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Solving Structural Optimization Problems via Semidefinite Programming

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Outline

- A structural optimization problem
- Problem formulation
- Semidefinite relaxation
- Preliminary numerical results
- Conclusions

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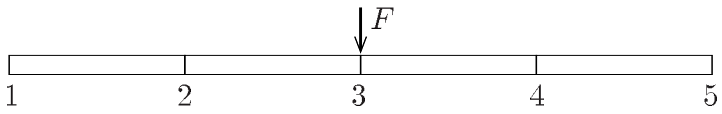
A Structural Optimization Problem

- Consider a beam with 5 nodes



- A force F is at the third node
- We would like to choose 2 nodes as supports from node 1, 2, 4, 5 such that mean compliance is the minimum
- At each node, there are two degrees of freedom: displacement and bending
- If node 1 and 4 are fixed, the stiffness equations and the stiffness matrix K are

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$$
\begin{bmatrix}
12 & 6L & -12 & 6L & & & & & & \\
6L & 4L^2 & -6L & 2L^2 & & & & & & \\
-12 & -6L & 24 & -12L & -12 & 6L & & & & \\
6L & 2L^2 & -12L & 8L^2 & -6L & 2L^2 & & & & \\
 & & -12 & -6L & \ddots & & & & & \\
 & & 6L & 2L^2 & & \ddots & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & & & 12 & -6L \\
 & & & & & & & & -6L & 4L^2
\end{bmatrix}
\begin{bmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \\ 0 \\ \theta_5 \end{bmatrix}
$$

$$
= \frac{L^3}{EI} \begin{bmatrix} f_1 & 0 & 0 & 0 & -F & 0 & 0 & 0 & f_2 & 0 \end{bmatrix}^T
$$

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- If node 1 and 5 are chosen, $\Rightarrow$ solve a system with 8 equations and 8 variables. $\Rightarrow$ Then obtain f_1 and f_2.
- Then $[v_1 \ldots v_5\ \theta_5]^T K [v_1 \ldots v_5\ \theta_5]$ is the objective value
- The general situation:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

- f_1: fixed, u: variables will be chosen

$$v = K_{11}^{-1}(f_1 - K_{12}u)$$
$$(K_{22} - K_{12}^T K_{11}^{-1} K_{12})u = f_2 - K_{12}^T K_{11}^{-1} f_1$$

- Let $S = K_{22} - K_{12}^T K_{11}^{-1} K_{12}, f = f_2 - K_{12}^T K_{11}^{-1} f_1 \Rightarrow Su = f$.

$$\text{the objective value} = [v_1\ v_2]^T K = f_1^T f_1 + u^T S u$$

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A quadratic mixed integer programming problem

$$\begin{aligned} \min \quad & u^T S u \\ & x_i(Su - f)_i = 0, i = 1, \ldots, N \\ & x_i(1 - x_i) = 0, \text{ for } i = 1, \cdots, N \\ & \sum_{i=1}^{N} x_i = N - n \ \ (n \ll N) \end{aligned}$$

- $x_i = 0$ or 1, $x_i = 0$: support
- There are $\binom{N}{n}$ combinations. Once an x is chosen, a linear system has to be solved.
- Linear programming relaxation: $0 \le x_i \le 1, i = 1, \ldots, N$.

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Semidefinite relaxation

- The original application is equivalent to

$$\min C \bullet X$$
$$A_i \bullet = b_i, i = 1, \ldots, m$$
$$X \succeq 0, X \text{ rank } 1,$$

  where $X \succeq 0$ means positive semidefinite, and $A \bullet B = \text{trace}(A^T B)$.
- Drop rank 1 constraint $\Rightarrow$ an SDP relaxation
- Semidefinite relaxation has been very useful for some difficult combinatorial problems
- $u^T S u = S \bullet (uu^T)$: all equations must be in quadratic forms.

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- Introducing a variable w_0:

$$\begin{array}{lrcll} \min & u^T S u & & & \\ & x_i(Su)_i + f_i x_i w_0 & = & 0 & \text{for } i = 1, \cdots, N \\ & x_i^2 - x_i w_0 & = & 0 & \text{for } i = 1, \cdots, N \\ & \sum_{i=1}^N x_i^2 & = & N - n & \\ & w_0^2 & = & 1 & \end{array}$$

with $y^T = (x^T, u^T, w_0)$, this gives the equivalent formulation:

$$\begin{array}{llcll} \min & C \bullet yy^T & & & \\ & A_i \bullet yy^T & = & 0 & i = 1, \cdots, N \\ & B_i \bullet yy^T & = & 0 & i = 1, \cdots, N \\ & A_0 \bullet yy^T & = & 1 & \\ & B_0 \bullet yy^T & = & N - n \ (n \ll N) & \end{array}$$

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where

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_i = \begin{bmatrix} 0 & \frac{1}{2}U_i & \frac{1}{2}f_i \\ \frac{1}{2}U_i^T & 0 & 0 \\ \frac{1}{2}f_i^T & 0 & 0 \end{bmatrix}$$

$$B_i = \begin{bmatrix} D_i & 0 & -\frac{1}{2}V_i \\ 0 & 0 & 0 \\ -\frac{1}{2}V_i^T & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $x^T V_i w_0 = x_i w_0$, $x^T D_i x = x_i^2$ and $x^T U_i u = x_i (Su)_i$.

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Thus, the semidefinite relaxation:

$$\begin{array}{llcll} \min & C \bullet X & & & \\ & A_i \bullet X & = & 0 & i = 1, \cdots, N \\ & B_i \bullet X & = & 0 & i = 1, \cdots, N \\ & A_0 \bullet X & = & 1 & \\ & B_0 \bullet X & = & N - n & \\ & X \succeq & 0 & & \end{array}$$

- Other constraints: $(1 - x_i) u_i = 0$

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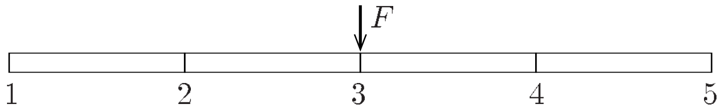
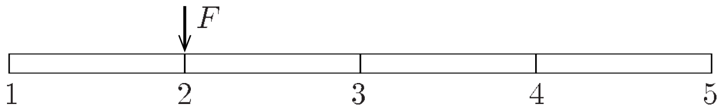
Numerical Example: Bean Elements

- Select two supports
  diag(X) = [0.7355, **0.2781**, **0.2781**, 0.7355].
  solution: node 2 and 3



- diag(X) = [**0.0148**, **0.1269**, 0.8856, 0.9861].
  solution: node 1 and 3



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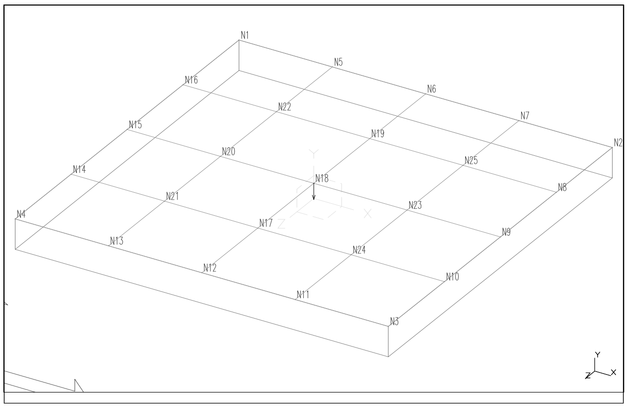
Numerical Example: Shell Elements



- Assumption: 5 degrees of freedom: no rotation of the normal direction
- A force at the center
  Try to find four supports on edges

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diag(X) = [0.9291, 0.9291, 0.9291, 0.9291, 0.9851, **0.1008**, 0.9851, 0.9851, **0.1008**, 0.9851, 0.9851, **0.1008**, 0.9851, 0.9851, **0.1008**, 0.9851]
solution = node 6, 9, 12, 15

- A force at node 20
  diag(X) = [0.9854, 0.9463, 0.9463, 0.9854, 0.9177, **0.3627**, 0.9921, 0.9910, 0.6330, 0.9910, 0.9921, **0.3627**, 0.9177, 0.9839, **0.0090**, 0.9839]
  solution = node 6, 12, 15

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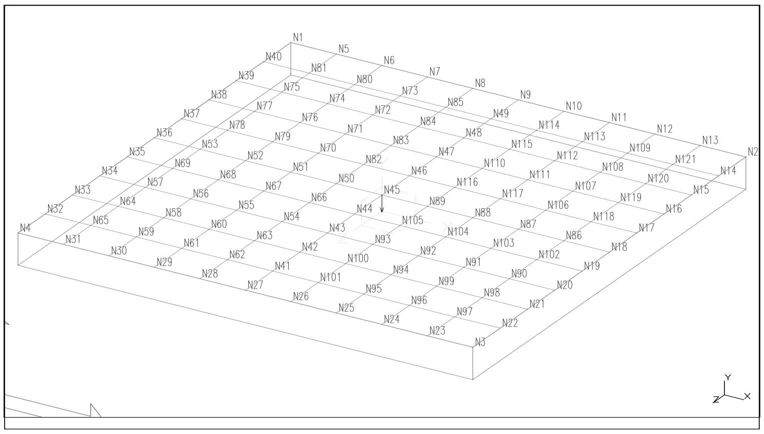
Numerical Example: Shell Elements



- A force at the center
  diag(X) = [**0.5605**, **0.5605**, **0.5621**, **0.5557**, 0.9969, 0.9755, 0.9142, 0.8939, 0.8772, 0.8941, 0.9161, 0.9753, 0.9969, 0.9968, 0.9754, 0.9151, 0.8935, 0.8787, 0.8921, 0.9166, 0.9748, 0.9966,

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0.9969, 0.9756, 0.9151, 0.8917, 0.8828, 0.8915, 0.9166, 0.9755, 0.9969, 0.9967, 0.9751, 0.9140, 0.8925, 0.8790, 0.8941, 0.9151, 0.9755, 0.9968]

- The SDP relaxation : 161 by 161 matrix as the variable, 282 constraints $\Rightarrow$ already a large SDP problem
- Need to add more combinatorial cuts or cuts from the problem itself.

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Conclusions and Future Research

- We have shown how to generate SDP relaxation for this particular structural optimization problem
- The current numerical results shows the potential but more tests need to be done
- Adding cuts to SDP relaxation is important
- How to generate solutions from SDP relaxation ?
- The comparison with existing method: penalty function method
- Issues related to large-scale SDP problems