Chapter 8

Polygonal Inequality

1. Introduction

The η-expansion is a powerful tool for the research of neural network's behavior. Its coefficients are exhibited as solution of simple algebra equations. Polygonal inequality is one of the most useful methods for calculating them.

2. Notation

For a nets of N-elements, state of the nets:

$\xi = (\xi_1 , \xi_2 , \dots , \xi_N)$ $\qquad \xi_h = \pm 1$

Assume for each neuron h, the function is:

$$\xi_{h,m+1} = \sigma [F_h (\xi_m)] \qquad \dots(1)$$

or $\xi_{m+1} = \sigma [\boldsymbol{F}(\xi_m)]$ in vector form.

Here, F_h is any real function such that $F_h \neq 0$ for any input.

σ is the signum function ($\sigma(F) = 1$ if $F > 0$ and $\sigma(F) = -1$ if $F < 0$)

The vector ξ (dimension N) can be rewritten in tensor power space, which is now a 2^N dimensional vector. One convenient definition is:

$$\eta = \begin{pmatrix} \eta_0 \\ \eta_0 \\ \vdots \\ \eta_\alpha \\ \vdots \\ \eta_{2^N} \end{pmatrix} \equiv \left(\frac{1}{\sqrt{2}}\right)^N \underbrace{\begin{pmatrix} 1 \\ \xi_N \end{pmatrix} \times \begin{pmatrix} 1 \\ \xi_N \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 \\ \xi_1 \end{pmatrix}}_{\substack{N-times \\ call\cdot direct\cdot product}} = \begin{pmatrix} 1 \\ \xi_1 \\ \xi_2 \\ \xi_1\xi_2 \\ \xi_3 \\ \xi_1\xi_3 \\ \vdots \\ \xi_1 \cdots \xi_N \end{pmatrix} \qquad \dots(2)$$

3. η-expansion

By A. De Luca (1965), The most general Boolean function can be expanded as

$$\sigma\left[F\left(\vec{\xi}\right)\right]=\sum_{\alpha=0}^{2^N-1} f_\alpha \eta_\alpha \qquad \ldots(3)$$

with $$f_\alpha = \ <\eta_\alpha \sigma[F(\vec{\xi})]> \ = \ <\sigma[\eta_\alpha F(\vec{\xi})]> \qquad \ldots(4)$$

where we define the trace operation :

$$<G(\vec{\xi})> \ = \ \frac{1}{2^N} \sum_{\xi^1=\pm1,\xi^2=\pm1,\ldots,\xi^N=\pm1} G(\xi^1,\xi^2,\ldots\xi^N) \qquad \ldots(5)$$

Beware that the trace operation (“< >”) stands for some kind of “average” of the function G. So it is not a function of particular $\vec{\xi}$. Because we just sum over all the possible states, but not particular one. The <G> (trace of G) reflects the structure of the function.

The (3) and (4) equations can be written as such is shown by A. De Luca (1965), which in general:

$$\begin{aligned}&\sigma[\sum_j^N w_j\xi_j - T]\\ &=\sum_{s=0}^{N}\sum_{C(i_1,\ldots i_s)}^{1..N}\{\sum_{k=1}^{2N}(\sigma_{i_1}^{(k)}\ldots\sigma_{i_s}^{(k)})\sigma[\sum_{p=1}^{N} w_p\sigma_p^{(k)}-T]\}(\xi_{i_1}\ldots\xi_{i_s})\end{aligned} \qquad \ldots(6)$$

The first 2 summations have C(N,0)+C(N,1)+C(N,2)+…..C(N,N) terms = 2^N terms. Clearly (3) together with (4) is an compact expression for (6).

4. General Nets and Self Duality

For a general net, it can be described by:

$$\begin{cases}\text{(I)} & u_h(t+r)=1\left[\displaystyle\sum_{\substack{k=1..N\\ r=1..L}} a_{hk}^{(r)}u_k(t-r\gamma)-s_h\right]\\ \text{(II)} & w_h(t+r)=\displaystyle\sum_{\substack{k=1..N\\ r=1..L}} a_{hk}^{(r)}1\left[w_k(t-r\gamma)\right]-s_h\end{cases}$$

or $$\begin{cases}\vec{u}_{m+1}=1[A\vec{u}_m-\vec{s}\,]\\ \vec{w}_{m+1}=A1[\vec{w}_m]-\vec{s}\end{cases}$$ in vector form.

where 1[ ] is the step function, and “s” is the threshold. With the condition AI = 2S, the vector form of (I) and (II) become: $\begin{cases}\vec{u}_{m+1} = \sigma[A\vec{u}_m] \\ \vec{w}_{m+1} = A\sigma[\vec{w}_m]\end{cases}$

which means self-duality. This condition can simplify the discussion below remarkably.

5. Coefficient of η-expansion

There are totally 2^N coefficients of the η-expansion, but of course, not all of them are independent. We can find constraints / equations for them to solve the problem. We did find the following three trivial but important remarks did help:

$$\begin{cases}(a) & \sigma[F(\vec{\xi})G(\vec{\xi})] = \sigma[F(\vec{\xi})]\sigma[G(\vec{\xi})] \\ (b) & \sigma\{\sigma[F(\vec{\xi})]\} = \sigma[F(\vec{\xi})] \\ (c) & \left\{\sigma[F(\vec{\xi})]\right\}^2 = +1\end{cases}$$

(a) gives the r.h.s equality of (4).

By (b), we know that $\sigma[\sum f_\alpha \eta_\alpha] = \sum f_\alpha \eta_\alpha$

By (c), $\left[\sum f_\alpha \eta_\alpha\right]^2 = \sum f_\beta f_\gamma \eta_\beta \eta_\gamma = \sum g_\alpha \eta_\alpha = 1$

$$\text{with } \begin{cases} g_0 = \sum f_\alpha^2 = 1 \\ g_\alpha = 0 \qquad \alpha = 1..2^N - 1\end{cases} \qquad \ldots(7)$$

so there are totally 2^N equations for 2^N unknown f_α.

In the above calculation, we can surely rearrange and merge the $\eta_\beta \eta_\gamma$ terms into a η_α terms that give the coefficients expression g_α. But here we do not concern much about that because they are quadratic in f_α, and so, not very useful (maybe except the g_0 term) in resolving the f_α.

As an example, consider $\sigma[F(\xi_1, \xi_2)]$ and $\sigma[F(\xi_1, \xi_2, \xi_3)]$, both odd in ξ :

$$\sigma[F(\xi_1, \xi_2)] = f_1\xi_1 + f_2\xi_2 \qquad \text{with } \begin{cases} f_1^2 + f_2^2 = 1 \\ f_1 \cdot f_2 = 0\end{cases}$$

$$\sigma[F(\xi_1, \xi_2, \xi_3)] = f_1\xi_1 + f_2\xi_2 + f_3\xi_3 + f_{123}\xi_{123}$$

$$\text{with } \begin{cases} f_1^2 + f_2^2 + f_3^2 + f_{123}^2 = 1 \\ f_h f_k + f_l f_{123} = 0 \qquad h \neq k \neq l = 1,2,3\end{cases}$$

Note that the last 4 equations can be rearranged as to become:

$$\begin{cases}(f_1+f_2+f_3+f_{123})^2=1\\(f_1+f_{123})(f_2+f_3)=0\\(f_2+f_{123})(f_1+f_3)=0\\(f_3+f_{123})(f_1+f_2)=0\end{cases}$$

In general, this kind of factorization can be done, illustrating as the following: Let Γ be the set $\{\xi_1, \xi_2, \ldots \xi_N\}$. Now partition it into 2 subsets A and B. So $A=\{\xi_i, \xi_j, \ldots \xi_l\}$ (can also be empty subset) and B=Γ-A. Make all the variables in A (in B) to be the same value $=\xi_A$ ($=\xi_B$). This partition can be done in 2^N ways, or 2^{N-1} ways if A and B are considered exchangeable.

Let $\begin{cases}f_A=\sum\limits_{\alpha'\in A} f_{\alpha'}\\ f_B=\sum\limits_{\alpha''\in B} f_{\alpha''}\end{cases}$ …(8)

with these notation and ξ_A, ξ_B, one finds the following equations:

$$\begin{cases}(f_A+f_B)^2=(\sum\limits_{\alpha} f_\alpha)^2=1\\ f_A\cdot f_B=0\end{cases} \quad \ldots(9)$$

(here the first equation obtains if A or B is empty), they are indeed equivalent to (7)'s $\alpha=0..2^N-1$ equations.

Their solutions: $\begin{cases}f_A=\pm1\\ f_B=0\end{cases}$ *or* $\begin{cases}f_A=0\\ f_B=\pm1\end{cases}$

So it totally gives $2^{N-1}*2=2^N$ equations, of course, the same as (7).

From now, we will only consider linearly separable (l.s.) function. That is, functions of the form

$$\sigma\left[\sum_{h=1}^{N} x_h\xi_h\right]=\sum f_\alpha\eta_\alpha \qquad (require\ \sum_{h=1}^{N} x_h\xi_h\neq 0 \quad for\,any\ \vec{\xi})$$

It's known that the f_α term vanishes if η_α is even (i.e., $\eta_0,\eta_{ij},\eta_{hklm}\ldots$). Only odd term is left. For example, N=2, from η-expansion given by (3), (4):

$$F(\xi_1,\xi_2)\equiv a_1\xi_1+a_2\xi_2$$
$$\Rightarrow \sigma\left[a_1\xi_1+a_2\xi_2\right]=f_0+f_1\xi_1+f_2\xi_2+f_{12}\xi_1\xi_2$$

$$\begin{cases} f_0 = \frac{1}{4}\left[\underset{(\xi_1,\xi_2)=(1,1)}{\sigma(a_1+a_2)} + \underset{(\xi_1,\xi_2)=(1,-1)}{\sigma(a_1-a_2)} + \underset{(\xi_1,\xi_2)=(-1,1)}{\sigma(-a_1+a_2)} + \underset{(\xi_1,\xi_2)=(-1,-1)}{\sigma(-a_1-a_2)}\right] = 0 \\ f_1 = \frac{1}{4}\left[\sigma(a_1+a_2) + \sigma(a_1-a_2) - \sigma(-a_1+a_2) - \sigma(-a_1-a_2)\right] = \frac{1}{2}\left[\sigma(a_1+a_2) + \sigma(a_1-a_2)\right] \\ f_2 = \frac{1}{4}\left[\sigma(a_1+a_2) - \sigma(a_1-a_2) + \sigma(-a_1+a_2) - \sigma(-a_1-a_2)\right] = \frac{1}{2}\left[\sigma(a_1+a_2) - \sigma(a_1-a_2)\right] \\ f_{12} = \frac{1}{4}\left[\sigma(a_1+a_2) - \sigma(a_1-a_2) - \sigma(-a_1+a_2) + \sigma(-a_1-a_2)\right] = 0 \end{cases}$$

$$\therefore \quad \sigma\left[a_1\xi_1 + a_2\xi_2\right] = f_1\xi_1 + f_2\xi_2 \qquad and \begin{cases} f_1 = \frac{1}{2}\left[\sigma(a_1+a_2) + \sigma(a_1-a_2)\right] \\ f_2 = \frac{1}{2}\left[\sigma(a_1+a_2) - \sigma(a_1-a_2)\right] \end{cases}$$

Interestingly, we find that, $f_2(a_1,a_2) = f_1(a_2,a_1)$ …(10)

This symmetry is indeed can be obtained from a more general conclusion for l.s. function.

Let's write $\sigma(x^T\xi) = \sum_\alpha f_\alpha(x)\eta_\alpha = f^T(x)\eta$ …(11)

If P is a permutation matrix on ξ, and Π is the corresponding permutation matrix on η.

then
$$\sigma(x^T P\xi) = f^T(x)\Pi\eta = \left(\Pi^T f(x)\right)^T \eta$$
$$and \quad \sigma(x^T P\xi) = \sigma\left((P^T x)^T \xi\right) = f^T(P^T x)\eta \qquad from(11)$$

$\therefore \quad \Pi^T f(x) = f(P^T x)$, clearly (9) is only the N=2 case.

Consider the partition A, B again:

Let's write $\begin{cases} x_A = \sum_{\alpha' \in A} x_{\alpha'} \\ x_B = \sum_{\alpha'' \in B} x_{\alpha''} \end{cases}$ …(12)

Let ξ_A=1, ξ_B =-1 , by substituting them into $\sigma[x_1\xi_1 + x_2\xi_2] = \sum f_\alpha\eta_\alpha$, together with (8)

and the fact that all the even terms vanish (so $\eta_\alpha = +1 \;\; (\alpha \in A),\;\; \eta_\beta = -1 \;\; (\beta \in B)$) ,

we get:

$\sigma[x_A - x_B] = f_A - f_B$ …(13)

This equation, together with (9), completely gives all conditions for the l.s. function.

Or you can use the equivalent expression: $(x_A - x_B)(f_A - f_B) > 0$

So together with (9), can be used to solve the problems:

$$x_h \Rightarrow f_\alpha \qquad (x_h \;\; known, \;\; find \;\; f_\alpha)$$

or $$f_\alpha \Rightarrow x_h \qquad (f_\alpha \;\; known, \;\; find \;\; x_h)$$

6. Another approach : Polygonal Inequality

The works above give the conditions for f_α, that is, a set of equations. But generally they may not be solved easily. We will try another method, the polygonal inequality, which seems that it might be helpful. And finally we will show it indeed can solve for f_α of arbitrary l.s. function, after this method has been generalized.

We only consider l.s. function ($F(\xi)=\sum a_h\xi_h$) with condition :

$$a_1 \ge a_2 \ge ......a_N > 0 \qquad \ldots(14)$$

This condition is indeed, the canonical form of self-duality. It simplifies the calculation by the way that it makes all the even terms in the η-expansion vanish.

Then one readily has (Dertouzos, 1965) :

$$f_1 \ge f_2 \ge ......f_N \ge 0 \qquad \ldots(15)$$

Let's introduce new notation :

$$F^{(i_1,i_2,\ldots i_h)}(\vec{\xi}) = \sum_{i\neq i_1,i_2,\ldots i_h} a_i\xi_i$$

or $$F^{(\alpha)}(\vec{\xi}) = \sum_{i\neq i_1,i_2,\ldots i_h} a_i\xi_i \qquad (with \;\; \alpha \equiv (i_1,i_2,\ldots i_h)\;) \qquad \ldots(16)$$

recall that only the odd term left, the η-expansion of (3) (4), subjects to condition (14), can be written as:

$$\begin{aligned}\sigma\left[\sum_{h=1}^{N} a_h\xi_h\right] &= \sum_{h=1}^{N} f_h\xi_h + \sum_{h_1<h_2<h_3}^{1..N} f_{h_1h_2h_3}\xi_{h_1}\xi_{h_2}\xi_{h_3} + ...f_{12\ldots N}\xi_1\xi_2\ldots\xi_N \\ &= \sum f^{(1)}\xi^{(1)} + \sum f^{(3)}\xi^{(3)} + \sum f^{(5)}\xi^{(5)}......f^{(N)}\xi^{(N)}\end{aligned} \qquad \ldots(17)$$

Consider first

$$\begin{aligned} f_h &= \left\langle \xi^h \sigma\left[F(\vec{\xi})\right]\right\rangle \\ &= \frac{1}{2}\left(\left\langle \sigma\left[a_h + F^{(h)}(\vec{\xi})\right]\right\rangle + \left\langle \sigma\left[a_h - F^{(h)}(\vec{\xi})\right]\right\rangle\right) \qquad \ldots \; as \;\; \pm 1 \, for \, \xi_h \qquad \ldots(18) \\ &= \left\langle \sigma\left[a_h + F^{(h)}(\vec{\xi})\right]\right\rangle \qquad ...totally \;\; 2^{N-1} \;\; terms \end{aligned}$$

From the second equations of (18), its clear that the non-zero contribution to f_h come only from the configurations $(\xi_1,\xi_2,\ldots\xi_{h-1},\xi_{h+1},\ldots\xi_N)$ such that

$$\left|F^{(h)}(\vec{\xi})\right| < a_h \qquad \ldots(19)$$

Otherwise the two terms in the second equations will be cancelled out.

So $f_h = \frac{1}{2^{N-1}} \square_h$

where $\square_h$ denotes the number of configurations for which (19) holds.

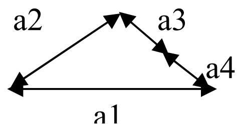
For example, If $a_1 > a_2 + a_3 + ... + a_N$ …(20)

then (19) always holds for h=1; thus, $\square_h = 2^{N-1}$, $f_1 = 1$, and all other $f_h = f_\alpha = 0$. (20) means that it is impossible to construct a polygon with sides $a_1, a_2, a_3, ..., a_N$. Otherwise, in general, $\square_h$ is twice the number of distinct triangles which one can construct by taking a_h as base side, and aligning along the two other sides with all the remaining $a_i \neq a_h$ (their ordering on each of these sides is immaterial).

To see how it is counted, let consider an example of N=4 :

First show all the possible configurations (here =8) of $F^{(1)}(\vec{\xi})$:

$$F^{(1)}(\xi) = \begin{array}{c} \begin{matrix} a_2 & a_3 & a_4 \end{matrix} \\ \left\{ \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{matrix} \right\} \end{array} \begin{matrix} \\ \\ \\ \\ \leftarrow a_1 > a_2 - (a_3 + a_4) \\ \leftarrow a_1 > (a_3 + a_4) - a_2 \\ \\ \\ \\ \end{matrix} \quad \text{…(21)}$$



for a triangle with a_1 as base side, assume we can construct a triangle like the one above, then we can write down their triangular inequalities :

$$\begin{cases} a_1 < a_2 + (a_3 + a_4) \\ a_2 < a_1 + (a_3 + a_4) \\ (a_3 + a_4) < (a_1 + a_2) \end{cases} \quad \text{rearrange as} \quad \begin{cases} a_1 < a_2 + (a_3 + a_4) \\ a_1 > a_2 - (a_3 + a_4) \\ a_1 > (a_3 + a_4) - a_2 \end{cases} \quad \text{…(22)}$$

The first equation of (22) is useless, but the second and third equations are identical to the configuration 4 and 5 in (21). So with a_i given, the number of states which satisfies (19) can be counted by the number of constructed triangles, then we can calculate the f_h.

For l.s. functions, the f_h or $\square_h$ are not independent from one another. For example:

If $a_1 \geq a_2 \geq a_3 > 0$, then $\sigma[a_1\xi_1 + a_2\xi_2 + a_3\xi_3] = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_3 - \frac{1}{2}\xi_1\xi_2\xi_3$

If $a_1 > a_2 + a_3$, then $\sigma[a_1\xi_1 + a_2\xi_2 + a_3\xi_3] = \xi_1$

7. Nonlinear coefficient

It's known that l.s. function is uniquely determined by its linear coefficients f_h. So all the higher terms should also be determined by them. Lets consider the trilinear terms of (17):

$$f_{hkl} = \left\langle \xi_h \xi_k \xi_l \sigma[F(\xi)] \right\rangle$$

together with lower terms, after cancellations, one finds:

$$f_{hkl} + f_h + f_k + f_l = \left\langle \sigma[a_h + a_k + a_l + F^{(h,k,l)}(\xi)] \right\rangle = \frac{1}{2^{N-3}} \square_{hkl} \qquad \ldots(23)$$

Again we use the trick and can show that the $\square_{hkl}$ is twice the number of distinct triangles by stretching on the base side the segments a_h, a_k, a_l as before. If no such a triangle is possible, that is, $a_h + a_k + a_l > \sum_{i \neq h,k,l} a_i$

then $\square_{hkl}$ = 2^{N-3}, $f_{hkl} + f_h + f_k + f_l = 1$. So, get higher terms if lower terms are already known.

The r.h.s. of (23) can also be obtained from the η-expansion of a l.s. function of N-2 variables:

$$\sigma\left[\sum_{i \neq h,k,l} a_i \xi_i + (a_h + a_k + a_l)\xi_m \right] \qquad \ldots(24)$$

now the l.h.s. of (23) is the coefficient the linear term ξ_m.

For all higher terms, E.R. Caianiello (1981) gives the equations:

$$f_{h_1 h_2 \ldots h_k} + \sum_{i_1 < i_2 \ldots < i_{k-2}}^{(h_1 \ldots h_k)} f_{i_1 i_2 \ldots i_{k-2}} + \ldots + \sum_{i_1 < i_2 < i_3}^{(h_1 \ldots h_k)} f_{i_1 i_2 i 3} + \sum_{i=h_1}^{h_k} f_i = \frac{1}{2^{N-k}} \square_{h_1 \ldots h_k} \qquad \ldots(25)$$

where $\square_{h_1 \ldots h_k} = 2^{N-k}$ if (20) hold with a_1 replace by $(a_{h_1} + a_{h_2} + \ldots + a_{h_k})$,or equals

twice the number of triangles with base side $(a_{h_1} + a_{h_2} + \ldots + a_{h_k})$.

8. General case

First let's extend to the case with threshold, that is, adding an extra constant terms a_0, and replacing the self dual condition (14) by simply $a_h \geq 0$ (for h>=0). Then the η-expansion can also include the even terms. But indeed nothing changes for the polygonal inequality principle, except that now $\square_{h_1 h_2 \ldots h_k}$ count the number of triangles (not twice now) which can be constructed by adding a_0 into the set $\{a_1,\ldots a_N\}$. If $a_0=0$, clearly reduce to the previous case.

Again, paper of E.R. Caianiello (1981) gives the whole equations (which is quite long …) for this general case (a_0 take into account) as an generalization of equation (25).

9. Calculating arbitrary l.s. functions

Previously we depend on condition (14) to save us a lots of labor. Although we have generalized it a little, but how to calculate an arbitrary l.s. function? In here show a transformation that help to transform any function into a special form (which then can be easily calculated).

Consider a special form of l.s. function :

$$\sigma\left[F(\xi)\right]=\sigma\left[\sum_{h=1}^{2n+1}\xi_h\right] \qquad \ldots(27)$$

where N=2n+1 guarantees that F≠0 always. And clearly

$$f_{h_1h_2\ldots h_{2n+1}}\equiv f_{1,2,\ldots 2n+1}\equiv f_{(2n+1)}$$

this reduces the number coefficients very much.

Then $\sigma\left[F(\xi)\right]=f_{(1)}\sum\xi^h+f_{(3)}\sum\limits_{h_1<h_2<h_3}\xi^{h_1}\xi^{h_2}\xi^{h_3}+\ldots+f_{(2n+1)}\xi^1\xi^2\ldots\xi^{2n+1}$ …(28)

It's coefficients equations given by (26) then can be much more simple, so that even it's solution can be write down immediately!!

$$\begin{cases} f_{(1)}=\dfrac{1}{2^{2n}}\dbinom{2n}{n}=\dfrac{(2n-1)!!}{(2n)!!} \\ f_{(3)}=\ldots \\ f_{(2h+1)=}(-1)^h\dfrac{(2n-2h-1)!!(2h-1)!!}{(2n)!!} \end{cases} \qquad \ldots(29)$$

and by $f_{(k)}=(-1)^{n+1}f_{(2n+2-k)}$ …(30)

even save half of the works of (28)!! So, η-expansion of l.s. functions of the form as (27) are indeed, can be easily solved.

For an arbitrary l.s. function $\sigma\left[\sum\limits_{h=1}^{N}a_h\xi_h\right]$ with a_h real and $\sum\limits_{h=1}^{N}a_h\xi_h\neq 0$ always.

There exists a ε such that : $0<N\varepsilon<\min\left|\sum\limits_{h=1}^{N}a_h\xi_h\right|$

and, for any a_h, two positive integer n_h, l_h such that $0 \le \dfrac{n_h}{l_h} - a_h = \varepsilon_h \le \varepsilon$

so $\left|\sum_{h=1}^{N} \varepsilon_h \xi_h\right| \le N\varepsilon < \min\left|\sum_{h=1}^{N} a_h \xi_h\right|$

The l.h.s. is so small that, add to the l.s. function , will not change the result, that is,

$$\begin{aligned}\sigma\left[\sum_{h=1}^{N} a_h \xi_h\right] &= \sigma\left[\sum_{h=1}^{N} a_h \xi_h + \sum_{h=1}^{N} \varepsilon_h \xi_h\right] \\ &\equiv \sigma\left[\sum_{h=1}^{N} \frac{n_h}{l_h} \xi_h\right] = \sigma\left[\sum_{h=1}^{N} m_h \xi_h\right]\end{aligned} \qquad \ldots(31)$$

here $m_h = n_h \prod_{k \ne h}^{1..N} l_k$ is an integer. So any function can be map to the corresponding m_h>=0 function.

Then we reduce the function farther as following:

$$\begin{cases} \square = m_1 + m_2 + \ldots + m_N \\ \xi^1 = \xi^{N+1} = \xi^{N+2} = \ldots = \xi^{N+m_1-1} \\ \xi^2 = \xi^{N+m_1} = \xi^{N+m_1+1} \ldots = \xi^{N+m_1+n_2-1} \\ \vdots \\ \xi^{\square} = \xi^{\square - m_N + 2} = \xi^{\square - m_N + 3} = \ldots = \xi^{\square} \end{cases} \qquad \ldots(32)$$

so
$$\begin{aligned}\sigma\left[\sum_{h=1}^{N} m_h \xi_h\right] &= \sigma\left[\sum_{i=1}^{\square} \xi_i\right]_{red.} \\ &= \left(f_{(1)}^{\square} \sum_{i=1}^{\square} \xi^i + f_{(3)}^{\square} \sum_{i_1<i_2<i_3}^{1..\square} \xi^{i_1}\xi^{i_2}\xi^{i_3} + \ldots + f_{(\square)}^{\square} \xi^1 \xi^2 \ldots \xi^{\square} \right)_{red.}\end{aligned}$$

as given by (28).

The general algorithms for calculating $\sigma\left[\sum_{h=1}^{N} m_h \xi_h\right]$ in terms of $\sigma\left[\sum_{i=1}^{\square} \xi_i\right]_{red.}$ is as follow:

$$\begin{aligned}\sigma\left[\sum_{h=1}^{N} m_h \xi_h\right] &= \underset{(N)}{f}^T \underset{(N)}{\eta} \\ &= \left(\underset{(\square)}{f}^T \underset{(\square)}{\eta} \right)_{red.} = \underset{(\square)}{f}^T \Delta^T \underset{(N)}{\eta}\end{aligned}$$

so $\underset{(N)}{f} = \Delta \underset{(\square)}{f}$

that is , from the coefficients of $\underset{(\square)}{f}$ to coefficients $\underset{(N)}{f}$.

The Λ is a $2^N \times 2^{\square}$ matrix, with element $\lambda_{\alpha,\beta}$ =1 or 0, with notation:

$\alpha \equiv (i_1, i_2 \ldots i_h)$ is index of $\underset{(N)}{f}$

$\beta \equiv (j_1, j_2 \ldots j_h)$ is index of $\underset{(\square)}{f}$

$\beta' \equiv (j'_1, j'_2 \ldots j'_h)$ is the reduce index from β given by rules (32)

Then $\lambda_{\alpha,\beta}$ =1 (if $\alpha=\beta'$) and $\lambda_{\alpha,\beta}$ =0 (if $\alpha\neq\beta'$).

So here are the steps for solving the most general case:

$$\sigma\left[\sum_{h=1}^{N} a_h \xi_h\right] \Rightarrow \sigma\left[\sum_{h=1}^{N} m_h \xi_h\right] \Rightarrow \sigma\left[\sum_{i=1}^{\square} \xi_i\right]_{red.} \Rightarrow \underset{(\square)}{f} \Rightarrow \Delta \Rightarrow \underset{(N)}{f}$$