Review of Linear Algebra

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Linear Algebra

- Linear algebra is the study of linear vector spaces and of linear operator on those vector spaces.

- Object: Descrive the standard notations (the Dirac notations) adopted for those comcepts in the study of Quantum mechanics.

review of Linear Algebra

- vector space
- bases and linear indepentence
- linear operator and matrices
- Pauli matrices
- inner product, outer product ,and tensor product
- eigenvalues, eigenvector, and singular value decomposition

Vector space

- axiom ......

Notation	Description
$\|\psi\rangle$	Vector. 'ket' vector
$\langle\psi\|$	Vector dual to $\|\psi\rangle$ 'bra' vector
$\langle\phi\|\psi\rangle$	inner product of $\|\phi\rangle$ and $\|\psi\rangle$
$\|\phi\rangle \otimes \|\psi\rangle$	tensor product of $\|\phi\rangle$ and $\|\psi\rangle$
$\|\phi\rangle\|\psi\rangle$	abbreviated notation for tensor product
$\mathbf{A}^*$	Complex conjugate of the $\mathbf{A}$matrix
$\mathbf{A}^\mathbf{T}$	Transpose of the $\mathbf{A}$matrix
$\mathbf{A}^\dagger$	Hermitian conjugate or adjoint. $\mathbf{A}^\dagger = (\mathbf{A}^\mathbf{T})^*$
$\langle\phi\|\mathbf{A}\|\psi\rangle$	Inner product between $\|\phi\rangle$ and $\mathbf{A}\|\psi\rangle$. Equivalently, inner product between $\mathbf{A}^\dagger\|\phi\rangle$ and $\|\psi\rangle$.

Dirac notation

Bases and linear independence

- $\mathbf{C^2}$ space

- $|v_1\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $|v_2\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Any vector $|v\rangle \equiv \begin{bmatrix} a \\ b \end{bmatrix}$ can be written as a linear combination $|v\rangle = a|v_1\rangle + b|v_2\rangle$

- $|v_1\rangle$ and $|v_2\rangle$ span the vector space $\mathbf{C^2}$

- $|v_1\rangle, \ldots, |v_n\rangle$ are linearly dependent if

$$a_1|v_1\rangle + a_2|v_2\rangle + \ldots + a_n|v_n\rangle = 0 \quad \text{, with at least } a_i \neq 0$$

- $|v_1\rangle$ and $|v_2\rangle$ are linearly independent and span the $\mathbf{C^2}$ space. We call them the basis of $\mathbf{C^2}$ space.

linear operator and matrices

- A linear operator is defined to be function $A : V \to W$ which is linear in its inputs

$$A(\sum_i a_i |v_i\rangle) = \sum_i a_i A|v_i\rangle$$

- linear operator $A : V \to W$
  - $|v_1\rangle, \ldots, |v_m\rangle$ is a basis for V
  - $|w_1\rangle, \ldots, |w_m\rangle$ is a basis for W

- For each j in the range $1, \ldots, m$, there exist complex numbers A_{1j} through A_{nj} such that

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

- the matrix whose entries are A_{ij} is the matrix representation of operator A.

Pauli matrices

- $$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- $$Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Inner products

- inner product $(|v\rangle, |w\rangle)$ is written $\langle v|w\rangle$

- a function $(\cdot,\cdot)$ from $V \times V$ to $\mathbf{C}$ is an inner porduct if
  - $(\cdot,\cdot)$ is linear in the second argument
  $$(|v\rangle, \sum_i a_i |w_i\rangle) = \sum a_i \langle v|w\rangle$$
  - $\langle v|w\rangle = (\langle w|v\rangle)^*$
  - $\langle v|v\rangle \geq 0$ with equality iff $|v\rangle = 0$

- for $\mathbf{C^2}$

$$\langle v|w\rangle \equiv v_1^* w_1 + v_2^* w_2 = [v_1^* \quad v_2^*] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- norm of a vector $|v\rangle$

$$\| |v\rangle \| \equiv \sqrt{\langle v|v\rangle}$$

Gram-Schmidt procedure

- $|w_1\rangle, \ldots, |w_n\rangle$ is a basis set for V

- to produce an orthonormal basis $|v_1\rangle, \ldots, |v_n\rangle$

- $|v_1\rangle \equiv |w_1\rangle / \||w_1\rangle\|$

- $$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle|v_i\rangle}{\||w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle|v_i\rangle\|}$$

outer product

- $|v\rangle$ is a vector in an inner product space V

- $|w\rangle$ is a vector in an inner product space W

- outer product $|w\rangle\langle v|$:

$$(|w\rangle\langle v|)(|v'\rangle) \equiv |w\rangle(\langle v|v'\rangle) = \langle v|v'\rangle|w\rangle$$

completeness relation

- suppose $|\cdot\rangle$ is an orthonormal basis for V, so an arbitrary vector $|v\rangle = \sum_i v_i|i\rangle$

$$(\sum_i |i\rangle\langle i|)|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i v_i|i\rangle = |v\rangle, \qquad \sum_i |i\rangle\langle i| = \mathbf{I}$$

- $A : V \to W$

$$A = \mathbf{I_W A I_V} = \sum_{\mathbf{ij}} |\mathbf{w_j}\rangle\langle\mathbf{w_j}|\mathbf{A}|\mathbf{v_i}\rangle\langle\mathbf{v_i}| = \sum_{\mathbf{ij}} \langle\mathbf{w_j}|\mathbf{A}|\mathbf{v_i}\rangle|\mathbf{w_j}\rangle\langle\mathbf{v_i}|$$

eigenvalue and eigenvector

- $A|v\rangle = v|v\rangle$, v : eigenvalue and $|v\rangle$: eigenvector

- characteristic function : $c(\lambda) \equiv \det|A - \lambda I|$

- A diagonal representation for an operator A on space V is a representation $A = \sum_i \lambda_i |i\rangle\langle i|$, ex.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Adjoints and hermitian operators

- Suppose A is any linear operator on a Hilbert space V.
  $A^\dagger$ is adjoint or Hermitian conjugate of A if for all vecrot $|v\rangle, |w\rangle \in V$, $\qquad (|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle)$.

- operator H is Hermitian or self-adjoint operator if $H^\dagger = H$.

- operator U is said to be unitary if $U^\dagger U = I$

- operator M is said to be normal if $MM^\dagger = M^\dagger M$

- any operator can be diagonalized iff it is normal

tensor product

- a tensor product is a larger vector space formed from two smaller ones simply bycombining elements from each in all possible ways theat preserve both linearity and scalar multiplication
  - If V is a vector space of dimension n $\quad |v\rangle$
  - and W is a vector space of dimension m $\quad |w\rangle$
  - then $V \otimes W$ is a vector space of dimension mn $\quad |v\rangle \otimes |w\rangle$
- e.g. $|0\rangle \otimes |0\rangle = |00\rangle \quad |1\rangle \otimes |0\rangle = |10\rangle$ are elements of $V \otimes V$ and so is $|00\rangle + |10\rangle$

tensor product(2)

- define the linear operator $A \otimes B$ on $V \otimes W$ by the equation

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$$

- the tensor product of the Pauli matrices X and Y is

$$X \otimes Y = \begin{bmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

The commutator and anti-commutator

- the commutator between two operators A and B is
$$[A, B] \equiv AB - BA$$

- the anti-commutator is
$$\{A, B\} \equiv AB + BA$$

- theorem:(Simultaneous diagonalization theorem) Suppose A and B are Hermitian operators. The $[A, B] = 0$ if and only if there exists an orthonormal basis such that both A and B are diagonal with respect to that basis. We say thet A and B are simultaneously diagonalizable.

The polar and singular value decomposition

- (Polar decomposition) For any linear operator acting on a vector space we can write
$$A = U\sqrt{A^T A} \qquad \text{(left polar decomposition)}$$
where U is a unitary matrix – it is unique if A has an inverse.

- Alternatively
$$A = \sqrt{AA^T}U \qquad \text{(right polar decomposition)}$$

- (Singular-value decomposition) For all square matrices can weite
$$A = U'J = U'(TDT^\dagger) = (U'T)DT^\dagger = UDV$$

where U and V are unitary matrices and D is diagonal matrix with non-negative entries