Introduction to quantum computer

October 7, 2003

Outline

- Quantum bits
  - single qubit
  - multiple qubit
- Quantum computation
  - Single qubit gates
  - Multiple qubit gates
  - Quantum circuits

Quantum bits – Single qubit

What is a Qubit ?

- a qubit is a vector in 2D complex vector space

- a classicl bit has a state - either 0 or 1

- a qubit can in a state other $|0\rangle$ or $|1\rangle$
  it can in a linear combination of state : superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- $|0\rangle$ and $|1\rangle$ are two orthenormal basis of the 2D vector space

$$(|0\rangle, |1\rangle) = \langle 0|1\rangle = 0$$

- matrix representation:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Measurement

- A measurement on the qubit

  $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

  would give EITHER
  $|0\rangle$ with the probability $|\alpha|^2$,or
  $|1\rangle$ with the probability $|\beta|^2$.

- normalization : $|\alpha|^2 + |\beta|^2 = 1$

- the state becomes what you measured after measurement

multiple qubit

How about 2 Qubits ?

- classically, 4 possible states 00, 01, 10, and 11

- QM: a superposition of 4 states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$

- assuming the state vector describing 2 qubits is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

- normalization: $\sum_{x\in\{0,1\}^2} |\alpha_x|^2 = 1$

- measuring the 1st qubit give 0 with the probability

$$|\alpha_{00}|^2 + |\alpha_{01}|^2$$

- the post-measurement state

$$|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

Bell state or EPR pair

An important two qubit state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- measuring 1st qubit gives 2 possible results
  - 0 with the probability 1/2, and the post-measurement state $|00\rangle$
  - 1 with the probability 1/2, and the post-measurement state $|11\rangle$
- measuring 2nd qubit ALWAYS gives the same result with the 1st measuement

N qubits

A superposition of the 2^n states

$$|\psi\rangle = \sum_{x_n=0,or1} \alpha_{\dots}|x_1x_2\cdots x_n\rangle$$

Quantum computation

Single qubit gates

- classical NOT gate:

$$0 \rightarrow 1 \quad , \text{ and } 1 \rightarrow 0$$

- quantum NOT gate:

$$|0\rangle \rightarrow |1\rangle \quad , \; |1\rangle \rightarrow |0\rangle$$

- how about superposition state ?

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \beta|0\rangle + \alpha|1\rangle$$

Matrix representation

- the matrix representation of quantum NOT gate is:

$$\mathbf{X} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- $$\mathbf{X}|\mathbf{0}\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |\mathbf{1}\rangle$$

- $$\mathbf{X}(\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \beta|\mathbf{0}\rangle + \alpha|\mathbf{1}\rangle$$

What kinds of matrix can be a quantum gate ?

- We requires the normalization condition

$$|\alpha|^2 + |\beta|^2 = 1 \quad , \text{ for } |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- This will be hold after acting of the quantum.

$$|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

- It turns out the matrix repersenting the gate is the unitary matrix U

$$U^\dagger U = I$$

Another single qubit gates

- $\mathbf{Z}$ gate

$$\mathbf{Z} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Hadamard gate

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- $$\mathbf{H}(|\mathbf{0}\rangle) = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

  $$\mathbf{H}(|\mathbf{1}\rangle) = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- $$\mathbf{H}(\mathbf{H}|\psi\rangle) = (\mathbf{HH})|\psi\rangle = \mathbf{I}|\psi\rangle = |\psi\rangle$$

multiple gubit gates
the controlled-NOT(CNOT) gate

- if the control qubit is set to 0, then the target qubit left alone.

- if the control qubit is set to 1, then the target qubit is flipped.

- $|00\rangle \rightarrow |00\rangle$; $|01\rangle \rightarrow |01\rangle$; $|10\rangle \rightarrow |11\rangle$; $|11\rangle \rightarrow |10\rangle$.

- $|A, B\rangle \rightarrow |A, B \oplus A\rangle$, where $\oplus$ is addition modulo two

Controlled-NOT gate

matrix representation

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$U_{CN}^{\dagger} U_{CN} = I$$

Measurement gate

- a measurement gate performs the measurement

- measurement of a sigle qubit in the state $\alpha|0\rangle + \beta|1\rangle$ yields the result 0 or 1

- the state after measurement becomes $|0\rangle$ or $|1\rangle$

- the respective probabilities is $|\alpha|^2$ and $|\beta|^2$

Quantum circuits

- swap circuit

$$
\begin{aligned}
|a, b\rangle &\rightarrow |a, a \oplus b\rangle \\
&\rightarrow |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle \\
&\rightarrow |b, (a \oplus b) \oplus b\rangle = |b, a\rangle
\end{aligned}
$$

Quantum circuits

- no clone theory
  - if gate U could clone any quantum state $|\alpha\rangle$

$$U(|\alpha\rangle|0\rangle) = |\alpha\rangle|\alpha\rangle$$

  - U did not depend on $|\alpha\rangle$ alone

$$U(|\beta\rangle|0\rangle) = |\beta\rangle|\beta\rangle$$

  - what will happen if we want to clone $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$?

$$U(|\gamma\rangle|0\rangle) = U(|\alpha\rangle + |\beta\rangle)|0\rangle = |\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle \neq |\gamma\rangle|\gamma\rangle$$

Quantum circuits

- example: circuit to creat Bell state

- target state

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|\beta_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
|\beta_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|\beta_{11}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned}
$$