of $|\psi\rangle$ on the z-axis, that is, a measurement in the basis $\{|0\rangle, |1\rangle\}$, the resulting state is described by the density operator

   $$\rho = |\langle\psi|0\rangle|^2|0\rangle\langle 0| + |\langle\psi|1\rangle|^2|1\rangle\langle 1|.$$

   Calculate the expected fidelity between the density operators before ($\rho_\psi = |\psi\rangle\langle\psi|$) and after measurement.

   (c) Briefly explain the meaning of the difference between the two fidelities calculated above.

2. Consider the two qubit state

   $$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle_A\left(\frac{1}{2}|0\rangle_B + \frac{\sqrt{3}}{2}|1\rangle_B\right) + \frac{1}{\sqrt{2}}|1\rangle_A\left(\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B\right),$$

   (a) Compute the partial traces $\rho_A = \mathrm{tr}_B\left(|\Phi\rangle_{AB\,AB}\langle\Phi|\right)$ and $\rho_B = \mathrm{tr}_A\left(|\Phi\rangle_{AB\,AB}\langle\Phi|\right)$.

   (b) Diagonalize both density matrices.

(c) Find the Schmidt decomposition of $|\Phi\rangle_{AB}$. (Hint: match each pair of eigenstates from the two density matrices that have the same eigenvalues, and use the square root of that eigenvalue as its coefficient.)

3. Describe the effect of decoherence using the Bloch sphere parametrization of the density matrix. The effect of environment phase-damping on qubit A is

$$|0\rangle_A|0\rangle_E \to \sqrt{1-p}|0\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E,$$

$$|1\rangle_A|0\rangle_E \to \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|1\rangle_A|2\rangle_E,$$

where the environment (in a 3-d Hilbert space) has standard basis $\{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$. The Kraus operators for the evolution of ρ_A is

$$\mathbf{M}_0 = \sqrt{1-p}\mathbf{I},\ \mathbf{M}_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{M}_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and the qubit A evolves to

$$\rho'_A = \mathbf{M}_0\rho_A\mathbf{M}_0 + \mathbf{M}_1\rho_A\mathbf{M}_1 + \mathbf{M}_2\rho_A\mathbf{M}_2.$$

(a) Calculate ρ'_A, using $\rho_A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(b) What would happen if qubit A is allowed to decohere further? (compare ρ_A and ρ'_A)

(c) The density matrix of qubit A corresponding to the point $\vec{p} \equiv (p_1, p_2, p_3)$ where $|\vec{p}| \le 1$ is

$$\rho_A = \frac{1}{2}(\mathbf{I} + \vec{p}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbf{I} + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3).$$

What is the point $\vec{p'}$ corresponding to the evolved density operator ρ'_A in terms of p_1, p_2, p_3?

(d) Which point in the Bloch ball would be reached asymptotically after further evolution?

4. To distinguish two non-orthogonal states $|\varphi_1\rangle_A$ and $|\varphi_2\rangle_A$, we could prepare an ancillary system B and apply the unitary transformation $\mathbf{U}$ that act as

$$|\varphi_1\rangle_A|\beta\rangle_B \to |\varphi_1\rangle_A|\beta_1\rangle_B,$$
$$|\varphi_2\rangle_A|\beta\rangle_B \to |\varphi_2\rangle_A|\beta_2\rangle_B,$$

then the state of A can be determined without disturbing the state by measuring system B alone.

(a) Discuss the validity of this procedure. (Hint: Derive the relation between $|\beta_1\rangle_B$ and $|\beta_2\rangle_B$ from the unitarity of $\mathbf{U}$)

(b) What if $|\varphi_1\rangle_A$ and $|\varphi_2\rangle_A$ are orthogonal?

5. Suppose there is a physical system that has two possible states: ρ_1 with probability p_1 and ρ_2 with probability p_2; that is, its state can be interpreted as $p_1\rho_1 + p_2\rho_2$. We wish to perform a measurement that can determine its state with minimum error.

Suppose our measurement has two possible outcomes, projections $\mathbf{E}_1$ and $\mathbf{E}_2 = \mathbf{I} - \mathbf{E}_1$; that is, if we measure ρ, we would get the state $\rho\mathbf{E}_1$ with probability $\mathrm{tr}\,(\rho\mathbf{E}_1)$, and $\rho\mathbf{E}_2$ with probability $\mathrm{tr}\,(\rho\mathbf{E}_2)$. We guess that the state is ρ_1 when $\mathbf{E}_1$ is measured and ρ_2 otherwise, then the probability of error is

$$p_{\text{error}} = p_1 \mathrm{tr}\,(\rho_1\mathbf{E}_2) + p_2 \mathrm{tr}\,(\rho_2\mathbf{E}_1)\,.$$

(a) Show that if the eigenstates and eigenvalues of $p_2\rho_2 - p_1\rho_1$ is $|i\rangle$ and λ_i, that is,

$$p_2\rho_2 - p_1\rho_1 = \sum_i \lambda_i |i\rangle\langle i|,$$

where the $|i\rangle$'s form an orthonormal basis, then the error probability is

$$p_{\text{error}} = p_1 + \sum_i \lambda_i \langle i|\mathbf{E}_1|i\rangle.$$

(b) Find the non-negative operator $\mathbf{E}_1$ that minimizes p_{error}, show that the resulting error probability from this projection is

$$(p_{\text{error}})_{\text{optimal}} = p_1 + \sum_{i:\lambda_i<0} \lambda_i.$$

(c) The trace norm of the operator $p_2\rho_2 - p_1\rho_1$ is defined as

$$\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}} = \mathrm{tr}\,(|p_2\rho_2 - p_1\rho_1|) = \sum_{i:\lambda_i>0} \lambda_i - \sum_{i:\lambda_i<0} \lambda_i.$$

Show that

$$(p_{\text{error}})_{\text{optimal}} = \frac{1}{2} - \frac{1}{2}\|p_2\rho_2 - p_1\rho_1\|_{\text{tr}}.$$

(Hint: $\operatorname{tr}(p_2\rho_2 - p_1\rho_1) = p_2 - p_1 = \sum_i \lambda_i$.)

6. Use any definition of $(p_{\text{error}})_{\text{optimal}}$ from the previous problem:

   (a) Calculate $(p_{\text{error}})_{\text{optimal}}$ for $\rho_1 = \rho_2 = \rho$. (Hint: consider the cases where $p_1 > p_2$ and $p_2 > p_1$.)

   (b) Calculate $(p_{\text{error}})_{\text{optimal}}$ for ρ_1 and ρ_2 with supports on orthogonal subspaces. That is $\rho_1\rho_2 = \mathbf{0} = \rho_2\rho_1$ where $\mathbf{0}$ is the zero matrix. (Hint: use the form of trace norm $\|\rho\|_{\text{tr}} = \operatorname{tr}\left(\left[\rho^\dagger\rho\right]^{\frac{1}{2}}\right)$.)

   (c) Briefly explain the reason for the value of $(p_{\text{error}})_{\text{optimal}}$ in the previous two situations.

Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

General Qubit State

Vector (Pure)

$$|\psi(a,b)\rangle = a|0\rangle + b|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = |\psi(\theta,\phi)\rangle$$

$$|0\rangle \equiv |\uparrow_z\rangle, \; |1\rangle \equiv |\downarrow_z\rangle$$

Density Operator (Pure)

$$\rho(\theta,\phi) = |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| = \frac{1}{2}(\mathbf{I}+\hat{n}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbf{I}+\sin\theta\cos\phi\sigma_1+\sin\theta\sin\phi\sigma_2+\cos\theta\sigma_3)$$

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Density Operator (General)

$$\rho(\vec{p}) = \frac{1}{2}(\mathbf{I} + \vec{p}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbf{I} + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3)$$

$$|\vec{p}| \le 1$$

Qubit Measurement

$$P(\uparrow_{\hat{n}}) = \mathrm{tr}\,(|\uparrow_{\hat{n}}\rangle\langle\uparrow_{\hat{n}}|\rho) = \mathrm{tr}\left(\frac{1}{2}(\mathbf{I} + \hat{n}\cdot\vec{\sigma})\rho\right)$$

Partial Trace

Two General Systems

$$|\psi\rangle_{AB} = \sum_{i\mu} a_{i\mu} |i\rangle_A |\mu\rangle_B$$

$$\rho_A = \mathrm{tr}_B \left(|\psi\rangle_{AB\,AB}\langle\psi| \right) = \sum_{ij\mu} a_{i\mu} a^*_{j\mu} |i\rangle_{A\,A}\langle j|$$

$$\rho_B = \mathrm{tr}_A \left(|\psi\rangle_{AB\,AB}\langle\psi| \right) = \sum_{i\mu\nu} a_{i\mu} a^*_{i\nu} |\mu\rangle_{B\,B}\langle \nu|$$

Two Qubits

$$|\Psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$\rho_A = (|a|^2 + |b|^2)|0\rangle\langle 0| + (ac^* + bd^*)|0\rangle\langle 1| + (a^* c + b^* d)|1\rangle\langle 0| + (|c|^2 + |d|^2)|1\rangle\langle 1|$$

$$\rho_B = (|a|^2 + |c|^2)|0\rangle\langle 0| + (ab^* + cd^*)|0\rangle\langle 1| + (a^* b + c^* d)|1\rangle\langle 0| + (|b|^2 + |d|^2)|1\rangle\langle 1|$$